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ITERATIVE METHODS FOR FIXED POINTS AND EQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, a new iterative scheme by hybrid method is constructed. Strong convergence of the scheme to a common element of the set of fixed points of an infinite family of relatively quasi-nonexpansive mappings and set of common solutions to a system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth is proved. Our results extend important recent results.

1. Introduction and preliminaries

Let E be a real Banach space and C be nonempty closed convex subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \ \forall x, y \in C.$$

A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is defined as $F(T) := \{x \in C : Tx = x\}$.

Let $F : C \times C$ into $\mathbb{R}$ be an equilibrium bifunction. The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0,$$

for all $y \in C$. We shall denote the set of solutions of this equilibrium problem by $EP(F)$. Thus

$$EP(F) := \{x^* \in C : F(x^*, y) \geq 0, \ \forall y \in C\}.$$

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The equilibrium include fixed point problems, optimization problems and variational inequality problems as special cases (see, for example, [3]). Some methods have been proposed to solve the equilibrium problem, see for example, [6, 13, 22].

In [11], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{w \in C : \phi(w, y_n) \le \phi(w, x_n)\}, \\ W_n = \{w \in C : \langle x_n - w, Jx_0 - Jx_n\rangle, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \ n \ge 0. \end{cases}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)}x_0$, where $F(T) \neq \emptyset$.

Recently, Takahashi and Zembayashi [19] introduced a hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mappings which is also a solution to equilibrium problem in a uniformly smooth real Banach space which is also uniformly convex. In particular, they proved the following theorem.

Theorem 1.1. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. Let F be a bifunction from $C \times C$ satisfying $(A1)-(A4)$ and let T be a relatively nonexpansive mappings of C into itself such that $F := F(T) \bigcap EP(F) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C,\ C_1 = C$*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \ n \ge 1, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \forall y \in C \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \le \phi(w, x_n)\}, \ n \ge 1, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ n \ge 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\}_{n=1}^{\infty} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Motivated by the results of Takahashi and Zembayashi [19] (Theorem 1.1 above) and Matsushita and Takahashi [11], we prove a strong convergence theorem for an infinite family of relatively quasi-nonexpansive mappings and system of equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth. Our results extend the results of Takahashi and Zembayashi [19] and Matsushita and Takahashi [11].

2. Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : ||x|| \le 1, ||y|| \le \tau\}.$$

E is uniformly smooth if and only if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Let $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left|\left| \frac{x+y}{2} \right|\right| : ||x|| = ||y|| = 1; \epsilon = ||x-y|| \right\}.$$

E is *uniformly convex* if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| \leq 1,\ ||y|| \leq 1$ and $||x-y|| \geq \epsilon$, then $||\frac{1}{2}(x+y)|| \leq 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A normed space E is called *strictly convex* if for all $x, y \in E,\ x \neq y,\ ||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1,\ \forall \lambda \in (0,1)$.

Let E^* be the dual space of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = ||x||^2 = ||f||^2\}.$$

The following properties of J are well known (The reader can consult [8, 16, 17] for more details):

(1) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.
(2) $J(x) \neq \emptyset,\ x \in E$.
(3) If E is reflexive, then J is a mapping from E onto E^*.
(4) If E is smooth, then J is single valued.

Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x,y) := ||x||^2 - 2\langle x, J(y) \rangle + ||y||^2,\ \forall x, y \in E.$$

Let C be a nonempty subset of E and let T be a mapping from K into E. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widetilde{F}(T)$. We say that a mapping T is *relatively nonexpansive* (see, for example, [4, 5, 7, 11, 15]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x),\ \forall x \in C,\ p \in F(T)$;
(R3) $F(T) = \widetilde{F}(T)$.

If T satisfies (R1) and (R2), then T is said to be *relatively quasi-nonexpansive*. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have studied the methods of approximating the fixed points of relatively quasi-nonexpansive mappings (see, for example, [12, 14, 20] the references contained therein).

A mapping $T : C \to C$ is called *quasi-nonexpansive* if

$$||Tx - x^*|| \leq ||x - x^*||, \ \forall x \in C, \ x^* \in F(T).$$

It is clear that every nonexpansive mapping with nonempty set of fixed points is quasi-nonexpansive. Clearly, in Hilbert space H, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same, for $\phi(x, y) = ||x - y||^2, \ \forall x, y \in H$ and this implies that

$$\phi(p, Tx) \leq \phi(p, x) \Leftrightarrow ||Tx - p|| \leq ||x - p||, \ \forall x \in C, \ p \in F(T).$$

Examples of relatively quasi-nonexpansive mappings are given in [14].

