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STABILITY OF A FUNCTIONAL EQUATION RELATED TO DISTANCE MEASURES - II

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ABSTRACT. The present work continues the study of the stability of the functional equations of the type $f(pr, qs) + f(ps, qr) = f(p, q)\, f(r, s)$ namely (i) $f(pr, qs)+f(ps, qr) = g(p, q)\, g(r, s)$, and (ii) $f(pr, qs)+f(ps, qr) = g(p, q)\, h(r, s)$ for all $p, q, r, s \in G$, where G is an abelian group. These functional equations arise in the characterization of symmetrically compositive sumform distance measures.

1. INTRODUCTION

Let G be an abelian group. Let I denote the open unit interval (0, 1). Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Further, let

$$\Gamma_n^o = \left\{ P = (p_1, p_2, ..., p_n) \,\Big|\, 0 < p_k < 1,\ \sum_{k=1}^{n} p_k = 1 \right\}$$

denote the set of all n-ary discrete complete probability distributions (without zero probabilities), that is Γ_n^o is the class of discrete distributions on a finite set Ω of cardinality n with $n \geq 2$. Over the years, many distance measures between discrete probability distributions have been proposed. Hellinger coeeficient, Jeffreys distance, Chernoff coefficient, directed divergence, and its symmetrization J-divergence are examples of such measures (see [1] and [8]).

Almost all similarity, affinity or distance measures $\mu_n : \Gamma_n^o \times \Gamma_n^o \to \mathbb{R}_+$ that have been proposed between two discrete probability distributions can be represented

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in the *sum form*

$$\mu_n(P,Q) = \sum_{k=1}^{n} \phi(p_k, q_k), \tag{1.1}$$

where $\phi : I \times I \to \mathbb{R}$ is a real-valued function on unit square, or a monotonic transformation of the right side of (1.1), that is

$$\mu_n(P,Q) = \psi\left(\sum_{k=1}^{n} \phi(p_k, q_k)\right),$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is an increasing function on $\mathbb{R}$. The function ϕ is called a *generating function*. It is also referred to as the *kernel* of $\mu_n(P,Q)$.

In information theory, for P and Q in Γ_n^o, the symmetric divergence of degree α is defined as

$$J_{n,\alpha}(P,Q) = \frac{1}{2^{\alpha-1}-1}\left[\sum_{k=1}^{n}\left(p_k^{\alpha} q_k^{1-\alpha} + p_k^{1-\alpha} q_k^{\alpha}\right) - 2\right].$$

It is easy to see that $J_{n,\alpha}(P,Q)$ is symmetric. That is $J_{n,\alpha}(P,Q) = J_{n,\alpha}(Q,P)$ for all $P, Q \in \Gamma_n^o$. Moreover it satisfies the composition law

$$\begin{aligned} &J_{nm,\alpha}(P * R, Q * S) + J_{nm,\alpha}(P * S, Q * R) \\ &\qquad = 2J_{n,\alpha}(P,Q) + 2J_{m,\alpha}(R,S) + \lambda\, J_{n,\alpha}(P,Q)\, J_{m,\alpha}(R,S) \end{aligned}$$

for all $P, Q \in \Gamma_n^o$ and $R, S \in \Gamma_m^o$ where $\lambda = 2^{\alpha-1} - 1$ and

$$P * R = (p_1 r_1, p_1 r_2, ..., p_1 r_m, p_2 r_1, ..., p_2 r_m, ..., p_n r_m).$$

In view of this, symmetrically compositive statistical distance measures are defined as follows. A sequence of symmetric measures $\{\mu_n\}$ is said to be *symmetrically compositive* if for some $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\mu_{nm}(P \star R,\, Q \star S) + \mu_{nm}(P \star S,\, Q \star R) \\ &\qquad = 2\,\mu_n(P,\, Q) + 2\,\mu_m(R,\, S) + \lambda\,\mu_n(P,\, Q)\,\mu_m(R,\, S) \end{aligned}$$

for all $P, Q \in \Gamma_n^o$, $S, R \in \Gamma_m^o$, where

$$P * R = (p_1 r_1, p_1 r_2, ..., p_1 r_m, p_2 r_1, ..., p_2 r_m, ..., p_n r_m).$$

Chung, Kannappan, Ng and Sahoo [1] characterized symmetrically compositive sumform distance measures with a measurable generating function. The following functional equation

$$f(pr, qs) + f(ps, qr) = f(p,q)\, f(r,s) \tag{FE}$$

holding for all $p, q, r, s \in I$ was instrumental in the characterization of symmetrically compositive sumform distance measures. They proved the following theorem giving the general solution of this functional equation (FE).

