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DIFFERENTIAL SUBORDINATIONS FOR CERTAIN ANALYTIC FUNCTIONS MISSING SOME COEFFICIENTS

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ABSTRACT. For a positive integer n, applying Schwarz's lemma related to analytic functions $w(z) = c_n z^n + \cdots$ in the open unit disk $\mathbb{U}$, some assertion in a certain lemma which is well-known as Jack's lemma proven by Miller and Mocanu [J. Math. Anal. Appl. 65 (1978), 289–305] is given. Further, by using a certain method of the proof of subordination relation which was discussed by Suffridge [Duke Math. J. 37 (1970), 775–777] and MacGregor [J. London Math. Soc. (2) 9 (1975), 530–536], some differential subordination property concerning with the subordination

$$p(z) \prec q(z^n) \qquad (z \in \mathbb{U})$$

for functions $p(z) = a + a_n z^n + \cdots$ and $q(z) = a + b_1 z + \cdots$ which are analytic in $\mathbb{U}$ is deduced, and an extension of some subordination relation is given.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For a positive integer n and a complex number a, let $\mathcal{H}[a, n]$ be the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

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Also, let $\mathcal{A}_n$ denote the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} A_k z^k$$

with $\mathcal{A}_1 = \mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$.

A function $f(z) \in \mathcal{H}$ is said to be convex in $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a convex domain (A domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex if the line segment joining any two points of $\mathbb{D}$ lies entirely in $\mathbb{D}$). It is well-known that the function $f(z)$ is convex in $\mathbb{U}$ if and only if $f'(0) \neq 0$ and

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{1.1}$$

The normalized class of convex functions denoted by $\mathcal{K}$ consists of the set of all functions $f(z) \in \mathcal{S}$ for which $f(\mathbb{U})$ is convex.

Further, a function $f(z) \in \mathcal{H}$ is said to be starlike in $\mathbb{U}$ if it is univalent in $\mathbb{U}$ and $f(\mathbb{U})$ is a starlike domain (A domain $\mathbb{D} \subset \mathbb{C}$ is said to be starlike with respect to the origin if $0 \in \mathbb{D}$ and the line segment joining 0 and any point of $\mathbb{D}$ lies entirely in $\mathbb{D}$). It is well-known that the function $f(z)$ is starlike in $\mathbb{U}$ if and only if $f(0) = 0$, $f'(0) \neq 0$ and

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}). \tag{1.2}$$

The normalized class of starlike functions denoted by $\mathcal{S}^*$ consists of the set of all functions $f(z) \in \mathcal{S}$ for which $f(\mathbb{U})$ is starlike.

The equivalent analytic descriptions of $\mathcal{K}$ and $\mathcal{S}^*$ are given respectively as follows.

Remark 1.1. A necessary and sufficient condition for $f(z) \in \mathcal{K}$ is that $f(z) \in \mathcal{A}$ satisfies the inequality (1.1). Also, $f(z) \in \mathcal{S}^*$ if and only if $f(z) \in \mathcal{A}$ satisfies the inequality (1.2).

From the definitions of $\mathcal{K}$ and $\mathcal{S}^*$, we know that

$$f(z) \in \mathcal{K} \quad \text{if and only if} \quad zf'(z) \in \mathcal{S}^*. \tag{1.3}$$

We next introduce the familiar principle of differential subordinations between analytic functions. Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

$$p(z) \prec q(z) \qquad (z \in \mathbb{U}), \tag{1.4}$$

if there exists a function $w(z) \in \mathcal{H}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q\big(w(z)\big) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.4) implies that

$$p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.5}$$

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.4) is equivalent to the condition (1.5).

In order to discuss our main results, we will make use of Schwarz's lemma [2, Vol. I, Theorem 11] (see also [4, Lemma 2.1]) related to $w(z) \in \mathcal{H}[0,n]$.

Lemma 1.2. *Let* $w(z) = \sum_{k=n}^{\infty} c_k z^k \in \mathcal{H}[0,n]$. *If* $w(z)$ *satisfies* $|w(z)| < 1 \quad (z \in \mathbb{U})$, *then*

$$|w(z)| \leqq |z|^n \tag{1.6}$$

for each point $z \in \mathbb{U}$. *Further, if equality occurs in the inequality* (1.6) *for one point* $z_0 \in \mathbb{U}\backslash\{0\}$, *then*

$$w(z) = c_n z^n \tag{1.7}$$

for some complex number c_n *with* $|c_n| = 1$, *and the equality in the inequality* (1.6) *holds for all* $z \in \mathbb{U}$. *Finally, we have* $|c_n| \leqq 1$, *and* $|c_n| = 1$ *if and only if* $w(z)$ *is given by the equation* (1.7).