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty closed convex subset of E. Following Alber [2], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) := arg\min_{y \in C} \phi(y, x) \quad (x \in E).$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 9, 10, 17]). If E is a Hilbert space, then Π_C is the metric projection of H onto C. From [10], in uniformly convex and uniformly smooth Banach spaces, we have

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E.$$

We know that the following lemmas hold for generalized projections.

Lemma 2.1. *(Alber [2], Kamimura and Takahashi [10]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \ \forall x \in C, \ \forall y \in E.$$

Lemma 2.2. *(Alber [2], Kamimura and Takahashi [10]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $x \in E$ and $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, J(x) - J(z) \rangle \leq 0, \ \forall y \in C.$$

The fixed points set $F(T)$ of a relatively quasi-nonexpansive mapping is closed convex as a consequence of the following lemma.

Lemma 2.3. *(Qin et al. [14], Nilsrakoo and Saejung [12]) Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let T be a closed relatively quasi- nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

Lemma 2.4. *(Kamimura and Takahashi [10]) Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in E such that either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded. If $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} ||x_n - y_n|| = 0$.*

Lemma 2.5. *(Xu, [21]) Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0,1]$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R},\ \ g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||).$$

Lemma 2.6. *(Zegeye et al., [23]) Let C be a nonempty closed and convex subset a real uniformly convex Banach space E, let $T_i : C \to E,\ i = 1, 2, ...$ be closed relatively quasi-nonexpansive mappings such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Then the mapping $T := J^{-1}\Big(\sum_{i=0}^{\infty} \zeta_i JT_i\Big) : C \to E$ is closed relatively quasi-nonexpansive mapping and $F(T) = \cap_{i=1}^{\infty} F(T_i)$, where $\sum_{i=0}^{\infty} \zeta_i = 1,\ \zeta_i > 0,\ \forall i \geq 0$ and $T_0 = I$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y, \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.7. *(Blum and Oettli, [3]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \ \text{ for all } y \in K.$$

Lemma 2.8. *(Takahashi and Zembayashi, [18]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r^F : E \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}$$

for all $z \in E$. Then, the following hold:
1. T_r^F is single-valued;
2. T_r^F is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,

$$\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle;$$

3. $F(T_r^F) = EP(F)$;
4. $EP(F)$ is closed and convex.

Observe that an operator T in a Banach space E is said to be *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

3. Main Results

We now state and prove the following theorem.

Theorem 3.1. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k = 1, 2, ..., m$, let F_k be a bifunction from $C \times C$ satisfying (A1) − (A4) and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{k=1}^{m} EP(F_k) \cap \Big(\cap_{i=1}^{\infty} F(T_i)\Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\ n \geq 1,\\ u_{1,n} = T^{F_1}_{r_{1,n}} y_n\\ u_{2,n} = T^{F_2}_{r_{2,n}} y_n\\ \vdots\\ u_{m,n} = T^{F_m}_{r_{m,n}} y_n\\ w_n = J^{-1}(\beta_{1,n} J u_{1,n} + \beta_{2,n} J u_{2,n} + ... + \beta_{m,n} J u_{m,n})\\ C_{n+1} = \{w \in C_n : \phi(w, w_n) \leq \phi(w, x_n)\},\ n \geq 1,\\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 1, \end{cases} \tag{3.1}$$

where J is the duality mapping on E and $T := J^{-1}\Big(\sum_{i=0}^{\infty} \zeta_i J T_i\Big)$ with $T_0 = I$ and $\sum_{i=0}^{\infty} \zeta_i = 1,\ \zeta_i > 0,\ \forall i \geq 0$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_{k,n}\}_{n=1}^{\infty},\ k = 1, 2, ..., m$ are sequences in $(0, 1)$ such that
(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$
(ii) $\sum_{k=1}^{m} \beta_{k,n} = 1,\ n \geq 1$
(iii) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty),\ (k = 1, 2, ..., m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0,\ k = 1, 2, ..., m.$
Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Proof. We first show that $C_n,\ \forall n \geq 1$ is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed convex for some $n > 1$. From the definition of C_{n+1}, we have that $z \in C_{n+1}$ implies $\phi(z, w_n) \leq \phi(z, x_n)$. This is equivalent to

$$2\Big(\langle z, Jx_n\rangle - \langle z, Jw_n\rangle\Big) \leq ||x_n||^2 - ||w_n||^2$$