Theorem 1.1. *Suppose $f : I^2 \to \mathbb{R}$ satisfies the functional equation (FE), that is*

$$f(pr, qs) + f(ps, qr) = f(p,q)\, f(r,s)$$

for all $p, q, r, s \in I$. *Then*

$$f(p,q) = M_1(p)\, M_2(q) + M_1(q)\, M_2(p)$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ *are multiplicative functions. Further, either* M_1 *and* M_2 *are both real or* M_2 *is the complex conjugate of* M_1. *The converse is also true.*

The stability of the functional equation (FE) and two generalizations of (FE) namely,

$$f(pr,qs) + f(ps,qr) = f(p,q)g(r,s) \qquad (FE_{fg})$$

$$f(pr,qs) + f(ps,qr) = g(p,q)f(r,s) \qquad (FE_{gf})$$

for all $p, q, r, s \in G$, were studied in [5]. In this paper, we study the stability of two more generalizations of (FE), namely

$$f(pr,qs) + f(ps,qr) = g(p,q)g(r,s) \qquad (FE_{gg})$$

$$f(pr,qs) + f(ps,qr) = g(p,q)h(r,s) \qquad (FE_{gh})$$

for all $p, q, r, s \in G$. For other functional equations similar to (FE), the interested reader should refer to [3], [4], [6] and [7]. For an account on stability of functional equations, the book [2] is an excellent source for reference.

2. Stability of functional equation (FE_{gg})

The following theorem states that an approximate equation of (FE_{gg}) with the boundedness of $f(p,q) - g(p,q)$ and $f(p,q) - f(q,p)$ also implies the functional equation (FE_{gg}).

Theorem 2.1. *Let* $f, g : G^2 \to \mathbb{R}$ *and* $\phi : G^2 \to \mathbb{R}$ *be a nonzero function satisfying*

$$|f(pr,qs) + f(ps,qr) - g(p,q)g(r,s)| \le \phi(p,q) \quad \forall\, p,q,r,s \in G \tag{2.1}$$

and $|f(p,q) - g(p,q)| \le M$, *and* $|f(p,q) - f(q,p)| \le M'$ *for all* $p, q \in G$ *and some constants* M, M'. *Then either* g *is bounded or* g *satisfy the equation* (FE), *that is*

$$g(pr,qs) + g(ps,qr) = g(p,q)g(r,s).$$

Proof. Let g be an unbounded solution of the inequality (2.1). Then we can choose a sequence $\{(x_n, y_n)\,|\, n \in \mathbb{N}\}$ in G^2 such that $0 \neq |g(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$ and $s = y_n$ in (2.1), we have

$$|f(px_n, qy_n) + f(py_n, qx_n) - g(p,q)g(x_n,y_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(px_n, qy_n) + f(py_n, qx_n)}{g(x_n, y_n)} - g(p,q) \right| \le \frac{\phi(p,q)}{|g(x_n,y_n)|}. \tag{2.2}$$

Taking the limit of the both sides of (2.2) as $n \to \infty$, we obtain

$$g(p,q) = \lim_{n\to\infty} \frac{f(px_n, qy_n) + f(py_n, qx_n)}{g(x_n, y_n)}. \tag{2.3}$$

Next, letting $r = rx_n$ and $s = sy_n$ in (2.1), we have

$$|f(prx_n, qsy_n) + f(psy_n, qrx_n) - g(p,q)g(rx_n, sy_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(prx_n, qsy_n) + f(psy_n, qrx_n)}{g(x_n, y_n)} - g(p,q)\frac{g(rx_n, sy_n)}{g(x_n, y_n)} \right| \le \frac{\phi(p,q)}{|g(x_n, y_n)|}. \tag{2.4}$$