Further, we need the following lemma which is well-known as Jack's lemma [3] proven by Miller and Mocanu [10] (see also [6]). For $0 < r_0 \leqq 1$, we let

$$\mathbb{U}_{r_0} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < r_0\}, \quad \partial\mathbb{U}_{r_0} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = r_0\}$$

and $\overline{\mathbb{U}_{r_0}} = \mathbb{U}_{r_0} \cup \partial\mathbb{U}_{r_0}$. In particular, we write $\mathbb{U}_1 = \mathbb{U}$. The Jack–Miller–Mocanu lemma (Jack's lemma) is contained in Lemma 1.3.

Lemma 1.3. *Let* n *be a positive integer, and let* $z_0 \in \mathbb{U}$ *with* $|z_0| = r$ *and* $0 < r < 1$. *Also, let* $w(z) = \sum_{k=n}^{\infty} c_k z^k$ *be continuous on* $\overline{\mathbb{U}_r}$ *and analytic on* $\mathbb{U}_r \cup \{z_0\}$ *with* $w(z) \not\equiv 0$. *If* $|w(z_0)| = \max_{z \in \overline{\mathbb{U}_r}} |w(z)|$, *then there exists a real number* k *with*

$$k \geqq n \tag{1.8}$$

such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \quad \text{and} \quad \mathrm{Re}\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geqq k.$$

For Lemma 1.3, by replacing the disk $\mathbb{U}_r$ with a more general region, Miller and Mocanu [10] (see also [7]) derived an extension of the previous lemma which is related to the subordination of two functions.

Lemma 1.4. *Let* $p(z) \in \mathcal{H}[a,n]$ *with* $p(z) \not\equiv a$. *Also, let* $q(z)$ *be analytic and univalent on the closed unit disk* $\overline{\mathbb{U}}$ *except for at most one pole on* $\partial\mathbb{U}$ *with* $q(0) = a$. *If* $p(z)$ *is not subordinate to* $q(z)$ *in* $\mathbb{U}$, *then there exist two points* $z_0 \in \partial\mathbb{U}_r$ *with* $0 < r < 1$, $\zeta_0 \in \partial\mathbb{U}$, *and a real number* k *with* $k \geqq n \geqq 1$ *for which* $p(\mathbb{U}_r) \subset q(\mathbb{U})$,

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$

and

(iii) $\operatorname{Re}\left(1+\dfrac{z_0 p''(z_0)}{p'(z_0)}\right) \geqq k \operatorname{Re}\left(1+\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

This lemma plays a crucial role in developing the theory of differential subordinations.

Let n be a positive integer, and let β and γ be complex numbers with $\beta \neq 0$. For the function $h(z) \in \mathcal{H}$ with $h(0) = a$, Miller and Mocanu [9] (see also [10]) have investigated the analytic solution $q(z)$ of the Briot–Bouquet differential equation

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \tag{1.9}$$

with $q(0) = h(0)$, and determined sufficient conditions for the univalence of the analytic solution $q(z)$ as follows.

Lemma 1.5. *Let n be a positive integer, and let β and γ be complex numbers with $\beta \neq 0$. Also, let $h(z) \in \mathcal{H}$ with $h(0) = a$, and suppose that $\operatorname{Re}\big(\beta h(z) + \gamma\big) > 0$ $(z \in \mathbb{U})$ with $\operatorname{Re}(\beta\, a + \gamma) > 0$. Then the solution $q(z)$ of the differential equation (1.9) with $q(0) = a$ is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\big(\beta q(z) + \gamma\big) > 0 \quad (z \in \mathbb{U})$. Further, if $h(z)$ is convex and univalent in $\mathbb{U}$, then the solution $q(z)$ of (1.9) is univalent in $\mathbb{U}$.*

Moreover, for the function $p(z) \in \mathcal{H}[a, n]$, Miller and Mocanu [9] (see also [10]) discussed the following subordination relation

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{U}) \quad \text{implies} \quad p(z) \prec q(z) \quad (z \in \mathbb{U}), \tag{1.10}$$

where $h(z)$ given in (1.9) is univalent in $\mathbb{U}$ with $h(0) = a$, and $q(z)$ is the univalent solution of the differential equation (1.9). Note that the extremal function of the subordination relation (1.10) is $p(z) = q(z^n)$.

