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OSCILLATIONS, QUASI-OSCILLATIONS AND JOINT CONTINUITY

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ABSTRACT. Parallel to the concept of quasi-separate continuity, we define a notion for quasi-oscillation of a mapping $f : X \times Y \to \mathbb{R}$. We also introduce a topological game on X to approximate the oscillation of f. It follows that under suitable conditions, every quasi-separately continuous mapping $f : X \times Y \to \mathbb{R}$ has the Namioka property. An illuminating example is also given.

1. INTRODUCTION

Throughout this paper, unless explicitly stated otherwise, we will assume that X and Y are topological spaces and Y is compact. Let $f : X \times Y \to \mathbb{R}$ be a mapping. Following [7], f is called *quasi-separately continuous* at $(x_0, y_0) \in X \times Y$ if the function $t \mapsto f(x_0, t)$ is continuous at y_0 and for every finite set F of Y and $\varepsilon > 0$, there is some open set $V \subset X$ such that $x_0 \in \overline{V}$ and $|f(x, y) - f(x_0, y)| < \varepsilon$ whenever $x \in V$ and $y \in F$. The function f is called *quasi-separately continuous* if f is quasi-separately continuous at each point of $X \times Y$. We define the quasi-oscillation of a mapping $f : X \times Y \to \mathbb{R}$ at $x_0 \in X$ as follows:

$$\mathcal{Q}(f, x_0) = \sup_{F \text{ is finite}} \{\inf\{ \sup_{(x,y) \in V \times F} |f(x, y) - f(x_0, y)| : V \text{ open}, x_0 \in \overline{V}\}\}.$$

It is easy to see that $f : X \times Y \to \mathbb{R}$ is quasi-separately continuous at (x_0, y_0) if and only if f is continuous with respect to second variable in y_0 and $\mathcal{Q}(f, x_0) = 0$.

Following [6], a mapping $f : X \times Y \to \mathbb{R}$ is said to have the Namioka property if there exists a dense in G_δ subset D of X such that f is jointly continuous at each point of $D \times Y$.

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In this paper, we are interested to the following problem:
Suppose that $f : X \times Y \to \mathbb{R}$ is a mapping. Under what conditions on X, there are constants c_1 and c_2 such that

$$\mathcal{O}(f;(x,y)) \leq c_1 \sup_{t\in X} \mathcal{Q}(f,t) + c_2 \sup_{(t,s)\in X\times Y} \mathcal{O}(f(t,.),s)$$

for each point $(x,y) \in D \times Y$, where

$$\mathcal{O}(f(t,.),s) = \inf\{\operatorname{diam}(f(\{t\}\times U)) : U \text{ is open in } Y \text{ and } s \in U\}$$

denotes the oscillation of $y \mapsto f(t,y)$ in s and D is a dense G_δ subset of X?

Problems of this type are considered by some authors (see e.g. [1, 2, 10, 11] and the references therein).

In this paper, inspired by [1, 5] and [9], we will introduce a topological game $\mathcal{G}(X)$ on X. Then we will show that for each mapping $f : X \times Y \to \mathbb{R}$, there exists a dense G_δ subset D of X such that the oscillation of f at each point of $D \times Y$ is less than $10 \sup_{x\in X} \mathcal{Q}(f,x) + 6 \sup_{(x,y)\in X\times Y} \mathcal{O}(f(x,\cdot),y)$ provided that the first player has no winning strategy in $\mathcal{G}(X)$.

It follows that under the above condition on X, every quasi-separately continuous mapping $f : X \times Y \to \mathbb{R}$ has the Namioka property. This can be considered as a generalization of the main result in [12].

2. Main results

The story of topological games goes back to Baire [4]. Since then several topological games were invented and applied by some authors [5, 8, 9, 12]. Here, we introduce a topological game as follows.

