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(δ, ε)-DOUBLE DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be an algebra and let $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be two linear mappings. A (δ, ε)-double derivation is a linear mapping $d : \mathcal{A} \to \mathcal{A}$ satisfying $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ $(a, b \in \mathcal{A})$. We study some algebraic properties of these mappings and give a formula for calculating $d^n(ab)$. We show that if $\mathcal{A}$ is a Banach algebra such that either is semi-simple or every derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is continuous then every (δ, ε)-double derivation on $\mathcal{A}$ is continuous whenever so are δ and ε. We also discuss the continuity of ε when d and δ are assumed to be continuous.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an algebra. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is said to be a derivation if it satisfies the Leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. Now suppose that δ, ε are two ordinary derivations. We see that $d = \delta\varepsilon$ satisfies

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad (a, b \in \mathcal{A}). \tag{1.1}$$

This can be assumed as a generalization of the concept of a derivation.

Now let $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be two linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *(δ, ε)-double derivation* if it satisfies (1.1). By a δ-double derivation we mean a (δ, δ)-double derivation. See [7] for an initial study of δ-double derivations. Clearly, if d is a derivation then d^2 is a d-double derivation, and also d is a 0-double derivation where 0 denotes the zero mapping. Moreover, if I denotes the identity mapping on $\mathcal{A}$, then each σ-derivation $d : \mathcal{A} \to \mathcal{A}$ is a $(\sigma - I, d)$-double derivation. Here by a σ-derivation we mean a linear mapping d on $\mathcal{A}$ satisfying $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$ $(a, b \in \mathcal{A})$, for some linear mapping σ on $\mathcal{A}$, see

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[4, 6] for more about σ-derivations. Also, every homomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ is a $(\frac{\varphi}{2} - I, \varphi)$-double derivation.

In Section 2, we study some algebraic properties of (δ, ε)-double derivations and give a formula to calculate $d^n(ab)$. Section 3 is devoted to the study of automatic continuity of (δ, ε)-double derivations on Banach algebras and to extension of some results of [7]. We will observe that under the assumption of continuity of any pair of the linear mappings d, δ and ε, what happens for the third one. Assuming that δ and ε are continuous on $\mathcal{A}$, we show that if every derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is continuous then every (δ, ε)-double derivation on $\mathcal{A}$ is continuous. Also, it is proved that every (δ, ε)-double derivation on a semi-simple Banach algebra is continuous whenever so are δ and ε. Next we assume that d and δ are continuous and obtain some results concerning the separating space of ε. We will show that if d is a continuous (δ, ε)-double derivation on a commutative unital prime Banach algebra, then ε is continuous whenever δ is nonzero and continuous. We also obtain some results concerning δ-double derivations.

2. ALGEBRAIC PROPERTIES

Let $\mathcal{A}$ be an algebra. Suppose that δ, ε are two linear mappings on $\mathcal{A}$, and $d : \mathcal{A} \to \mathcal{A}$ is a (δ, ε)-double derivation, that is

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad (a, b \in \mathcal{A}).$$

For simplicity, we consider a bilinear mapping $\lambda : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$\lambda(a, b) = \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad (a, b \in \mathcal{A}).$$

Proposition 2.1. *Let $\mathcal{A}$ be an algebra and let δ, ε be two linear mappings on $\mathcal{A}$. Suppose that $d : \mathcal{A} \to \mathcal{A}$ is a (δ, ε)-double derivation.*

(i) For each idempotent $e \in \mathcal{A}$, $ed(e)e = -e\lambda(e, e)e$.

Moreover, if $\mathcal{A}$ is unital, then

(ii) $\lambda(a, 1) = -ad(1), \lambda(1, a) = -d(1)a$ for all $a \in \mathcal{A}$, and $d(1) = -\lambda(1, 1)$;

(iii) $\lambda(ab, 1) = a\lambda(b, 1), \lambda(1, ab) = \lambda(1, a)b$ for all $a, b \in \mathcal{A}$;

(iv) $d(1) = 0$ if and only if $\lambda(a, 1) = 0 = \lambda(1, a)$ for all $a \in \mathcal{A}$.

Proof. (i) Let e be an idempotent in $\mathcal{A}$. Then

$$d(e) = d(e^2) = ed(e) + d(e)e + \lambda(e, e). \tag{2.1}$$

Multiplying (2.1) by e gives the result.

(ii) For each $a \in \mathcal{A}$,

$$d(a) = ad(1) + d(a)1 + \lambda(a, 1). \tag{2.2}$$

Hence $\lambda(a, 1) = -ad(1)$. Similarly $\lambda(1, a) = -d(1)a$. The last assertion is now obvious.

