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ON MINKOWSKI AND HERMITE–HADAMARD INTEGRAL INEQUALITIES VIA FRACTIONAL INTEGRATION

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ABSTRACT. In this paper, we use the the Riemann–Liouville fractional integral to develop some new results related to the Hermite–Hadamard inequality. Other integral inequalities related to the Minkowsky inequality are also established. Our results have some relationships with [E. Set, M. E. Ozdemir and S.S. Dragomir, J. Inequal. Appl. **2010**, Art. ID 148102, 9 pp.] and [L. Bougoffa, J. Inequal. Pure and Appl. Math. **7** (2006), no. 2, Article 60, 3 pp.]. Some interested inequalities of these references can be deduced as some special cases.

1. INTRODUCTION AND PRELIMINARIES

In recent years, inequalities are playing a very significant role in all fields of mathematics, and present a very active and attractive field of research. As example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals [2, 9, 10]. One of the famous integral inequalities is

$$\frac{f(a+b)}{2} \leq \frac{2}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where f is a convex function [7].

The history of this inequality begins with the paper of Ch. Hermite [8] and J. Hadamard [7] in the years 1883-1893, see C.P. Niculescu and L.E. Persson [11] and the references therein for some historical notes of Hermite–Hadamard inequality. Many researchers have given considerable attention to (1) and a number of extensions and generalizations have appeared in the literature, see [1, 4, 5].

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The aim of this paper is to establish several new integral inequalities for nonnegative and integrable functions that are related to the Hermite–Hadamard result using the Riemann–Liouville fractional integral. Other integral inequalities related to the Minkowski inequality are also established. Our results have some relationships with [3, 12]. Some interested inequalities of these references can be deduced as some special cases.
We shall introduce the following definitions and properties which are used throughout this paper.

Definition 1.1. A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0, \infty[)$.

Definition 1.2. A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in \mathbb{R}$, if $f^{(n)} \in C_\mu$.

Definition 1.3. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau; \quad \alpha > 0, t > 0,$$
$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.
For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0.$$

More details, one can consult [6].

2. Main Results

Our first result is the following reverse Minkowski fractional integral inequality

Theorem 2.1. *Let* $\alpha > 0, p \geq 1$ *and let* f, g *be two positive functions on* $[0, \infty[$, *such that for all* $t > 0, J^\alpha f^p(t) < \infty, J^\alpha g^p(t) < \infty$. *If* $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0, t]$, *then we have*

$$\Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}} + \Big[J^\alpha g^p(t)\Big]^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \Big[J^\alpha (f+g)^p(t)\Big]^{\frac{1}{p}}. \tag{2.1}$$

Proof. Using the condition $\frac{f(\tau)}{g(\tau)} < M, \tau \in [0, t], t > 0$, we can write

$$(M+1)^p f^p(\tau) \leq M^p (f+g)^p(\tau). \tag{2.2}$$

Multiplying both sides of (2.2) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0, t)$, then integrating the resulting inequalities with respect to τ over $(0, t)$, we obtain

$$\frac{(M+1)^p}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f^p(\tau) d\tau$$

$$\leq \tfrac{M^p}{\Gamma(\alpha)} \textstyle\int_0^t (t-\tau)^{\alpha-1} (f+g)^p(\tau) d\tau \,,$$

which is equivalent to

$$J^\alpha f^p(t) \leq \frac{M^p}{(M+1)^p} J^\alpha (f+g)^p(t).$$

Hence, we can write

$$\Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}} \leq \frac{M}{M+1}\Big[J^\alpha (f+g)^p(t)\Big]^{\frac{1}{p}}. \tag{2.3}$$

On the other hand, using the condition $mg(\tau) \leq f(\tau)$, we can write

$$(1+\frac{1}{m})g(\tau) \leq \frac{1}{m}(f(\tau)+g(\tau)).$$

Therefore,

$$\Big(1+\frac{1}{m}\Big)^p g^p(\tau) \leq \Big(\frac{1}{m}\Big)^p\Big(f(\tau)+g(\tau)\Big)^p. \tag{2.4}$$

Now, multiplying both sides of (2.4) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0,t)$, then integrating the resulting inequalities with respect to τ over $(0,t)$, we obtain

$$\Big[J^\alpha g^p(t)\Big]^{\frac{1}{p}} \leq \frac{1}{m+1}\Big[J^\alpha (f+g)^p(t)\Big]^{\frac{1}{p}}. \tag{2.5}$$

Adding the inequalities (2.3) and (2.5), we obtain the inequality (2.1). □

Remark 2.2. Applying Theorem 2.1 for $\alpha = 1$, we obtain [3, Theorem 1.2] on $[0,t]$.