This implies that C_{n+1} is closed convex for the same $n > 1$. Hence, C_n is closed and convex $\forall n \geq 1$. This shows that $\Pi_{C_{n+1}} x_0$ is well defined for all $n \geq 0$.
We next show that $F \subset C_n,\ \forall n \geq 1$. From Lemma 2.8, one has that $T^{F_k}_{r_{k,n}},\ k = 1, 2, ..., m$ is relatively quasi-nonexpansive mapping. For $n = 1$, we have $F \subset C =$

C_1. Then for each $x^* \in F$, we obtain

$$\begin{aligned}\phi(x^*, w_n) &= \phi(x^*, J^{-1}(\beta_{1,n}Ju_{1,n} + \beta_{2,n}Ju_{2,n} + ... + \beta_{m,n}Ju_{m,n})) \\ &= ||x^*||^2 - 2\beta_{1,n}\langle x^*, Ju_{1,n}\rangle - 2\beta_{2,n}\langle x^*, Ju_{2,n}\rangle \\ &\quad -2\beta_{3,n}\langle x^*, Ju_{3,n}\rangle - \ldots - 2\beta_{m,n}\langle x^*, Ju_{m,n}\rangle \\ &\quad +||\beta_{1,n}Ju_{1,n} + \beta_{2,n}Ju_{2,n} + ... + \beta_{m,n}Ju_{m,n}||^2 \\ &\leq ||x^*||^2 - 2\beta_{1,n}\langle x^*, Ju_{1,n}\rangle - 2\beta_{2,n}\langle x^*, Ju_{2,n}\rangle \\ &\quad -2\beta_{3,n}\langle x^*, Ju_{3,n}\rangle - \ldots - 2\beta_{m,n}\langle x^*, Ju_{m,n}\rangle \\ &\quad +\beta_{1,n}||Ju_{1,n}||^2 + \beta_{2,n}||Ju_{2,n}||^2 + \beta_{3,n}||Ju_{3,n}||^2. \end{aligned} \tag{3.2}$$

Furthermore, using (3.2), we have

$$\begin{aligned}\phi(x^*, w_n) &= \beta_{1,n}\phi(x^*, T^{F_1}_{r_{1,n}}y_n) + \beta_{2,n}\phi(x^*, T^{F_2}_{r_{2,n}}y_n) \\ &\quad +\beta_{3,n}\phi(x^*, T^{F_3}_{r_{3,n}}y_n) + \ldots + \beta_{m,n}\phi(x^*, T^{F_m}_{r_{m,n}}y_n) \\ &\leq \beta_{1,n}\phi(x^*, y_n) + \beta_{2,n}\phi(x^*, y_n) \\ &\quad +\beta_{3,n}\phi(x^*, y_n) + \ldots + \beta_{m,n}\phi(x^*, y_n) \\ &\leq \phi(x^*, y_n). \end{aligned} \tag{3.3}$$

Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 2.5 and (3.1), we have

$$\begin{aligned}\phi(x^*, y_n) &= \phi(x^*, J^{-1}(\alpha_nJx_n + (1-\alpha_n)JTx_n)) \\ &= ||x^*||^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, JTx_n\rangle \\ &\quad +||\alpha_nJx_n + (1-\alpha_n)JTx_n||^2 \\ &\leq ||x^*||^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, JTx_n\rangle \\ &\quad +\alpha_n||Jx_n||^2 + (1-\alpha_n)||JTx_n||^2 - \alpha_n(1-\alpha_n)g(||Jx_n - JTx_n||) \\ &= \alpha_n\phi(x^*, x_n) + (1-\alpha_n)\phi(x^*, Tx_n) - \alpha_n(1-\alpha_n)g(||Jx_n - JTx_n||) \\ &\leq \phi(x^*, x_n) - \alpha_n(1-\alpha_n)g(||Jx_n - JTx_n||). \end{aligned} \tag{3.4}$$

So, $x^* \in C_n$. This implies that $\emptyset \neq F \subset C_n,\ \forall n \geq 1$ and the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (3.1) is well defined.