In the present paper, we will discuss the following subordination relation

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{U}) \quad \text{implies} \quad p(z) \prec q(z^n) \quad (z \in \mathbb{U}) \tag{1.11}$$

for $p(z) \in \mathcal{H}[a, n]$ and $h(z) \in \mathcal{H}[a, 1]$, where $h(z)$ is convex in $\mathbb{U}$, and $q(z)$ is the univalent solution of the differential equation (1.9), because the extremal function of the subordination relation (1.10) is $p(z) = q(z^n)$. But, since the function $q(z^n)$ is not univalent in $\mathbb{U}$ for univalent function $q(z)$ in $\mathbb{U}$, we can not discuss the subordination

$$p(z) \prec q(z^n) \qquad (z \in \mathbb{U}) \tag{1.12}$$

by applying Lemma 1.4. Hence, in order to discuss the subordination relation (1.11), we need to consider some different property for the subordination (1.12). In our investigation, by using a certain method of the proof of subordination relation which was discussed by Suffridge [11], MacGregor [5], and Kuroki and Owa [4], we deduce some differential subordination property concerning with the subordination (1.12) for $p(z) \in \mathcal{H}[a, n]$ and $q(z) \in \mathcal{H}[a, 1]$, where $q(z)$ is univalent in $\mathbb{U}$. Further, applying the subordination property, and by making use of several lemmas (cf. Eenigenburg, Miller, Mocanu and Reade [1], and Kuroki and Owa [4]), we will discuss the subordination relation (1.11).

2. Note for the Jack-Miller-Mocanu lemma

To considering our main results, we need to discuss some property in Lemma 1.3.

Lemma 2.1. *In Lemma 1.3, equality occurs in the inequality (1.8) if and only if $w(z) = c_n z^n$ for some complex number c_n with $c_n \neq 0$.*

Proof. From Lemma 1.3, we know that there exists a real number k with $k \geqq n$ such that $\dfrac{z_0 w'(z_0)}{w(z_0)} = k$, where $z_0 \in \mathbb{U}$ with $|z_0| = r$ and $0 < r < 1$. In order to prove our assertion, we need to observe the inequality (1.8) in the proof of Lemma 1.3 (see [10]).

If we let

$$\phi(z) = \frac{w(z_0 z)}{w(z_0)} = \frac{c_n z_0^n}{w(z_0)} z^n + \frac{c_{n+1} z_0^{n+1}}{w(z_0)} z^{n+1} + \cdots \qquad (z \in \overline{\mathbb{U}}),$$

then $\phi(z)$ is continuous on $\overline{\mathbb{U}}$ and analytic on $\mathbb{U} \cup \{1\}$. From

$$\max_{z \in \mathbb{U} \cup \{1\}} \left| w(z_0 z) \right| = \max_{\zeta \in \mathbb{U}_r \cup \{z_0\}} |w(\zeta)| = |w(z_0)| \quad (\zeta = z_0 z),$$

we have

$$|\phi(z)| \leqq \max_{z \in \mathbb{U} \cup \{1\}} |\phi(z)| = \max_{z \in \mathbb{U} \cup \{1\}} \left| \frac{w(z_0 z)}{w(z_0)} \right| \leqq \frac{|w(z_0)|}{|w(z_0)|} = 1, \qquad (z \in \mathbb{U} \cup \{1\}).$$

Since $\phi(0) = 0$, by employing Lemma 1.2, we obtain

$$|\phi(z)| = \left| \frac{w(z_0 z)}{w(z_0)} \right| \leqq |z|^n \qquad (z \in \mathbb{U}). \tag{2.1}$$

Further, if equality occurs in the inequality (2.1) for one point $z \in \mathbb{U}\backslash\{0\}$, then we have

$$\phi(z) = \frac{w(z_0 z)}{w(z_0)} = \frac{c_n z_0^n}{w(z_0)} z^n,$$

and since $w(z_0 z) = c_n (z_0 z)^n$, we know that $w(z) = c_n z^n$ and

$$\left| \frac{c_n z_0^n}{w(z_0)} \right| = \left| \frac{c_n z_0^n}{c_n z_0^n} \right| = 1.$$

Therefore, we see that the equality in the inequality (2.1) for $z \in \mathbb{U}\backslash\{0\}$ if and only if $w(z) = c_n z^n$ for some complex number c_n with $c_n \neq 0$.

In the inequality (2.1), at the point $z = r$ with $0 < r < 1$, we have

$$\operatorname{Re}\left\{ \frac{w(z_0 r)}{w(z_0)} \right\} \leqq \left| \frac{w(z_0 r)}{w(z_0)} \right| \leqq r^n, \tag{2.2}$$

and the equality in the inequality (2.2) holds for at least $w(z) = c_n z^n$, where $c_n \neq 0$. Specially, we note that

$$\frac{w(z_0 r)}{w(z_0)} = \frac{c_n z_0^n r^n}{c_n z_0^n} = r^n,$$

that is

$$\mathrm{Re}\left\{\frac{w(z_0 r)}{w(z_0)}\right\} = \left|\frac{w(z_0 r)}{w(z_0)}\right| = r^n$$

for $w(z) = c_n z^n$, where $c_n \neq 0$.