Further, letting $r = sx_n$ and $s = ry_n$ in (2.1), we have

$$|f(psx_n, qry_n) + f(pry_n, qsx_n) - g(p,q)g(sx_n, ry_n)| \le \phi(p,q)$$

which is

$$\left| \frac{f(psy_n, qry_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} - g(p,q)\frac{g(sx_n, ry_n)}{g(x_n, y_n)} \right| \le \frac{\phi(p,q)}{|g(x_n, y_n)|}. \tag{2.5}$$

Thus from (2.3), (2.4), (2.5), boundedness by M, M', and $0 \neq |g(x_n, y_n)| \to \infty$, we obtain

$$\begin{aligned}
&g(pr, qs) + g(ps, qr) \\
&\quad = \lim_{n\to\infty} \frac{f(prx_n, qsy_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(psx_n, qry_n) + f(psy_n, qrx_n)}{g(x_n, y_n)} \\
&\quad = \lim_{n\to\infty} \left(\frac{f(prx_n, qsy_n) + f(psy_n, qrx_n)}{g(x_n, y_n)} + \frac{f(psx_n, qry_n) + f(pry_n, qsx_n)}{g(x_n, y_n)} \right) \\
&\quad = g(p,q) \lim_{n\to\infty} \frac{g(rx_n, sy_n) + g(sx_n, ry_n)}{g(x_n, y_n)} \\
&\quad = g(p,q)\Big[g(r,s) + \lim_{n\to\infty} \frac{g(rx_n, sy_n) - f(rx_n, sy_n)}{g(x_n, y_n)} \\
&\qquad\qquad + \lim_{n\to\infty} \frac{g(sx_n, ry_n) - f(sx_n, ry_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(sx_n, ry_n) - f(ry_n, sx_n)}{g(x_n, y_n)} \Big] \\
&\quad = g(p,q)g(r,s) + g(p,q)\Big[\lim_{n\to\infty} \frac{g(rx_n, sy_n) - f(rx_n, sy_n)}{g(x_n, y_n)} \\
&\qquad\qquad + \lim_{n\to\infty} \frac{g(sx_n, ry_n) - f(sx_n, ry_n)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(sx_n, ry_n) - f(ry_n, sx_n)}{g(x_n, y_n)} \Big] \\
&\quad = g(p,q)g(r,s).
\end{aligned}$$

The proof of the theorem is now complete. ☐

Theorem 2.2. *Let $f, g : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p,q)g(r,s)| \le \phi(r,s) \quad \forall\, p, q, r, s \in G. \tag{2.6}$$

Then g is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $|f(p,q) - g(p,q)| \le M$ for all $p, q \in G$ and some nonnegative constant M.

Proof. Suppose g is unbounded. We would like to show that g satisfies (FE) for all $p, q, r, s \in G$ if and only if $|f(p,q) - g(p,q)| \le M$ for all $p, q \in \mathbb{R}$ and for some $M \ge 0$.

Since g is unbounded, because of (2.6), f must be unbounded. Now we prove that the if part of the proof. Suppose g satisfies (FE) for all $p, q, r, s \in G$. Letting $p = q = r = s = 1$ in (FE), we see that that $g(1, 1) = 0$ or $g(1, 1) = 2$. We claim that $g(1, 1) = 2$. Suppose not. Then $g(1, 1) = 0$. Taking $r = s = 1$ in (2.6), we obtain

$$|2\, f(p, q)| = |2\, f(p, q) - g(p, q)\, g(1, 1)| \leq \phi(1, 1),$$

that is f is bounded contrary to the fact that f is unbounded. Hence $g(1, 1) = 2$ and therefore

$$|2f(p, q) - g(p, q)\, g(1, 1)| \leq \phi(1, 1)$$

which implies

$$|f(p, q) - g(p, q)| \leq \frac{\phi(1, 1)}{2} = M.$$

Next, let us prove the only if part. Since g is the unbounded solution of the inequality (2.6), therefore, there exists a sequence $\{(x_n, y_n) \,|\, n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |g(x_n, y_n)\,| \to \infty$ as $n \to \infty$.