Our second result is the following

Theorem 2.3. *Let* $\alpha > 0, p \geq 1$ *and let* f, g *be two positive functions on* $[0,\infty[$, *such that for all* $t > 0, J^\alpha f^p(t) < \infty, J^\alpha g^p(t) < \infty$. *If* $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0,t]$, *then we have*

$$\Big[J^\alpha f^p(t)\Big]^{\frac{2}{p}} + \Big[J^\alpha g^p(t)\Big]^{\frac{2}{p}} \geq \Big(\frac{(M+1)(m+1)}{M} - 2\Big)\Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}}\Big[J^\alpha g^p(t)\Big]^{\frac{1}{p}}. \tag{2.6}$$

Proof. Multiplying the inequalities (2.3) and (2.5), we obtain

$$\frac{(M+1)(m+1)}{M}\Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}}\Big[J^\alpha g^p(t)\Big]^{\frac{1}{p}} \leq \Big(\Big[J^\alpha (f(t)+g(t))^p\Big]^{\frac{1}{p}}\Big)^2. \tag{2.7}$$

Applying Minkowski inequality to the right hand side of (2.7), we get

$$\Big(\Big[J^\alpha (f(t)+g(t))^p\Big]^{\frac{1}{p}}\Big)^2 \leq \Big(\Big[J^\alpha f^p(t)\Big)^{\frac{1}{p}} + \Big(J^\alpha g^p(t)\Big]^{\frac{1}{p}}\Big)^2.$$

It follows then that,

$$\Big[J^\alpha (f(t)+g(t))^p\Big]^{\frac{2}{p}} \leq \Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}} + \Big[J^\alpha g^p(t)\Big]^{\frac{2}{p}} + 2\Big[J^\alpha f^p(t)\Big]^{\frac{1}{p}}\Big[J^\alpha g^p(t)\Big]^{\frac{1}{p}}. \tag{2.8}$$

Using (2.7) and (2.8), we obtain (2.6).
Theorem 2.3 is thus proved.

□

Remark 2.4. Applying Theorem 2.3 for $\alpha = 1$, we obtain [12, Theorem 2.2] on $[0, t]$.

We further have

Theorem 2.5. *Let $\alpha > 0, p > 1, q > 1$ and let f, g be two positive functions on $[0, \infty[$. If f^p, g^q are two concave functions on $[0, \infty[$, then we have*

$$\begin{aligned} &2^{-p-q} \left(f(0) + f(t)\right)^p \left(g(0) + g(t)\right)^q \left(J^\alpha \left(t^{\alpha-1}\right)\right)^2 \\ &\leq J^\alpha \left(t^{\alpha-1} f^p(t)\right) J^\alpha \left(t^{\alpha-1} g^q(t)\right). \end{aligned} \tag{2.9}$$

To prove this theorem, we need the following lemma.

Lemma 2.6. *Let h be a concave function on $[a, b]$. Then we have*

$$h(a) + h(b) \leq h(b + a - x) + h(x) \leq 2h(\frac{a+b}{2}). \tag{2.10}$$

Proof. Let h be a concave function on $[a, b]$. Then we can write

$$h(\frac{a+b+-x}{2}) = h(\frac{a+b}{2}) \geq \frac{h(b+a-x) + h(x)}{2}. \tag{2.11}$$

If we choose $x = \lambda a + (1 - \lambda)b$, then we have

$$\begin{aligned} &\frac{1}{2}\Big(h(a + b - \lambda a - (1 - \lambda)b) + h(\lambda a + (1 - \lambda)b)\Big) \\ &= \tfrac{1}{2}\Big(h(\lambda b - (1 - \lambda)a) + h(\lambda a + (1 - \lambda)b)\Big). \end{aligned}$$

Using the concavity of h, we obtain

$$\frac{1}{2}\Big(h(\lambda b - (1 - \lambda)a) + h(\lambda a + (1 - \lambda)b)\Big) \geq \frac{1}{2}\Big(h(a) + h(b)\Big) \tag{2.12}$$

By (2.11) and (2.12), we get (2.10).