Since $k = \dfrac{z_0 w'(z_0)}{w(z_0)}$, we have

$$k = \frac{d}{dr}\left\{\frac{w(z_0 r)}{w(z_0)}\right\}\bigg|_{r=1} = \lim_{r\to 1^-}\frac{w(z_0 r) - w(z_0)}{(r-1)w(z_0)} = \lim_{r\to 1^-}\left\{1 - \frac{w(z_0 r)}{w(z_0)}\right\}\frac{1}{1-r}.$$

Taking real parts and using (2.2), we obtain

$$k = \lim_{r\to 1^-}\left\{1 - \mathrm{Re}\frac{w(z_0 r)}{w(z_0)}\right\}\frac{1}{1-r} \geqq \lim_{r\to 1^-}\frac{1-r^n}{1-r} = n,$$

and equality $k = n$ if and only if $w(z) = c_n z^n$ for some complex number c_n with $c_n \neq 0$. □

3. Some subordination properties for certain analytic functions missing some coefficients

By making use of Lemma 1.3, and applying an assertion which was discussed in the previous section, we develop some differential subordination property related to the subordination (1.12).

Theorem 3.1. *Let* $p(z) = a + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{H}[a,n]$ *with* $p(z) \not\equiv a$. *Also, let* $q(z) = a + \sum_{k=1}^{\infty} b_k z^k \in \mathcal{H}[a,1]$ *and suppose that* $q(z)$ *is univalent in* $\mathbb{U}$ *with* $|a_n| < |b_1|$. *If* $p(z)$ *is not subordinate to* $q(z^n)$ *in* $\mathbb{U}$, *then there exists a radius* r *with* $0 < r < 1$ *such that*
(i) $\quad p(z_1) = q({z_2}^n)$

for some z_1, $z_2 \in \partial\mathbb{U}_r$, *and that* $p(\mathbb{U}_r) \subset q(\mathbb{U}_{r^n})$. *Further, there exists a real number* k *with* $k > 1$ *such that*

(ii) $\quad z_1 p'(z_1) = kn\, {z_2}^n q'({z_2}^n)$

and

(iii) $\quad \mathrm{Re}\left(1 + \dfrac{z_1 p''(z_1)}{p'(z_1)}\right) \geqq kn\,\mathrm{Re}\left(1 + \dfrac{{z_2}^n q''({z_2}^n)}{q'({z_2}^n)}\right).$

Proof. If we let

$$q_n(z) = q(z^n) = a + \sum_{k=1}^{\infty} b_k (z^n)^k \in \mathcal{H}[a, n],$$

then $|a_n| < |b_1|$ implies that $p(\mathbb{U}_\varepsilon) \subset q_n(\mathbb{U}_\varepsilon)$ for all sufficiently small values ε. Hence, noting that $p(z) \in \mathcal{H}[a, n]$ with $p(z) \not\equiv a$, we may define a function $w(z)$ by

$$w(z) = \Big(q^{-1}\big(p(z)\big)\Big)^{\frac{1}{n}} = c_1 z + \cdots \qquad (z \in \mathbb{U}) \tag{3.1}$$

and $w(z)$ is analytic in $\mathbb{U}_\varepsilon$ with $w(z) \not\equiv 0$ and satisfies $|w(z)| < \varepsilon \quad (z \in \mathbb{U}_\varepsilon)$, and that

$$p(z) = q\big(\big(w(z)\big)^n\big) = q_n\big(w(z)\big). \tag{3.2}$$

These assertions hold for all small values of ε. Since $p(z)$ is not subordinate to $q_n(z)$ in $\mathbb{U}$, there is no $w(z) \in \mathcal{H}[0, 1]$ with $|w(z)| < 1 \quad (z \in \mathbb{U})$ such that $p(z) = q_n\big(w(z)\big)$. Hence, it follows from this fact that there exists a radius r with $0 < r < 1$ such that the function $w(z)$ defined by (3.1) is analytic in $\mathbb{U}_r$ with

$$w(z_1) = z_2 \tag{3.3}$$

for some $z_1,\ z_2 \in \partial\mathbb{U}_r$, and that $|w(z)| < r \quad (z \in \mathbb{U}_r)$. Then, it is clear that

$$p(z_1) = q_n\big(w(z_1)\big) = q_n(z_2) = q({z_2}^n) \qquad (z_1,\ z_2 \in \partial\mathbb{U}_r)$$

and

$$p(\mathbb{U}_r) = q_n\big(w(\mathbb{U}_r)\big) \subset q(\mathbb{U}_{r^n}).$$

Moreover, noting that $q'(z) \neq 0 \quad (z \in \mathbb{U})$ if $q(z)$ is univalent in $\mathbb{U}$, then we have

$$\lim_{z \to z_1} \frac{w(z) - w(z_1)}{z - z_1} = \lim_{z \to z_1} \frac{p(z) - p(z_1)}{z - z_1} \, \frac{w(z) - w(z_1)}{q_n\big(w(z)\big) - q_n\big(w(z_1)\big)}$$

$$= \frac{p'(z_1)}{q_n'\big(w(z_1)\big)} = \frac{p'(z_1)}{n\,\big(w(z_1)\big)^{n-1} q'\big(w(z_1)\big)} \;\Big(= w'(z_1)\Big).$$