(iii) By (ii), for $a, b \in \mathcal{A}$ we have $\lambda(ab, 1) = -abd(1) = a\lambda(b, 1), \lambda(1, ab) = -d(1)ab = \lambda(1, a)b$.

(iv) It follows from (2.2). □

If δ,ε are derivations on an algebra $\mathcal{A}$, it is easy to see that $\delta\varepsilon$ is a (δ,ε)-double derivation. Now let δ,ε be derivations and let d be a (δ,ε)-double derivation. What can we say about d?

Proposition 2.2. *Let δ,ε be derivations and let d be a (δ,ε)-double derivation on an algebra $\mathcal{A}$. Then there exists a derivation D on $\mathcal{A}$ such that $d=\delta\varepsilon+D$.*

Proof. Straightforward. □

It is well known that every derivation on a commutative Banach algebra maps it into its radical, see [8]. As a consequence of Proposition 2.2, every (δ,ε)-double derivation d on a commutative Banach algebra $\mathcal{A}$, for which δ,ε are derivations, maps into the radical. If moreover, $\mathcal{A}$ is semi-simple, then $d=0$.

Now we are going to find a formula for $d^n(ab)$, where d is a (δ,ε)-double derivation. This is not as simple as the one for an ordinary derivation. In fact what we give here is something such as an algorithm to calculate $d^n(ab)$.

Let δ,ε be arbitrary linear mappings on an algebra $\mathcal{A}$. We construct a family of linear mappings $\{\phi^{\delta,\varepsilon}_{n,k}\}$, $(n\in\mathbb{N},\ 0\le k\le 2^n-1)$, which is called the *binary family* for the ordered pair of linear mappings (δ,ε), as follows.

Write the non-negative integer k in base 2 with exactly n digits, and put δ in place of 1's and ε in place of 0's. For example, if $n=4$ then $6=(0110)_2$, $10=(1010)_2$, $\phi^{\delta,\varepsilon}_{4,6}=\varepsilon\delta\delta\varepsilon=\varepsilon\delta^2\varepsilon$ and $\phi^{\delta,\varepsilon}_{4,10}=\delta\varepsilon\delta\varepsilon$. When there is no risk of ambiguity, we simply write $\phi_{n,k}$ instead of $\phi^{\delta,\varepsilon}_{n,k}$. The following lemma is stated in [6]. We give its proof for the sake of convenience.

Lemma 2.3. *Let $n\in\mathbb{N}$ and let k be a non-negative integer such that $0\le k\le 2^n-1$. Then*

(i) $\delta\phi_{n,k}=\phi_{n+1,k+2^n}$;
(ii) $\varepsilon\phi_{n,k}=\phi_{n+1,k}$;
(iii) $\phi_{n,k}\delta=\phi_{n+1,2k+1}$;
(iv) $\phi_{n,k}\varepsilon=\phi_{n+1,2k}$.

Proof. Write k in the base 2 as $(c_n\dots c_2c_1)_2$, where $c_j\in\{0,1\}$ for $j=1,\dots,n$. Then

(i) $\delta\phi_{n,k}=\phi_{n+1,(1c_n\dots c_2c_1)_2}=\phi_{n+1,k+2^n}$;
(ii) $\varepsilon\phi_{n,k}=\phi_{n+1,(0c_n\dots c_2c_1)_2}=\phi_{n+1,k}$;
(iii) $\phi_{n,k}\delta=\phi_{n+1,(c_n\dots c_2c_11)_2}=\phi_{n+1,2k+1}$;
(iv) $\phi_{n,k}\varepsilon=\phi_{n+1,(c_n\dots c_2c_10)_2}=\phi_{n+1,2k}$. □

Now consider the algebraic tensor product $\mathcal{A}\otimes\mathcal{A}$. Let δ,ε and d be arbitrary linear mappings on $\mathcal{A}$. Consider two bilinear mappings $(a,b)\mapsto d(a)\otimes b+a\otimes d(b)$ and $(a,b)\mapsto\delta(a)\otimes\varepsilon(b)+\varepsilon(a)\otimes\delta(b)$ from $\mathcal{A}\times\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{A}$. Then we have two linear mappings $\alpha,\beta:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ satisfying

$$\alpha(a\otimes b)=d(a)\otimes b+a\otimes d(b), \tag{2.3}$$

$$\beta(a\otimes b)=\delta(a)\otimes\varepsilon(b)+\varepsilon(a)\otimes\delta(b) \tag{2.4}$$

for $a,b\in\mathcal{A}$.