□

Proof of Theorem 2.5. Since the f^p and g^q are concave functions on $[0, \infty[$, then by Lemma 2.6, for any $t > 0$, we have

$$f^p(0) + f^p(t) \leq f^p(t - \tau) + f^p(\tau) \leq 2f^p(\frac{t}{2}) \tag{2.13}$$

and

$$g^q(0) + g^q(t) \leq f^q(t - \tau) + g^q(\tau) \leq 2g^q(\frac{t}{2}). \tag{2.14}$$

Multiplying both sides of (2.13) and (2.14) by $\frac{(t-\tau)^{\alpha-1}\tau^{\alpha-1}}{\Gamma(\alpha)}$; $\tau \in (0,t)$, then integrating the resulting inequalities with respect to τ over $(0,t)$, we obtain

$$\begin{aligned} &\frac{f^p(0)+f^p(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau \\ &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}f^p(t-\tau)d\tau + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}f^p(\tau)d\tau \\ &\leq \tfrac{2f^p(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\frac{g^q(0)+g^q(t)}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau \\ &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}g^q(t-\tau)d\tau + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}g^q(\tau)d\tau \\ &\leq \tfrac{2g^q(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau. \end{aligned} \tag{2.16}$$

Using the change of variables $t-\tau=y$, we can write

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}f^p(t-\tau)d\tau = J^\alpha\left(t^{\alpha-1}f^p(t)\right) \tag{2.17}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}g^q(t-\tau)d\tau = J^\alpha\left(t^{\alpha-1}g^q(t)\right). \tag{2.18}$$

Now, using (2.15) and (2.17), we get

$$\left(f^p(0)+f^p(t)\right)\left(J^\alpha\left(t^{\alpha-1}\right)\right) \leq 2J^\alpha\left(t^{\alpha-1}f^p(t)\right) \leq 2f^p(\frac{t}{2})\left(J^\alpha\left(t^{\alpha-1}\right)\right). \tag{2.19}$$

For g, we use (2.16) and (2.18). We obtain

$$\left(g^q(0)+g^q(t)\right)\left(J^\alpha\left(t^{\alpha-1}\right)\right) \leq 2J^\alpha\left(t^{\alpha-1}g^q(t)\right) \leq 2g^q(\frac{t}{2})\left(J^\alpha\left(t^{\alpha-1}\right)\right). \tag{2.20}$$

The inequalities (2.19) and (2.20) imply that

$$\left(f^p(0)+f^p(t)\right)\left(g^q(0)+g^q(t)\right)\left(J^\alpha\left(t^{\alpha-1}\right)\right)^2 \leq 4J^\alpha\left(t^{\alpha-1}f^p(t)\right)J^\alpha\left(t^{\alpha-1}g^q(t)\right). \tag{2.21}$$

On the other hand, since f and g are positive functions, then for any $t>0, p\geq 1, q\geq 1$, we have

$$\left(\frac{(f^p(0)+f^p(t))}{2}\right)^{\frac{1}{p}} \geq 2^{-1}(f(0)+f(t))$$

and

$$\left(\frac{(g^q(0)+g^q(t))}{2}\right)^{\frac{1}{q}} \geq 2^{-1}(g(0)+g(t)).$$

Hence, we obtain

$$\frac{(f^p(0)+f^p(t))}{2}J^\alpha\left(t^{\alpha-1}\right)\geq 2^{-p}(f(0)+f(t))^pJ^\alpha\left(t^{\alpha-1}\right) \tag{2.22}$$

and

$$\frac{(g^q(0)+g^q(t))}{2}J^\alpha\left(t^{\alpha-1}\right)\geq 2^{-q}(g(0)+g(t))^qJ^\alpha\left(t^{\alpha-1}\right). \tag{2.23}$$

The inequalities (2.22) and (2.23) imply that

$$\begin{gathered}\frac{(f^p(0)+f^p(t))(g^q(0)+g^q(t))}{4}\Big[J^\alpha(t^{\alpha-1})\Big]^2\\ \geq 2^{-p-q}(f(0)+f(t))^p(g(0)+g(t))^q\Big[J^\alpha(t^{\alpha-1})\Big]^2.\end{gathered} \tag{2.24}$$

Combining (2.21) and (2.24), we obtain the desired inequality (2.9).

Remark 2.7. Applying Theorem 2.5 for $\alpha=1$, we obtain [12, Theorem 2.3] on $[0,t]$.