Therefore, since $|w(z_1)| = |z_2| = r \quad (z_1,\ z_2 \in \partial\mathbb{U}_r)$, the function $w(z)$ defined by (3.1) is analytic on $\mathbb{U}_r \cup \{z_1\}$ and satisfies $|w(z)| \leqq r \quad (z \in \mathbb{U}_r \cup \{z_1\})$. Then, it is clear that

$$|w(z_1)| = |z_2| = r = \max_{z \in \mathbb{U}_r \cup \{z_1\}} |w(z)| \qquad (z_1,\ z_2 \in \partial\mathbb{U}_r),$$

that is that the modulus $|w(z)|$ takes the maximum value r at a point $z = z_1 \in \mathbb{U}_r \cup \{z_1\}$.

Thus, according to Lemma 1.3 and Lemma 2.1 for $n = 1$, there exists a real number k with $k \geqq 1$ such that

$$\frac{z_1 w'(z_1)}{w(z_1)} = k \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{z_1 w''(z_1)}{w'(z_1)}\right) \geqq k, \tag{3.4}$$

and equality occurs in the inequality $k \geqq 1$ if and only if

$$w(z) = c_1 z \qquad (z \in \mathbb{U}_r) \tag{3.5}$$

for some complex number c_1 with $c_1 = \dfrac{z_2}{z_1}$ $(z_1,\, z_2 \in \partial\mathbb{U}_r)$, because $w(z_1) = c_1 z_1 = z_2$ for some $z_1,\, z_2 \in \partial\mathbb{U}_r$. Now, applying Lemma 1.2 for $n = 1$, we will show that $w(z)$ defined by (3.1) does not have the form (3.5). If we define the function

$$\phi(z) = \frac{1}{r} w(rz) = c_1 z + \cdots \qquad (z \in \mathbb{U}),$$

then $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies $|\phi(z)| < 1 \quad (z \in \mathbb{U})$. By Lemma 1.2 for $n = 1$, we find $|c_1| \leqq 1$, and $|c_1| = 1$ if and only if $\phi(z) = c_1 z \quad (z \in \mathbb{U})$, which implies that $w(z)$ is given by the equation (3.5) for some complex number c_1 with $|c_1| = 1$. Then, by comparing the coefficients of z^n in the both sides of equality (3.2), we obtain $a_n = b_1 {c_1}^n$, and hence from $|c_1| \leqq 1$, we have

$$|a_n| = |b_1||c_1|^n \leqq |b_1|,$$

and equality occurs if and only if equation (3.5) for some c_1 with $|c_1| = 1$ holds true. Therefore, since $|a_n| < |b_1| \quad (|c_1| < 1)$ implies that $w(z)$ does not have the form (3.5) with $|c_1| = 1$, it follows that $k = \dfrac{z_1 w'(z_1)}{w(z_1)} > 1$ according to Lemma 2.1. From the above-mentioned, we deduce that there is a real number k so that $k > 1$ and (3.4). Equation (3.2) implies that

$$zp'(z) = n\, zw'(z)\big(w(z)\big)^{n-1} q'\Big(\big(w(z)\big)^n\Big)$$

and

$$\frac{zp''(z)}{p'(z)} = \frac{zw''(z)}{w'(z)} + \frac{zw'(z)}{w(z)}\left((n-1) + \frac{n\big(w(z)\big)^n q''\Big(\big(w(z)\big)^n\Big)}{q'\Big(\big(w(z)\big)^n\Big)}\right).$$

If we use these relations at $z = z_1 \in \partial\mathbb{U}_r$ and (3.4), then from equation (3.3), we see that

$$\begin{aligned} z_1 p'(z_1) &= n\, z_1 w'(z_1)\big(w(z)\big)^{n-1} q'\Big(\big(w(z_1)\big)^n\Big) \\ &= kn\, \big(w(z_1)\big)^n q'\Big(\big(w(z_1)\big)^n\Big) \\ &= kn\, {z_2}^n q'({z_2}^n) \qquad (k > 1). \end{aligned}$$

Moreover, since $q'(z_2{}^n) \neq 0 \quad (z_2 \in \partial\mathbb{U}_r)$, we obtain