Lemma 2.4. *If δ, ε and d are linear mappings on an algebra $\mathcal{A}$ and α, β are defined as above, then for each positive integer n*

(i) $\alpha^n(a \otimes b) = \sum_{k=0}^{n} \binom{n}{k} d^k(a) \otimes d^{n-k}(b)$;

(ii) $\beta^n(a \otimes b) = \sum_{k=0}^{2^n-1} \phi_{n,k}(a) \otimes \phi_{n,2^n-1-k}(b)$.

Proof. (i) We proceed by induction. Clearly the equality in (i) holds for $n = 1$. Assume that the result is true for the positive integer n. Then form (2.3) we have

$$\begin{aligned}
&\alpha^{n+1}(a \otimes b) = \alpha(\sum_{k=0}^{n} \binom{n}{k} d^k(a) \otimes d^{n-k}(b)) \\
&= \sum_{k=0}^{n} \binom{n}{k} d^{k+1}(a) \otimes d^{n-k}(b) + \sum_{k=0}^{n} \binom{n}{k} d^k(a) \otimes d^{n+1-k}(b) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} d^{k+1}(a) \otimes d^{n+1-(k+1)}(b) + \binom{n}{n} d^{n+1}(a) \otimes b \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k+1} d^{k+1}(a) \otimes d^{n+1-(k+1)}(b) + \binom{n}{0} a \otimes d^{n+1}(b) \\
&= \sum_{k=0}^{n-1} (\binom{n}{k+1} + \binom{n}{k}) d^{k+1}(a) \otimes d^{n+1-(k+1)}(b) \\
&\quad + \binom{n}{0} a \otimes d^{n+1}(b) + \binom{n}{n} d^{n+1}(a) \otimes b \\
&= \sum_{k=1}^{n} \binom{n+1}{k} d^k(a) \otimes d^{n+1-k}(b) + \binom{n+1}{0} a \otimes d^{n+1}(b) + \binom{n+1}{n+1} d^{n+1}(a) \otimes b \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} d^k(a) \otimes d^{n+1-k}(b).
\end{aligned}$$

(ii) Obviously, the result is true for $n = 1$. Let (ii) hold for n. Then from (2.4) and Lemma 2.3, we have

$$\begin{aligned}
&\beta^{n+1}(a \otimes b) = \beta\left(\sum_{k=0}^{2^n-1} \phi_{n,i}(a) \otimes \phi_{n,2^n-1-j}(b)\right) \\
&= \sum_{k=0}^{2^n-1} \delta\phi_{n,k}(a) \otimes \varepsilon\phi_{n,2^n-1-k}(b) + \sum_{k=0}^{2^n-1} \varepsilon\phi_{n,k}(a) \otimes \delta\phi_{n,2^n-1-k}(b) \\
&= \sum_{k=0}^{2^n-1} \phi_{n+1,k+2^n}(a) \otimes \phi_{n+1,2^n-1-k}(b) + \sum_{k=0}^{2^n-1} \phi_{n+1,k}(a) \otimes \phi_{n+1,2^{n+1}-1-k}(b)
\end{aligned}$$

$$= \sum_{k=2^n}^{2^{n+1}-1} \phi_{n+1,k}(a) \otimes \phi_{n+1,2^{n+1}-1-k}(b) + \sum_{k=0}^{2^n-1} \phi_{n+1,k}(a) \otimes \phi_{n+1,2^{n+1}-1-k}(b)$$

$$= \sum_{k=0}^{2^{n+1}-1} \phi_{n+1,k}(a) \otimes \phi_{n+1,2^{n+1}-1-k}(b)$$

□

Suppose that $\delta, \varepsilon, d, \alpha$ and β are as above. Let $\{\psi_{n,j}\}$ $(n \in \mathbb{N},\ 0 \le j \le 2^n - 1)$, be the binary family for (α, β). We calculate $\{\psi_{n,j}\}$ for $n = 3$.