Theorem 2.8. *Let $\alpha>0,\beta>0,p>1,q>1$ and let f,g be two positive functions on $[0,\infty[$. If f^p,g^q are two concave functions on $[0,\infty[$, then we have*

$$\begin{gathered}2^{2-p-q}(f(0)+f(t))^p(g(0)+g(t))^q\Big[J^\alpha(t^{\beta-1})\Big]^2\\ \leq\Big[\tfrac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}f^p(t)\right)+J^\alpha\left(t^{\beta-1}f^p(t)\right)\Big]\Big[\tfrac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}g^q(t)\right)+J^\alpha\left(t^{\beta-1}g^q(t)\right)\Big].\end{gathered} \tag{2.25}$$

Proof. Multiplying both sides of (2.13) and (2.14) by $\frac{(t-\tau)^{\alpha-1}\tau^{\beta-1}}{\Gamma(\alpha)}$; $\tau\in(0,t)$, then integrating the resulting inequalities with respect to τ over $(0,t)$, we obtain

$$\begin{gathered}\frac{f^p(0)+f^p(t)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau\\ \leq\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(t-\tau)d\tau+\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(\tau)d\tau\\ \leq\tfrac{2f^p(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau\end{gathered} \tag{2.26}$$

and

$$\begin{gathered}\frac{g^q(0)+g^q(t)}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau\\ \leq\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(t-\tau)d\tau+\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(\tau)d\tau\\ \leq\tfrac{2g^q(\frac{t}{2})}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\beta-1}d\tau.\end{gathered} \tag{2.27}$$

Using the change of variables $t-\tau=y$, we obtain

$$\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}f^p(t-\tau)d\tau=\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}f^p(t)\right) \tag{2.28}$$

and

$$\frac{\Gamma(\beta)}{\Gamma(\beta)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}g^q(t-\tau)d\tau=\frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}g^q(t)\right) \tag{2.29}$$

By the relations (2.26) and (2.28), we can state that

$$\begin{aligned}(f^p(0)+f^p(t))\left(J^\alpha\left(t^{\beta-1}\right)\right) &\leq \frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}f^p(t)\right)+J^\alpha\left(t^{\beta-1}f^p(t)\right)\\ &\leq 2f^p(\frac{t}{2})\left(J^\alpha\left(t^{\beta-1}\right)\right) \qquad (2.30)\end{aligned}$$

and with (2.27) and (2.29), we can write

$$\begin{aligned}(g^q(0)+g^q(t))\left(J^\alpha\left(t^{\beta-1}\right)\right) &\leq \frac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}g^q(t)\right)+J^\alpha\left(t^{\beta-1}g^q(t)\right)\\ &\leq 2g^q(\frac{t}{2})\left(J^\alpha\left(t^{\beta-1}\right)\right). \qquad (2.31)\end{aligned}$$

The inequalities (2.30) and (2.31) imply that

$$\begin{aligned}&(f^p(0)+f^p(t))\,(g^q(0)+g^q(t))\,(J^\alpha\left(t^{\beta-1}\right))^2\\ &\leq \left[\tfrac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}f^p(t)\right)+J^\alpha\left(t^{\beta-1}f^p(t)\right)\right]\left[\tfrac{\Gamma(\beta)}{\Gamma(\alpha)}J^\beta\left(t^{\alpha-1}g^q(t)\right)+J^\alpha\left(t^{\beta-1}g^q(t)\right)\right].\end{aligned} \tag{2.32}$$

As before, since f and g are positive functions, then for any $t>0, p\geq 1, q\geq 1$, we have

$$\frac{(f^p(0)+f^p(t))}{2}J^\alpha\left(t^{\beta-1}\right)\geq 2^{-p}(f(0)+f(t))^pJ^\alpha\left(t^{\beta-1}\right) \tag{2.33}$$

and

$$\frac{(g^q(0)+g^q(t))}{2}J^\alpha\left(t^{\beta-1}\right)\geq 2^{-q}(g(0)+g(t))^qJ^\alpha\left(t^{\beta-1}\right). \tag{2.34}$$

The inequalities (2.33) and (2.34) imply that

$$\begin{aligned}&\frac{(f^p(0)+f^p(t))(g^q(0)+g^q(t))}{4}\left[J^\alpha(t^{\beta-1})\right]^2\\ &\geq 2^{-p-q}(f(0)+f(t))^p(g(0)+g(t))^q\left[J^\alpha(t^{\beta-1})\right]^2.\end{aligned} \tag{2.35}$$

Combining (2.32) and (2.35), we obtain the desired inequality (2.25). □

Remark 2.9. Applying Theorem 2.8 for $\alpha=\beta$, we obtain Theorem 2.5.

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