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{z_1 p''(z_1)}{p'(z_1)}\right)\\
&=\operatorname{Re}\left\{1+\frac{z_1 w''(z_1)}{w'(z_1)}+k\left((n-1)+\frac{n\big(w(z_1)\big)^n q''\big(\big(w(z_1)\big)^n\big)}{q'\big(\big(w(z_1)\big)^n\big)}\right)\right\}\\
&=\operatorname{Re}\left(1+\frac{z_1 w''(z_1)}{w'(z_1)}\right)+k\operatorname{Re}\left((n-1)+\frac{n z_2{}^n q''(z_2{}^n)}{q'(z_2{}^n)}\right)\\
&\geqq kn\operatorname{Re}\left(1+\frac{z_2{}^n q''(z_2{}^n)}{q'(z_2{}^n)}\right) \qquad (k>1).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

In order to discuss the subordination relation (1.11) by applying Theorem 3.1, we need the following two lemmas related to the subordination $P(z) \prec h(z)$ $(z \in \mathbb{U})$ for $P(z) \in \mathcal{H}[a,n]$ and $h(z) \in \mathcal{H}[a,1]$ proven by Kuroki and Owa [4].

Lemma 3.2. *Let* $P(z) = a + \sum_{k=n}^{\infty} a_k z^k \in \mathcal{H}[a,n]$ *and* $h(z) = a + \sum_{k=1}^{\infty} b_k z^k \in \mathcal{H}[a,1]$. *If* $P(z) \prec h(z) \quad (z \in \mathbb{U})$, *then*

$$|a_n| \leqq |b_1|,$$

and equality occurs if and only if $P(z) = h(xz^n)$ *for some complex number* x *with* $|x| = 1$.

Lemma 3.3. *Let* $P(z) \in \mathcal{H}[a,n]$ *and* $h(z) \in \mathcal{H}[a,1]$. *If* $P(z) \prec h(z) \quad (z \in \mathbb{U})$, *then*

$$P(\mathbb{U}_r) \subset h(\mathbb{U}_{r^n})$$

for each r *with* $0 < r < 1$. *Further, if* $P(z_0)$ *is on the boundary of* $h(\mathbb{U}_{r^n})$ *for one point* $z_0 \in \partial\mathbb{U}_r$, *then there is a complex number* x *with* $|x| = 1$ *such that* $P(z) = h(xz^n)$, *and* $P(z)$ *is on the boundary of* $h(\mathbb{U}_{r^n})$ *for every point* $z \in \partial\mathbb{U}_r$.

Note that this lemma is a slight extension of the Lindelöf principle (cf. [2]).

4. An extension of some subordination relation

Suffridge [11] independently discovered some particular case of Lemma 1.3, and proved the following subordination relation

$$zp'(z) \prec zq'(z) \quad (z \in \mathbb{U}) \quad \text{implies} \quad p(z) \prec q(z) \quad (z \in \mathbb{U}) \tag{4.1}$$

for $p(z) \in \mathcal{H}[0,1]$ and $q(z) \in \mathcal{K}$.

In this section, applying some subordination properties which were considered in the previous section, we first discuss an extension of the subordination relation (4.1).

Theorem 4.1. *Let $q(z) \in \mathcal{H}[a,1]$, and suppose that $q(z)$ is convex in $\mathbb{U}$. If $p(z) \in \mathcal{H}[a,n]$ with $p(z) \not\equiv a$ satisfies $zp'(z) \prec nzq'(z) \quad (z \in \mathbb{U})$, then $p(z) \prec q(z^n) \quad (z \in \mathbb{U})$.*

Proof. If we let
$$P(z) = zp'(z) \quad \text{and} \quad h(z) = nzq'(z),$$
then the assumption $zp'(z) \prec nzq'(z) \quad (z \in \mathbb{U})$ can be rewritten by
$$P(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{4.2}$$
Moreover, if we set
$$p(z) = a + \sum_{k=n}^{\infty} a_k z^k \quad \text{and} \quad q(z) = a + \sum_{k=1}^{\infty} b_k z^k,$$
then
$$P(z) = \sum_{k=n}^{\infty} k a_k z^k \quad \text{and} \quad h(z) = n\sum_{k=1}^{\infty} k b_k z^k.$$
It follows that the subordination (4.2) implies $|a_n| \leqq |b_1|$ by Lemma 3.2. Since $|a_n| = |b_1|$ if and only if $P(z) = h(xz^n)$, where $|x| = 1$, we have $zp'(z) = nxz^n q'(xz^n)$ which implies that $p(z) = q(xz^n) = q\left(\left(x^{\frac{1}{n}}z\right)^n\right)$, where $|x| = 1$, and this means that $p(z) \prec q(z^n) \quad (z \in \mathbb{U})$. Therefore, we may continue the argument assuming that $|a_n| < |b_1|$.