Example 2.5. Take $n = 3$. By the definition of $\{\psi_{n,j}\}$ and Lemma 2.4 we have

$$0 = (000)_2, \psi_{3,0}(a \otimes b) = \beta^3(a \otimes b) = \sum_{i=0}^{2^3-1} \phi_{3,i}(a) \otimes \phi_{3,2^3-1-i}(b)$$

$$1 = (001)_2, \psi_{3,1}(a \otimes b) = \beta^2\alpha(a \otimes b) = \sum_{i=0}^{2^2-1} \sum_{k=0}^{1} \binom{1}{k} \phi_{2,i}(d^k(a)) \otimes \phi_{2,2^2-1-i}(d^{1-k}(b))$$

$$2 = (010)_2, \psi_{3,2}(a \otimes b) = \alpha\beta\alpha(a \otimes b)$$

$$= \sum_{r=0}^{2^1-1} \sum_{k=0}^{1} \sum_{i=0}^{2^1-1} \binom{1}{k} \phi_{1,r}(d^k(\phi_{1,i}(a))) \otimes \phi_{1,2^1-1-r}(d^{1-k}(\phi_{1,2^1-1-i}(b)))$$

$$3 = (011)_2, \psi_{3,3}(a \otimes b) = \beta\alpha^2(a \otimes b) = \sum_{i=0}^{2^1-1} \sum_{k=0}^{2} \binom{2}{k} \phi_{1,i}(d^k(a)) \otimes \phi_{1,2^1-1-i}(d^{1-k}(b))$$

$$4 = (100)_2, \psi_{3,4}(a \otimes b) = \alpha\beta^2(a \otimes b) = \sum_{k=0}^{1} \sum_{i=0}^{2^2-1} \binom{1}{k} d^k(\phi_{2,i}(a)) \otimes d^{1-k}(\phi_{2,2^2-1-i}(b))$$

$$5 = (101)_2, \psi_{3,5}(a \otimes b) = \alpha\beta\alpha(a \otimes b)$$

$$= \sum_{k=0}^{1} \sum_{i=0}^{2^1-1} \sum_{s=0}^{1} \binom{1}{k} \binom{1}{s} d^k(\phi_{1,i}(d^s(a)) \otimes d^{1-k}\phi_{1,2^1-1-i}(d^{1-s}(b))$$

$$6 = (110)_2, \psi_{3,6}(a \otimes b) = \alpha^2\beta(a \otimes b) = \sum_{k=0}^{2} \sum_{i=0}^{2^1-1} \binom{2}{k} d^k(\phi_{1,i}(a)) \otimes d^{2-k}(\phi_{1,2^1-1-i}(b))$$

$$7 = (111)_2, \psi_{3,7}(a \otimes b) = \alpha^3(a \otimes b) = \sum_{k=0}^{3} \binom{3}{k} d^k(a) \otimes d^{3-k}(b).$$

Lemma 2.6. $(\alpha + \beta)^n = \sum_{j=0}^{2^n-1} \psi_{n,j}.$

Proof. The equality holds for $n = 1$. Suppose that we have the equality for n. Then

$$\begin{aligned}(\alpha+\beta)^{n+1}(a\otimes b) &= (\alpha+\beta)((\alpha+\beta)^n(a\otimes b)) = (\alpha+\beta)(\sum_{j=0}^{2^n-1}\psi_{n,j})(a\otimes b)\\ &= \alpha(\sum_{j=0}^{2^n-1}\psi_{n,j}(a\otimes b)) + \beta(\sum_{j=0}^{2^n-1}\psi_{n,j}(a\otimes b))\\ &= \sum_{j=0}^{2^n-1}\psi_{n+1,j+2^n}(a\otimes b).\end{aligned}$$

□

Let $\mathcal{A}$ ba an algebra and d a (δ,ε)-double derivation on $\mathcal{A}$. Suppose that $\sigma : \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, is the linear mapping defined by $\sigma(a\otimes b) = ab \quad (a,b\in\mathcal{A})$. If α,β are defined as above, then it is easy to see that $d(ab) = \sigma((\alpha+\beta)(a\otimes b))$. In other words, $d(\sigma(a\otimes b) = \sigma((\alpha+\beta)(a\otimes b))$, that is $d\sigma = \sigma(\alpha+\beta)$.

Theorem 2.7. *Let d be a (δ,ε)-double derivation on an algebra $\mathcal{A}$. Then*

$$d^n(ab) = \sigma((\alpha+\beta)^n(a\otimes b)) = \sigma(\sum_{j=0}^{2^n-1}\psi_{n,j}(a\otimes b)). \tag{2.5}$$

Proof. We apply an induction argument. The result is clear for $n = 1$. Let (2.5) hold for n. Then

$$\begin{aligned}d^{n+1}(ab) &= d(d^n(ab)) = d(\sigma(\alpha+\beta)^n(a\otimes b)) = \sigma(\alpha+\beta)(\alpha+\beta)^n(a\otimes b))\\ &= \sigma((\alpha+\beta)^{n+1}(a\otimes b)).\end{aligned}$$

The last equality follows from Lemma 2.6. □

3. Automatic continuity

Let $\mathcal{A}$ be a Banach algebra and d a (δ,ε)-double derivation on $\mathcal{A}$. We recall that for a linear mapping $T : \mathcal{A}\to\mathcal{A}$, the separating space of T is the set

$$\mathcal{S}(T) = \{a\in\mathcal{A} : \exists\{a_n\}\subseteq\mathcal{A} \text{ s.t. } a_n\to 0, T(a_n)\to a\}.$$

By the closed graph theorem T is continuous if and only if $\mathcal{S}(T) = \{0\}$.