$\mathcal{G}(X)$ is played by two players β and α as follows: β starts a game by choosing a non-empty open set $U_1 \subset X$. α answers by selecting a couple (V_1, x_1), where $V_1 \subset U_1$ and $x_1 \in X$. In step n, β's move is a non-empty open $U_n \subset V_{n-1}$. Then α's n-th move is a pair (V_n, x_n) where V_n is a non-empty open subset of U_n and $x_n \in X$. The player α wins the game $\mathcal{G}(X)$ if there is some $z \in \bigcap_{i=1}^{\infty} V_n$ such that for every open subset G in X with $z \in \overline{G}$,

$$G \cap \{x_1, x_2, \dots\} \neq \emptyset.$$

A strategy s for α in the game $\mathcal{G}(X)$ is a rule which determines α's move at each stage. X is called β-favorable for the play $\mathcal{G}(X)$ if β has a winning strategy in this play, otherwise X is said to be β- *unfavorable* for this play. Clearly every separable Baire space X is β-unfavorable for the game $\mathcal{G}(X)$.

A similar topological game, with a different winning rule, was introduced in [5].

Let Z be a metric space and $r > 0$, a family $\mathfrak{F} \subset Z^X$ is said to be r-equicontinuous if there is an open neighborhood W of Δ, the diagonal of $X \times X$, such that

$$d(f(x), f(x')) < r \quad \text{for all } f \in \mathfrak{F} \text{ and } (x,x') \in W.$$

Theorem 2.1. *Let X be a β-unfavorable space and $f : X\times Y \to \mathbb{R}$ be a mapping. Then there is a dense G_δ subset D of X such that*

$$\mathcal{O}\big(f,(x,y)\big) \leq 10\sup_{t\in X}\mathcal{Q}(f,t)+6\sup_{(s,t)\in X\times Y}\mathcal{O}(f(t,\cdot),s) \quad \text{for all} \quad (x,y)\in D\times Y.$$

In particular, if $f : X \times Y \to \mathbb{R}$ is quasi-separately continuous, then it has the Namioka property.

Let

$$a=\sup_{x\in X}\mathcal{Q}(f,x), \quad b=\sup_{(x,y)\in X\times Y}\mathcal{O}(f(x,\cdot),y).$$

In order to prove the above theorem, we need to some auxiliary results.

Lemma 2.2. *Suppose that $\{f(x,.) : x\in U\}$ is r-equicontinuous for some $r > 0$ and a non-empty open subset U of X. Then for each $\varepsilon > 0$, there exist a non-empty open subset U' of U and a finite open cover $\{V_1,\dots,V_n\}$ of Y such that $diam\Big(f(U'\times V_i)\Big) \leq 2(r+a)+\varepsilon$ for each $1\leq i\leq n$.*

Proof. Since $\{f(x,.) : x\in U\}$ is r-equicontinuous, there is a neighborhood W of Δ such that

$$|f(x,y)-f(x,y')|<r \quad x\in U,(y,y')\in W.$$

For each $y\in Y$, put $W_y=\{y' : (y,y')\in W\}$. Then $\{W_y : y\in Y\}$ is an open cover for Y. Since Y is compact, there are points $y_1,\dots,y_n\in Y$ such that $Y=\bigcup_{i=1}^{n}W_{y_i}$. Write $V_i=W_{y_i}$ for each $1\leq i\leq n$. Fix some $x_1\in U$. Since $\mathcal{Q}(f,x_1)<a+\varepsilon/2$, there is some non-empty open subset $U_1\subset U$ such that

$$|f(x_1,y_1)-f(x,y_1)|<a+\varepsilon/2 \quad (x\in U_1).$$

Suppose that for $1\leq k<n$ points $x_1,\dots,x_k$ and open subsets $U_1,\dots,U_k$ of U have been selected. Then choose some arbitrary point $x_{k+1}\in U_K$. By our assumption, $\mathcal{Q}(f,x_k)<a+\varepsilon/2$, therefore we can find some non-empty open subset $U_{k+1}\subset U_k$ such that

$$|f(x_k,y_k)-f(x,y_k)|<a+\varepsilon/2 \quad (x\in U_{k+1}).$$

In this way by (finite) induction on k, points $x_1,\dots,x_n\in U$ and $U_1\supset\cdots\supset U_n$ are determined. Put $U'=U_n$, then for each $1\leq i\leq k$, $y\in V_i$ and $x\in U'$ we have

$$\begin{aligned}|f(x,y)-f(x_i,y_i)|\leq|f(x,y)-f(x,y_i)| &+ |f(x_i,y_i)-f(x,y_i)|\\ &< r+a+\varepsilon/2.\end{aligned}$$