Letting $p = x_n$ and $q = y_n$ in (2.6), we have

$$|f(x_n r, y_n s) + f(x_n s, y_n r) - g(x_n, y_n) g(r, s)| \leq \phi(r, s).$$

Taking the limit as $n \to \infty$, we obtain that

$$g(r, s) = \lim_{n\to\infty} \frac{f(x_n r, y_n s) + f(x_n s, y_n r)}{g(x_n, y_n)} \tag{2.7}$$

Letting $p = x_n p$ and $q = y_n q$ in (2.6), we have

$$|f(x_n pr, y_n qs) + f(x_n ps, y_n qr) - g(x_n p, y_n q) g(r, s)| \leq \phi(r, s)$$

which is

$$\left| \frac{f(x_n pr, y_n qs) + f(x_n ps, y_n qr)}{g(x_n, y_n)} - \frac{g(x_n p, y_n q)}{g(x_n, y_n)} g(r, s) \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{2.8}$$

Letting $p = x_n q$ and $q = y_n p$ in (2.6), dividing $g(x_n, y_n)$, passing to the limit as $n \to \infty$, we have

$$\left| \frac{f(x_n qr, y_n ps) + f(x_n qs, y_n pr)}{g(x_n, y_n)} - g(r, s) \frac{g(x_n q, y_n p)}{g(x_n, y_n)} \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{2.9}$$

Thus from (2.7), (2.8), (2.9), boundedness by M, and $0 \neq |g(x_n, y_n)| \to \infty$, we obtain

$$
\begin{aligned}
&g(pr, qs) + g(ps, qr) \\
&= \lim_{n\to\infty} \frac{f(x_n pr, y_n qs) + f(x_n qs, y_n pr)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(x_n ps, y_n qr) + f(x_n qr, y_n ps)}{g(x_n, y_n)} \\
&= \lim_{n\to\infty} \left(\frac{f(x_n pr, y_n qs) + f(x_n ps, y_n qr)}{g(x_n, y_n)} + \frac{f(x_n qs, y_n pr) + f(x_n qr, y_n ps)}{g(x_n, y_n)} \right) \\
&= \lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p)}{g(x_n, y_n)}\, g(r, s) \\
&= \left[g(p, q) + \lim_{n\to\infty} \frac{g(x_n p, y_n q) - f(x_n p, y_n q) + g(x_n q, y_n p) - f(x_n q, y_n p)}{g(x_n, y_n)} \right] g(r, s) \\
&= g(p, q)\, g(r, s) \\
&\qquad + \left[\lim_{n\to\infty} \frac{g(x_n p, y_n q) - f(x_n p, y_n q) + g(x_n q, y_n p) - f(x_n q, y_n p)}{g(x_n, y_n)} \right] g(r, s) \\
&= g(p, q)\, g(r, s).
\end{aligned}
$$

This completes the proof of the theorem. $\square$

The following corollary follows from the Theorem 2.2.

Corollary 2.3. *Let $f, g : G^2 \to \mathbb{R}$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)| \leq \varepsilon \quad \forall\, p, q, r, s \in G$$

for some $\epsilon \geq 0$. Then the function g is bounded or it satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some nonnegative constant M.

3. Stability of functional equation (FE_{gh})

Theorem 3.1. *Let $f, g, h : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}$ be a nonzero function satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \leq \phi(r, s) \tag{3.1}$$

for all $p, q, r, s \in G$. If $|f(p, q) - g(p, q)| \leq M$ for all $p, q \in G$ and some constant M, then g is bounded or h satisfies (FE) for all $p, q, r, s \in G$.

Proof. Let g be unbounded. Then we can choose a sequence $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ in $\mathbb{R}^2$ such that $0 \neq |g(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $p = x_n$ and $q = y_n$ in (3.1), we have

$$|f(x_n r, y_n s) + f(x_n s, y_n r) - g(x_n, y_n)h(r, s)| \leq \phi(r, s),$$

which is

$$\left| \frac{f(x_n r, y_n s) + f(x_n s, y_n r)}{g(x_n, y_n)} - h(r, s) \right| \leq \frac{\phi(r, s)}{|g(x_n, y_n)|}.$$

Taking the limit as $n \to \infty$, we obtain

$$h(r, s) = \lim_{n\to\infty} \frac{f(x_n r, y_n s) + f(x_n s, y_n r)}{g(x_n, y_n)} \tag{3.2}$$

Letting $p = x_n p$ and $q = y_n q$ in (3.1), we have

$$f(x_n pr, y_n qs) + f(x_n ps, y_n qr) - g(x_n p, y_n q)h(r, s)| \le \phi(r, s),$$

which is

$$\left| \frac{f(x_n pr, y_n qs) + f(x_n ps, y_n qr)}{g(x_n, y_n)} - \frac{g(x_n p, y_n q)}{g(x_n, y_n)} h(r, s) \right| \le \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{3.3}$$