If we assume that $p(z)$ is not subordinate to $q(z^n)$ in $\mathbb{U}$, then by Theorem 3.1, there exist a radius r $(0 < r < 1)$ and a real number k $(k > 1)$ such that $p(z_1) = q({z_2}^n)$ and $z_1 p'(z_1) = kn\, {z_2}^n q'({z_2}^n)$ for some $z_1,\ z_2 \in \partial\mathbb{U}_r$, and hence we have
$$P(z_1) = z_1 p'(z_1) = kn\, {z_2}^n q'({z_2}^n) = k\, h({z_2}^n),$$
where $z_1 \in \partial\mathbb{U}_r$, ${z_2}^n \in \partial\mathbb{U}_{r^n}$ and $k > 1$.

Since $q(z) = a + \sum\limits_{k=1}^{\infty} b_k z^k$ is convex in $\mathbb{U}$ with $b_1 \neq 0$, it is clear that $\dfrac{q(z)-a}{b_1} \in \mathcal{K}$. From the relation (1.3), we find that $z\left(\dfrac{q(z)-a}{b_1}\right)' \in \mathcal{S}^*$, which implies that $h(z) = nzq'(z)$ is starlike and univalent in $\mathbb{U}$. Thus, since $h(\mathbb{U}_{r^n})$ is starlike with respect to the origin for $0 < r < 1$, we deduce that
$$P(z_1) \notin \overline{h(\mathbb{U}_{r^n})} \tag{4.3}$$
for $z_1 \in \partial\mathbb{U}_r$. According to Lemma 3.3, since the relation (4.3) contradicts the assumption (4.2) of the theorem, and hence we must have $p(z) \prec q(z^n) \quad (z \in \mathbb{U})$. This completes the proof of Theorem 4.1. □

Remark 4.2. Since $zq'(z)$ is starlike in $\mathbb{U}$ for a convex function $q(z) \in \mathcal{H}[a,1]$, $zp'(z) \prec zq'(z) \quad (z \in \mathbb{U})$ implies $zp'(z) \prec nzq'(z) \quad (z \in \mathbb{U})$ for $p(z) \in \mathcal{H}[a,n]$ and $q(z) \in \mathcal{H}[a,1]$ which is convex in $\mathbb{U}$. Therefore, it follows from $q(z^n) \prec q(z)$ $(z \in \mathbb{U})$ that if $p(z)$ and $q(z)$ satisfy the relation of Theorem 4.1, then the relation (4.1) which was proven by Suffridge [11] holds.

5. Some applications to the Briot–Bouquet differential subordinations

To considering the subordination relation (1.11), we need the following lemma concerning the Briot–Bouquet differential subordinations given by Eenigenburg, Miller, Mocanu and Reade [1] (see also [9, 10]).

Lemma 5.1. *Let β and γ be complex numbers with $\beta \neq 0$, and let $h(z)$ be convex and univalent in $\mathbb{U}$ with $\mathrm{Re}\big(\beta h(z) + \gamma\big) > 0 \quad (z \in \mathbb{U})$. If $q(z) \in \mathcal{H}$ with $q(0) = h(0)$ satisfies the Briot–Bouquet differential subordination*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \qquad (z \in \mathbb{U}), \tag{5.1}$$

then $q(z) \prec h(z) \quad (z \in \mathbb{U})$.

Applying Theorem 3.1, and by making use of several lemmas, we deduce the assertion related to the subordination relation (1.11) bellow.

Theorem 5.2. *Let n be a positive integer, and let β and γ be complex numbers with $\beta \neq 0$. Also, let $h(z) \in \mathcal{H}[a,1]$ be convex in $\mathbb{U}$ and satisfies $\mathrm{Re}\big(\beta h(z) + \gamma\big) > 0$ $(z \in \mathbb{U})$ with $\mathrm{Re}(\beta\, a + \gamma) > 0$. Suppose that $q(z)$ with $q(0) = a$ is the solution of the Briot–Bouquet differential equation*

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z). \tag{5.2}$$

If $p(z) \in \mathcal{H}[a,n]$ with $p(z) \not\equiv a$ satisfies the Briot–Bouquet differential subordination

$$P(z) = p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \qquad (z \in \mathbb{U}), \tag{5.3}$$

then $p(z) \prec q(z^n) \prec q(z) \prec h(z) \quad (z \in \mathbb{U})$.