It follows that for each $1\leq i\leq k$, $\operatorname{diam}\Big(f(U'\times V_i)\Big)\leq 2(r+a)+\varepsilon.$ □

Lemma 2.3. *For each non-empty open subset U of X and $\varepsilon>0$, there is a non-empty open subset U' of U such that $\{f(t,\cdot) : t\in U'\}$ is $(4a+3b+\varepsilon)$-equicontinuous.*

Proof. Suppose that for some $\varepsilon > 0$, there is a non-empty open subset U of X such that $\{f(x,\cdot) : x \in U'\}$ is not $(4a+3b+\varepsilon)$-equicontinuous for each non-empty open subset U' of U. We will define inductively a strategy for the player β in $\mathcal{G}(X)$. Put $U_1 = U$ as the first move of β. Let $n > 1$ and $(V_1, x_1), \dots, (V_n, x_n)$ be selected by α and $\delta = \varepsilon/20$. Since for each $x \in X$, $\sup_{y\in Y} \mathcal{O}(f(x,\cdot), y) \leq b$, by [3, Proposition 1.18], we can find some $g_x \in C(Y)$ such that $|g_x(y) - f(x,y)| < b/2 + \delta$ for all $y \in Y$. Let

$$W_n = \Big\{(y,y') \in Y \times Y : |g_{x_i}(y) - g_{x_i}(y')| < \frac{1}{n}, 1 \leq i \leq n\Big\}.$$

Thanks to continuity of g_{x_i}'s, W_n is an open neighborhood of Δ. Let $r = 4a + 3b + \varepsilon$. Since $\{f(x,\cdot) : x \in V_n\}$ is not r-equicontinuous, we can find some $t_n \in V_n$ and $(y_n, y'_n) \in W_n$ such that $|f(t_n, y_n) - f(t_n, y'_n)| \geq r$. Since $Q(f, t_n) \leq a$, there is a non-empty subset $U_{n+1} \subset V_n$ such that for each $t \in U_{n+1}$,

$$|f(t_n, y_n) - f(t, y_n)| < a + \delta \quad \text{and} \quad |f(t_n, y'_n) - f(t, y'_n)| < a + \delta.$$

Let U_{n+1} be the answer of β to $\big((V_1, x_1), \dots, (V_n, x_n)\big)$. Therefore a strategy for the player β is inductively defined. Since this strategy is not winning for β, some play $\big\{(U_n, (V_n, x_n))\big\}$ is won by α. Therefore, there is some $z \in \bigcap_{n\geq 1} V_n$ such that for each open subset G of X with $z \in \overline{G}$, $G \cap \{x_1, x_2, \dots\} \neq \emptyset$. Let (y_∞, y'_∞) be a cluster point of $\{(y_n, y'_n)\}$ in $Y \times Y$. Then for each $n \geq i \geq 1$, we have $|g_{x_i}(y_n) - g_{x_i}(y'_n)| < \frac{1}{n}$. Since g_{x_i} is continuous, it follows that $g_{x_i}(y_\infty) = g_{x_i}(y'_\infty)$. Moreover, for each n we have

$$\begin{aligned} r &\leq |f(t_n, y_n) - f(t_n, y'_n)| \\ &\leq |f(t_n, y_n) - f(z, y_n)| + |f(z, y_n) - f(z, y'_n)| + |f(z, y'_n) - f(t_n, y'_n)| \\ &< 2a + 2\delta + |f(z, y_n) - g_z(y_n)| + |g_z(y_n) - g_z(y'_n)| + |g_z(y_n) - f(z, y'_n)| \\ &< 2a + b + 4\delta + |g_z(y_n) - g_z(y'_n)|. \end{aligned}$$