Letting $p = x_n q$ and $q = y_n p$ in (3.1) and proceeding as above, we have

$$|f(x_n qr, y_n ps) + f(x_n qs, y_n pr) - g(x_n q, y_n p)h(r, s)| \le \phi(r, s). \tag{3.4}$$

From the last inequality (3.4), we obtain

$$\left| \frac{f(x_n qr, y_n ps) + f(x_n qs, y_n pr)}{g(x_n, y_n)} - \frac{g(x_n q, y_n p)}{g(x_n, y_n)} h(r, s) \right| \le \frac{\phi(r, s)}{|g(x_n, y_n)|}. \tag{3.5}$$

Using (3.2), (3.3), and (3.5), we obtain

$$\begin{aligned} & h(pr, qs) + h(ps, qr) \\ &= \lim_{n\to\infty} \frac{f(x_n pr, y_n qs) + f(x_n qs, y_n pr)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(x_n ps, y_n qr) + f(x_n qr, y_n ps)}{g(x_n, y_n)} \\ &= \lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p)}{g(x_n, y_n)} h(r, s) \\ &= \lim_{n\to\infty} \Big[\frac{g(x_n p, y_n q) - f(x_n p, y_n q) + g(x_n q, y_n p) - f(x_n q, y_n p)}{g(x_n, y_n)} + h(p, q) \Big] h(r, s) \\ &= h(p, q)h(r, s). \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $f, g, h : G^2 \to \mathbb{R}$ and $\phi : G^2 \to \mathbb{R}_+$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \le \phi(p, q)$$

for all $p, q, r, s \in G$. If $|f(p, q) - h(p, q)| \le M$ for all $p, q \in G$ and some constant M then h is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

Proof. Let h be unbounded. Then we can choose a sequence $\{(x_n, y_n) \,|\, n \in \mathbb{N}\}$ in G^2 such that $0 \ne |h(x_n, y_n)| \to \infty$ as $n \to \infty$.

Proceeding as similar to the derivation of (3.2), we obtain

$$g(p, q) = \lim_{n\to\infty} \frac{f(pr_n, qs_n) + f(ps_n, qr_n)}{h(x_n, y_n)}.$$

The rest of the proof runs similar to the Theorem 3.1 and we see that g satisfies (FE) for all $p, q, r, s \in G$. □

Corollary 3.3. *Let $f, g, h : G^2 \to \mathbb{R}$ be functions satisfying*

$$|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)| \le \varepsilon$$

for all $p, q, r, s \in G$ and for some $\epsilon \ge 0$.
(a) If $|f(p, q) - g(p, q)| \le M$ for all $p, q \in G$ and some constant M then g is bounded or h satisfies (FE) for all $p, q, r, s \in G$.

(b) If $|f(p,q) - h(p,q)| \leq M$ for all $p, q \in G$ and some constant M then h is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

Remark 3.4. (i) Choosing g and h appropriately in Theorem 3.1 one can obtain the stability for the functional equations (FE_{fg}), (FE_{gf}) and (FE). For example, by letting first g to be f and then h to be g, the stability of (FE_{fg}) can be obtained which was studied in [5].

(ii) Theorems 2.1 and 3.1 hold if one replaces the domain of the functions f, g, h, ϕ by S^2, where S is an abelian semigroup.

4. Extension of the results to Banach spaces

In this section, let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. All results in the Section 2 and the Section 3 can be extended to the superstability on the Banach space. For simplicity, we will combine the two theorems of the same functional equation in Section 2 and Section 3 into the one theorem, respectively.

Theorem 4.1. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p,q)h(r,s)\| \leq \begin{cases} (i) & \phi(r,s) \\ (ii) & \phi(p,q) \end{cases} \tag{4.1}$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) If $\|f(p,q) - g(p,q)\| \leq M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ g$ is bounded or h satisfies (FE) for all $p, q, r, s \in G$ in the case (i) of (4.1).
(b) If $\|f(p,q) - h(p,q)\| \leq M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ h$ is bounded or g satisfies (FE) for all $p, q, r, s \in G$ in the case (ii) of (4.1).