We now show that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists. From (3.1), we have $x_n = \Pi_{C_n}x_0$ which implies that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0,\ \forall z \in C_n \tag{3.5}$$

and in particular

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0,\ \forall p \in F.$$

By Lemma 2.1, we have

$$\begin{aligned}\phi(x_n, x_0) &= \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \\ &\leq \phi(p, x_0)\end{aligned}$$

for each $p \in F \subset C_n,\ n \geq 1$. Hence, the sequence $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ is bounded. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0),\ \forall n \geq 0.$$

Therefore, $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}_{n=0}^{\infty}$ exists.

Now, we show that $\{x_n\}_{n=0}^{\infty}$ is Cauchy. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It then follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

It then follows from Lemma 2.4 that $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since E is a Banach space and C is closed convex, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

We next show that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Now since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ we have in particular that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$ and this further implies that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, w_n) \leq \phi(x_{n+1}, x_n), \ \forall n \geq 0.$$

Therefore,

$$\lim_{n\to\infty} \phi(x_{n+1}, w_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 2.4 that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 = \lim_{n\to\infty} ||x_{n+1} - w_n||.$$

So,

$$||x_n - w_n|| \leq ||x_{n+1} - x_n|| + ||x_{n+1} - w_n||.$$

Hence,

$$\lim_{n\to\infty} ||x_n - w_n|| = 0.$$

Since $x_n \to p$ as $n \to \infty$ and $||x_n - w_n|| \to 0$ as $n \to \infty$, we have $w_n \to p$ as $n \to \infty$. Furthermore, since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} ||x_n - w_n|| = 0$, we obtain

$$\lim_{n\to\infty} ||Jx_n - Jw_n|| = 0.$$

Let $r := \sup_{n\geq 1}\{||x_n||, ||Tx_n||\}$. Substituting (3.3) into (3.4), we obtain

$$\alpha_n(1 - \alpha_n) g(||Jx_n - JTx_n||) \leq \phi(x^*, x_n) - \phi(x^*, w_n).$$

But

$$\begin{aligned} \phi(x^*, x_n) - \phi(x^*, w_n) &= ||x_n||^2 - ||w_n||^2 - 2\langle x^*, Jx_n - Jw_n\rangle \\ &\leq \Big| ||x_n||^2 - ||w_n||^2 \Big| + 2\Big|\langle x^*, Jx_n - Jw_n\rangle\Big| \\ &\leq \Big| ||x_n|| - ||w_n|| \Big| (||x_n|| + ||w_n||) + 2||x^*|| ||Jx_n - Jw_n|| \\ &\leq ||x_n - w_n|| (||x_n|| + ||w_n||) + 2||x^*|| ||Jx_n - Jw_n||. \end{aligned}$$

From $\lim_{n\to\infty}||x_n - w_n|| = 0$ and $\lim_{n\to\infty}||Jx_n - Jw_n|| = 0$, we obtain

$$\phi(x^*, x_n) - \phi(x^*, w_n) \to 0,\ n \to \infty.$$

Using the condition $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} g(||Jx_n - JTx_n||) = 0.$$

By property of g, we have $\lim_{n\to\infty}||Jx_n - JTx_n|| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}||x_n - Tx_n|| = 0.$$

Since T is closed and $x_n \to p$, we have $p \in F(T) = \cap_{i=1}^{\infty}F(T_i)$.

Next, we show that $p \in \cap_{k=1}^{m}EP(F_k)$. Since $x_n \to p$, we obtain from (3.3), (3.4) and Lemma 2.4 that $y_n \to p,\ n \to \infty$. Furthermore, since $T_{r_{k,n}}^{F_k}$ is relatively nonexpansive for each $k = 1, 2, ..., m$, we obtain

$$0 \le \phi(p, u_{k,n}) \;=\; \phi(p, T_{r_{k,n}}^{F_k}y_n) \le \phi(p, y_n) \to 0.$$

Then we have from Lemma 2.4 that $\lim_{n\to\infty}||p - u_{k,n}|| = 0,\ k = 1, 2, ..., m$. Consequently, we have that

$$||u_{k,n} - y_n|| \le ||u_{k,n} - p|| + ||y_n - p|| \to 0. \tag{3.6}$$

Also, since J is uniformly norm-to-norm continuous on bounded sets and using (3.6), we obtain

$$\lim_{n\to\infty}||Ju_{k,n} - Jy_n|| = 0.$$

Since $\liminf_{n\to\infty} r_{k,n} > 0$, then

$$\lim_{n\to\infty}\frac{||Ju_{k,n} - Jy_n||}{r_{k,n}} = 0. \tag{3.7}$$

By Lemma 2.8, we have that

$$F_k(u_{k,n}, y) + \frac{1}{r_{k,n}}\langle y - u_{k,n}, Ju_{k,n} - Jy_n\rangle \ge 0,\ \forall y \in C.$$