Thanks to continuity of g_z,

$$r \leq 2a + b + 4\delta + |g_z(y_\infty) - g_z(y'_\infty)|. \tag{2.1}$$

Since $Q(f, z) \leq a$, there is an open subset G of X such that $z \in \overline{G}$ and

$$|f(z, y_\infty) - f(t, y_\infty)| < a + \delta \text{ and } |f(z, y'_\infty) - f(t, y'_\infty)| < a + \delta$$

for each $t \in G$. Take some $i \geq 1$ such that $x_i \in G$, then we have

$$\begin{aligned} |g_z(y_\infty) - g_z(y'_\infty)| \leq |g_z(y_\infty) - g_{x_i}(y_\infty)| &+ |g_{x_i}(y_\infty) - g_{x_i}(y'_\infty)| \\ &+ |g_{x_i}(y'_\infty) - g_z(y'_\infty)| \\ \leq |g_z(y_\infty) - f(z, y_\infty)| &+ |f(z, y_\infty) - f(x_i, y_\infty)| \\ +|f(x_i, y_\infty) - g_{x_i}(y_\infty)| &+ 0 + |g_{x_i}(y'_\infty) - f(x_i, y'_\infty)| \\ +|f(x_i, y'_\infty) - f(z, y'_\infty)| &+ |f(z, y'_\infty) - g_z(y'_\infty)| \\ \leq 2b + 4\delta + 2a + 2\delta &= 2a + 2b + 6\delta. \end{aligned}$$

It follows from the above inequality and (2.1) that

$$r \leq 2a + b + 4\delta + 2a + 2b + 6\delta = 4a + 3b + 10\delta = r - \varepsilon/2.$$

This contradiction proves our result. ∎

Proof of Theorem 2.1. Let $r = 10a + 6b$ and

$$A_n = \Big\{x \in X : \mathcal{O}\big(f,(x,y)\big) < r + \frac{1}{n} \text{ for all } y \in Y\Big\} \quad (n \in \mathbb{N}).$$

Since Y is compact and oscillation is upper semi-continuous, A_n is open for each $n \in \mathbb{N}$. We will show that A_n is dense in X for each $n \in \mathbb{N}$. Let U be an arbitrary non-empty open subset of X. By Lemma 2.3, there is a non-empty open subset U' of U such that $\{f(t,\cdot) : t \in U'\}$ is $(4a + 3b + \frac{1}{8n})$-equicontinuous. According to Lemma 2.2, there exits a non-empty open subset U'' of U' and a finite cover $\{V_1, \ldots, V_m\}$ such that

$$diam(U'' \times V_i) \leq 2\big((4a + 3b + \frac{1}{8n}) + a\big) + \frac{1}{4n} < r + \frac{1}{n}.$$

This means that $U'' \subset A_n \cap U$. Therefore A_n is dense in X for each $n \in \mathbb{N}$. Define $D = \bigcap_{n \geq 1} A_n$. Then for each $(x, y) \in D \times Y$, we have $\mathcal{O}\big(f,(x,y)\big) \leq 10a + 6b$. This completes the proof of the Theorem. □

Remark 2.4. (1) Saint-Raymond [12] proved that every separately continuous mapping $f : X \times Y \to \mathbb{R}$, where X is a separable Baire space has the Namioka property. Since every separable Baire space is α-favorable for the game $\mathcal{G}(X)$, by Theorem 2.1 this result is also true when f is quasi-separately continuous.
(2) Let X be a β-unfavorable space and $g : X \to \mathbb{R}$ be a quasi-continuous mapping which is not continuous. For example, let $g(x) = [x]$ for each $x \in \mathbb{R}$. Define $f : X \times Y \to \mathbb{R}$ by $f(x, y) = g(x)$. Since f is not separately continuous, the results on joint continuity of separate continuous mappings can not be applied. However, f is quasi-separately continuous. Therefore, by Theorem 2.1, f has the Namioka property.

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