Proof. First we show (a). Assume that (i) of (4.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As well known we have $\|x^*\| = 1$ hence, for every $x, y \in G$, we have

$$\begin{aligned} \phi(r,s) &\geq \|f(pr, qs) + f(ps, qr) - g(p,q)h(r,s)\| \\ &= \sup_{\|y^*\|=1} \left| y^*\big(f(pr, qs) + f(ps, qr) - g(p,q)h(r,s)\big) \right| \\ &\geq \left| x^*\big(f(pr, qs)\big) - x^*\big(f(ps, qr)\big) - x^*\big(g(p,q)\big)x^*\big(h(r,s)\big) \right|, \end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield solutions of inequality (3.1). Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 3.1 shows that the function $x^* \circ h$ solves the equation (FE). In other words, bearing the linear multiplicativity of x^* in mind, for all $p, q, r, s \in G$, the difference

$$DFE := h(pr, qs) + h(ps, qr) - h(p,q)h(r,s)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$DFE(p,q,r,s) \in \bigcap \left\{ \ker x^* \mid x^* \text{ is a multiplicative member of } E^* \right\}$$

for all $p, q, r, s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h(pr, qs) + h(ps, qr) - h(p, q)h(r, s) = 0 \qquad \text{for all} \quad p, q, r, s \in G,$$

as claimed. The other case (b) is similar, so its proof will be omitted. This completes the proof. □

Corollary 4.2. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)h(r, s)\| \leq \varepsilon$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) If $\|f(p, q) - g(p, q)\| \leq M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ g$ is bounded or h satisfies (FE) for all $p, q, r, s \in G$.
(b) If $\|f(p, q) - h(p, q)\| \leq M$ for all $p, q \in G$ and some constant M then the superposition $x^* \circ h$ is bounded or g satisfies (FE) for all $p, q, r, s \in G$.

The proof of the following theorem follows similar to the proof of Theorem 4.1.

Theorem 4.3. *Let $f, g, h : G^2 \to E$ and $\phi : G^2 \to \mathbb{R}$ be functions satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)\| \leq \begin{cases} (i) & \phi(r, s) \\ (ii) & \phi(p, q) \end{cases}$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$:
(a) In case (i), the superposition $x^* \circ g$ is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $\|f(p, q) - g(p, q)\| \leq M$ for all $p, q \in G$ and some nonnegative constant M.
(b) In case (ii), if $\|f(p, q) - g(p, q)\| \leq M$ and $\|f(p, q) - f(q, p)\| \leq M'$ for all $p, q \in G$ and for some nonnegative constants M, M', then either the superposition $x^* \circ g$ is bounded or g satisfy the equation (FE).

Corollary 4.4. *Let $f, g, h : G^2 \to E$ be a function satisfying*

$$\|f(pr, qs) + f(ps, qr) - g(p, q)g(r, s)\| \leq \varepsilon$$

for all $p, q, r, s \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$, the superposition $x^* \circ g$ is bounded or g satisfies the equation (FE) for all $p, q, r, s \in G$ if and only if $\|f(p, q) - g(p, q)\| \leq M$ for all $p, q \in G$ and some nonnegative constant M.

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References

1. J.K. Chung, Pl. Kannappan, C.T. Ng and P.K. Sahoo, *Measures of distance between probability distributions*, J. Math. Anal. Appl. **138** (1989), 280–292.
2. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.
3. Pl. Kannappan and P.K. Sahoo, *Sum form distance measures between probability distributions and functional equations*, Int. J. of Math. & Stat. Sci. **6** (1997), 91–105.
4. Pl. Kannappan, P.K. Sahoo and J.K. Chung, *On a functional equation associated with the symmetric divergence measures*, Utilitas Math. **44** (1993), 75–83.
5. G.H. Kim and P.K. Sahoo, *Stability of a functional equation related to distance measures-I*, preprint.
6. T. Riedel and P.K. Sahoo, *On a generalization of a functional equation associated with the distance between the probability distributions*, Publ. Math. Debrecen **46** (1995), 125–135.
7. T. Riedel and P.K. Sahoo, *On two functional equations connected with the characterizations of the distance measures*, Aequationes Math. **54** (1998), 242–263.
8. P.K. Sahoo, *On a functional equation associated with stochastic distance measures*, Bull. Korean Math. Soc. **36** (1999), 287–303.

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