Furthermore, using $(A2)$ in the last inequality, we obtain

$$\frac{1}{r_{k,n}}\langle y - u_{k,n}, Ju_{k,n} - Jy_n\rangle \ge F_k(y, u_{k,n}).$$

By $(A4)$, (3.7) and $u_{k,n} \to p$, we have

$$F_k(y, p) \le 0,\ \forall y \in C.$$

Let $z_t := ty + (1-t)p$ for all $t \in (0, 1]$ and $y \in K$. This implies that $z_t \in K$. This yields that $F_k(z_t, p) \le 0$. It follows from $(A1)$ and $(A4)$ that

$$\begin{aligned} 0 &= F_k(z_t, z_t) \le tF_k(z_t, y) + (1-t)F_k(z_t, p) \\ &\le tF_k(z_t, y) \end{aligned}$$

and hence

$$0 \le F_k(z_t, y).$$

From condition $(A3)$, we obtain

$$F_k(p, y) \geq 0,\ \forall y \in C.$$

This implies that $p \in EP(F_k),\ k = 1, 2, \ldots, m$. Thus, $p \in \cap_{k=1}^{m} EP(F_k)$. Hence, we have $p \in F = \cap_{k=1}^{m} EP(F_k) \cap F(T)$.

Finally, we show that $p = \Pi_F x_0$. Now by taking the limit in (3.5), we have

$$\langle p - z, Jx_0 - Jp \rangle \geq 0,\ \forall z \in F.$$

By Lemma 2.2, we have $p = \Pi_F x_0$. □

Corollary 3.2. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k = 1, 2$, let F_k be a bifunction from $C \times C$ satisfying $(A1)-(A4)$ and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of closed relatively-quasi nonexpansive mappings of C into itself such that $F := \cap_{k=1}^{2} EP(F_k) \cap \Big(\cap_{i=1}^{\infty} F(T_i) \Big) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0$,*

$$\left\{ \begin{array}{l} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTx_n),\ n \geq 1, \\ u_n = S_{r_{1,n}}^{F_1} y_n \\ v_n = S_{r_{2,n}}^{F_2} y_n \\ w_n = J^{-1}(\beta_n Ju_{1,n} + (1-\beta_n) Ju_{2,n}) \\ C_{n+1} = \{w \in C_n : \phi(w, w_n) \leq \phi(w, x_n)\},\ n \geq 1, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 1, \end{array} \right.$$

where J is the duality mapping on E and $T := J^{-1}\Big(\sum_{i=0}^{\infty} \zeta_i JT_i \Big)$ with $T_0 = I$ and $\sum_{i=0}^{\infty} \zeta_i = 1,\ \zeta_i > 0,\ \forall i \geq 0$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that
(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$
(ii) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty),\ (k = 1, 2)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0,\ k = 1, 2$.
Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Corollary 3.3. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k = 1, 2, ..., m$, let F_k be a bifunction from $C \times C$ satisfying $(A1)-(A4)$ such that $F := \cap_{k=1}^{m} EP(F_k) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in C,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0$,*

$$\left\{ \begin{array}{l} u_{1,n} = S_{r_{1,n}}^{F_1} x_n \\ u_{2,n} = S_{r_{2,n}}^{F_2} x_n \\ \vdots \\ u_{m,n} = S_{r_{m,n}}^{F_m} x_n \\ w_n = J^{-1}(\beta_{1,n} Ju_{1,n} + \beta_{2,n} Ju_{2,n} + \ldots + \beta_{m,n} Ju_{m,n}) \\ C_{n+1} = \{w \in C_n : \phi(w, w_n) \leq \phi(w, x_n)\},\ n \geq 1, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 1, \end{array} \right.$$

where J is the duality mapping on E and $\sum_{i=0}^{\infty} \zeta_i = 1,\ \zeta_i > 0,\ \forall i \geq 0$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_{k,n}\}_{n=1}^{\infty},\ k = 1, 2, ..., m$ are sequences in $(0, 1)$ such that *(i)* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$

(ii) $\sum_{k=1}^{m} \beta_{k,n} = 1,\ n \geq 1$

(iii) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty),\ (k = 1,2,...,m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0,\ k = 1,2,...,m.$

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_F x_0$*.*

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