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A SIMPLE OBSERVATION ABOUT COMPACTNESS AND FAST DECAY OF FOURIER COEFFICIENTS

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ABSTRACT. Let X be a Banach space and suppose $Y \subseteq X$ is a Banach space compactly embedded into X, and (a_k) is a weakly null sequence of functionals in X^*. Then there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that $|a_n(y)| \leq \varepsilon_n \|y\|_Y$ for every $n \in \mathbb{N}$ and every $y \in Y$. We prove this result and we use it for the study of fast decay of Fourier coefficients in $L^p(\mathbb{T})$ and frame coefficients in the Hilbert setting.

1. Motivation

One of the classical problems in Harmonic Analysis is the study of the relationship that exists between decay properties of the Fourier coefficients $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ of a 2π-periodic function $f : \mathbb{T} \to \mathbb{C}$ and its membership to several function spaces. Just to mention a few well known examples, we show the following list of results:

- Riemann-Lebesgue Lemma sates that for $f \in L^1(\mathbb{T})$ the Fourier coefficients satisfy $\lim_{n\to\pm\infty} |c_n(f)| = 0$.
- Parseval's identity states that $f \in L^2(\mathbb{T})$ if and only if $\{c_n(f)\} \in \ell^2(\mathbb{Z})$.
- For $1 < p \leq 2$, Hausdorff-Young's inequality states that if $f \in L^p(\mathbb{T})$ then $\{c_n(f)\} \in \ell^q(\mathbb{Z})$, where $\frac{1}{p} + \frac{1}{q} = 1$, and $\|\{c_n(f)\}\|_{\ell^q(\mathbb{Z})} \leq \|f\|_{L^p(\mathbb{T})}$.
- If $p > 2$ and $f \in L^p(\mathbb{T})$, then $f \in L^2(\mathbb{T})$ and $\{c_n(f)\} \in \ell^2(\mathbb{Z})$.
- De Leeuw, Kahane and Katznelson [1, Theorem 2.1, page 278] proved that for any sequence $\{c_n\} \in \ell^2(\mathbb{Z})$ there exists a continuous function $f \in \mathbf{C}(\mathbb{T})$ such that $|c_n(f)| \geq c_n$ for all $n \in \mathbb{Z}$.

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Moreover, other related results stand up the relationship that exists between fast decay of Fourier coefficients of a function f and its smoothness properties:

- If f is absolutely continuous and periodic then $c_n(f) = \mathbf{o}(1/n)$.
- If f is periodic, k-times differentiable and $f^{(k-1)}$ is absolutely continuous then $c_n(f) = \mathbf{o}(1/n^k)$.
- If f is of bounded variation on $\mathbb{T}$ then $c_n(f) \leq \frac{V_{[0,2\pi]}(f)}{2\pi|n|}$ for all $n \neq 0$.
- If $f \in \mathbf{Lip}_\alpha(\mathbb{T})$ then $c_n(f) = \mathbf{O}(n^{-\alpha})$.

In this note we prove a quite general result about functionals which implies that, for $1 \leq p < \infty$, associated to any Banach space Y compactly embedded into $L^p(\mathbb{T})$ there is a decreasing sequence $\{\varepsilon_n\} \searrow 0$ such that $|c_n(y)| \leq \varepsilon_n \|y\|_Y$ for all $n \in \mathbb{Z}$ and $y \in Y$. We also prove an analogous result for frames in the Hilbert space setting.

2. The main result

Proposition 2.1. *Let X be a Banach space and suppose $Y \subseteq X$ is a Banach space compactly embedded into X, and (a_k) is a weakly null sequence of functionals in X^*. Then there exists a sequence $\{\varepsilon_n\} \searrow 0$ such that $|a_n(y)| \leq \varepsilon_n \|y\|_Y$ for every $n \in \mathbb{N}$ and every $y \in Y$.*

Proof. It follows from Banach-Steinhaus Theorem and the hypothesis of (a_n) being weakly null, that $\sup_n \|a_n\| = M < \infty$. For each $m \in \mathbb{N}$, find a finite set $S_m \subset \{y \in Y : \|y\|_Y \leq 1\}$, such that, for every $y \in Y$ with $\|y\|_Y \leq 1$, there exists $z \in S_m$ with $\|y - z\|_X < 1/(2Mm)$. Then there exists a sequence $N_1 < N_2 < \dots$ such that $|a_k(z)| < 1/(2m)$ for any $z \in S_m$ and $k \geq N_m$. By the triangle inequality, $|a_k(y)| < 1/m$ for any $k \geq N_m$, and any y in the unit ball of Y. Now define $\varepsilon_i = 1/m$ for $N_m \leq i < N_{m+1}$. For $i < N_1$, let $\varepsilon_i = M$. We have shown that $|a_i(y)| < \varepsilon_i \|y\|_Y$ for any i. □

Remark 2.2. Note that existence of $\{\varepsilon_n\} \searrow 0$ such that $|a_n(y)| \leq \varepsilon_n \|y\|_Y$ for every $n \in \mathbb{N}$ and every $y \in Y$ is equivalent to claim that $\lim_{n\to\infty} \|(a_n)_{|Y}\|_{Y^*} = 0$.

Corollary 2.3. *Let $1 \leq p < \infty$ and let Y be a Banach space compactly embedded into $L^p(\mathbb{T})$. Then there exists a decreasing sequence $\{\varepsilon_n\} \searrow 0$ such that $|c_n(y)| \leq \varepsilon_n \|y\|_Y$ for all $n \in \mathbb{Z}$ and $y \in Y$.*

Proof. Hölder's inequality implies that, for each $n \in \mathbb{Z}$, the coefficient functionals $c_n : L^p(\mathbb{T}) \to \mathbb{C}$, $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$, are well defined and uniformly bounded with norm $\|c_n\| \leq 1$. Moreover, the results stated at the introductory section of this note show that the sequence $\{c_n\}$ is weakly null. □

Corollary 2.4. *Let H be a Hilbert space and let $\{\phi_n\}_{n=0}^\infty$ be a frame on H with constants $A, B > 0$. Then for every subspace Y compactly embedded into H there exists a decreasing sequence $\{\varepsilon_n\} \searrow 0$ such that $|\langle y, \phi_n\rangle| \leq \varepsilon_n \|y\|_Y$ for every $n \in \mathbb{N}$ and $y \in Y$.*

Proof. By definition of frame, we have that, for all $x \in H$,

$$A\|x\|_H^2 \leq \sum_{n=0}^{\infty} |\langle x, \phi_n\rangle|^2 \leq B\|x\|_H^2.$$

This shows, in particular, that the sequence of coefficient functionals $c_n : H \to \mathbb{C}$, $c_n(x) =< x, \phi_n >$ is uniformly bounded with norm $\|c_n\| \leq \sqrt{B}$, and weakly null. $\square$

References

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