Proof. Since $h(z) \in \mathcal{H}[a,1]$ satisfies $\mathrm{Re}\big(\beta h(z) + \gamma\big) > 0 \quad (z \in \mathbb{U})$, it follows from Lemma 1.5 that the solution $q(z)$ of the differential equation (5.2) is analytic in $\mathbb{U}$ with $q(0) = a$, which implies that $q(z) \in \mathcal{H}[a,1]$. Further, since $h(z)$ is convex and univalent in $\mathbb{U}$, we know that $q(z) \in \mathcal{H}[a,1]$ is univalent in $\mathbb{U}$ from Lemma 1.5. If we let

$$p(z) = a + \sum_{k=n}^{\infty} a_k z^k \quad \text{and} \quad q(z) = a + \sum_{k=1}^{\infty} b_k z^k,$$

then

$$P(z) = p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = a + \left(1 + \frac{n}{\beta\, a + \gamma}\right) a_n z^n + \cdots$$

and

$$h(z) = q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = a + \left(1 + \frac{n}{\beta\, a + \gamma}\right) b_1 z + \cdots .$$

Thus, by Lemma 3.2, it follows that the subordination (5.3) implies $|a_n| \leqq |b_1|$, and $|a_n| = |b_1|$ if and only if $P(z) = h(xz^n)$, that is,

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} = q(xz^n) + \frac{nxz^n q'(xz^n)}{\beta q(xz^n) + \gamma}$$

for some complex number x with $|x| = 1$. Then, a certain calculation (see [8]) yields that

$$p(z) = q(xz^n) = q\Big(\big(x^{\frac{1}{n}} z\big)^n\Big),$$

where $|x| = 1$, and this means that $p(z) \prec q(z^n) \quad (z \in \mathbb{U})$. Therefore, we may continue the argument assuming that $|a_n| < |b_1|$.

If we assume that $p(z)$ is not subordinate to $q(z^n)$ in $\mathbb{U}$, then by Theorem 3.1, there exist a radius r $(0 < r < 1)$ and a real number k $(k > 1)$ such that $p(z_1) = q({z_2}^n)$ and $z_1 p'(z_1) = kn\, {z_2}^n q'({z_2}^n)$ for some $z_1,\, z_2 \in \partial\mathbb{U}_r$, and hence we have

$$\begin{aligned} P(z_1) = p(z_1) + \frac{z_1 p'(z_1)}{\beta p(z_1) + \gamma} &= q({z_2}^n) + \frac{kn\, {z_2}^n q'({z_2}^n)}{\beta q({z_2}^n) + \gamma} \\ &= q({z_2}^n) + k\big(h({z_2}^n) - q({z_2}^n)\big), \end{aligned}$$

where $z_1 \in \partial\mathbb{U}_r$, ${z_2}^n \in \partial\mathbb{U}_{r^n}$ and $k > 1$, by using the equation (5.2).

Since equation (5.2), we know that $q(z)$ clearly satisfies the subordination (5.1). Thus, by making use of Lemma 5.1, we have the following subordination

$$q(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{5.4}$$

Further, according to Lemma 3.3 for $n = 1$, it follows from the subordination (5.4) that $q(\mathbb{U}_\rho) \subset h(\mathbb{U}_\rho)$ for each ρ with $0 < \rho < 1$, and taking $\rho = r^n$ with $0 < r < 1$, we find that

$$q(\mathbb{U}_{r^n}) \subset h(\mathbb{U}_{r^n}) \tag{5.5}$$

for $0 < r < 1$. Using the relation (5.5) with the fact that $h(\mathbb{U}_{r^n})$ is convex domain and $k > 1$, we deduce that

$$P(z_1) = q({z_2}^n) + k\big(h({z_2}^n) - q({z_2}^n)\big) \notin \overline{h(\mathbb{U}_{r^n})} \tag{5.6}$$

for $z_1 \in \partial\mathbb{U}_r$, where ${z_2}^n \in \partial\mathbb{U}_{r^n}$ and $k > 1$. According to Lemma 3.3, the relation (5.6) contradicts the assumption (5.3) of the theorem, and hence we must have $p(z) \prec q(z^n) \quad (z \in \mathbb{U})$. Therefore, combining $q(z^n) \prec q(z) \quad (z \in \mathbb{U})$ and the subordination (5.4), we conclude that

$$p(z) \prec q(z^n) \prec q(z) \prec h(z) \qquad (z \in \mathbb{U}),$$

which completes the proof of Theorem 5.2. $\square$

By taking $\beta = 1$ and $\gamma = 0$ in Theorem 5.2, and letting

$$p(z) = \frac{zf'(z)}{f(z)} \quad \text{and} \quad h(z) = \frac{n(1+z)}{n-z}$$

for a positive integer n and $f(z) \in \mathcal{A}_n$, we find the following subordination relation.

Corollary 5.3. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{n(1+z)}{n-z} \qquad (z \in \mathbb{U}),$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{n}{n-z^n} \qquad (z \in \mathbb{U}).$$

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