We are going to find out under which conditions the continuity of any pair of the linear mappings d, δ and ε, implies the continuity of the third one. First we assume that δ and ε are continuous and observe what happens for d. In the second step we assume continuity of d and one of δ or ε, say δ, and observe what happens for the third one. We also prove some results concerning δ-double derivations. For the first step we need some preliminaries.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a Banach $\mathcal{A}$-bimodule. A linear mapping $S : \mathcal{A}\longrightarrow\mathcal{X}$ is said to be left-intertwining if the mapping

$$b\longmapsto aS(b) - S(ab),\ \mathcal{A}\longrightarrow\mathcal{X},$$

is continuous for each $a \in \mathcal{A}$, and right-intertwining if the mapping

$$a \longmapsto S(a)b - S(ab),\ \mathcal{A} \longrightarrow \mathcal{X},$$

is continuous for all $b \in \mathcal{A}$. A linear mapping $S : \mathcal{A} \longrightarrow \mathcal{X}$ is intertwining if it is both left- and right-intertwining. For more about these objects see [1, Section 2.7].

Theorem 3.1. [2, Theorem 2.1] *Let $\mathcal{A}$ be a Banach algebra. Suppose that each derivation from $\mathcal{A}$ to a Banach $\mathcal{A}$-bimodule is continuous. Then each left intertwining map from $\mathcal{A}$ to each Banach $\mathcal{A}$-bimodule is continuous.*

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra. Suppose that each derivation from $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule is continuous. Then each (δ,ε)-double derivation d on $\mathcal{A}$ with continuous δ and ε is continuous.*

Proof. Since δ and ε are continuous, it is easy to see that d is an intertwining map when we consider $\mathcal{A}$ as a Banach $\mathcal{A}$-bimodule in a natural way. Thus, by Theorem 3.1, d is continuous. □

It is a well known result due to B. E. Johnson and A. M. Sinclair [5] that every derivation on a semi-simple Banach algebra is continuous. Here we give a similar result for double derivations.

Theorem 3.3. *Let $\mathcal{A}$ be a semi-simple Banach algebra and let δ,ε be continuous linear mappings on $\mathcal{A}$. Then every (δ,ε)-double derivation on $\mathcal{A}$ is continuous.*

Proof. Consider $\mathcal{A}$ as a Banach $\mathcal{A}$-bimodule with it's own product. Let d be a (δ,ε)-double derivation on $\mathcal{A}$. Thus d is an intertwining map and the separating space $\mathcal{S}(d)$ of d is a separating ideal of $\mathcal{A}$, see [1, Theorem 5.2.24]. Therefore by [1, Lemma 5.2.25], $\mathcal{S}(d)$ is finite dimensional and hence it contains a nonzero idempotent e, whenever $\mathcal{S}(d) \neq \{0\}$, [1, Corollary 5.2.26]. Let $a_n \to 0$ and $d(a_n) \to e$. Then

$$d(ea_n) = ed(a_n) + d(e)a_n + \lambda(e, a_n)$$

which tends to e as $n \to \infty$. But $ea_n \in S(d)$ and d is continuous on the finite dimensional Banach algebra $\mathcal{S}(d)$. Hence $e = 0$, a contradiction. □

In [7, Theorem 3.7] it is proved that every $*$-(δ,ε)-double derivation on a C^*-algebra, with continuous δ and ε, is continuous. Also in [7, Theorem 3.8] it is proved that a (δ,ε)-double derivation on a C^*-algebra is continuous whenever δ and ε are continuous linear $*$-mappings. The next Corollary is a more general result.

Corollary 3.4. *Let δ,ε be continuous linear mappings on a C^*-algebra $\mathcal{A}$. Then every (δ,ε)-double derivation on $\mathcal{A}$ is continuous.*

Now we begin the second step.
Let $\mathcal{B}$ and $\mathcal{C}$ be subsets of $\mathcal{A}$. By $\mathcal{BC}$ we mean the set $\{bc : b \in \mathcal{B},\quad c \in \mathcal{C}\}$. We recall that, the left (resp. right) ideal of $\mathcal{A}$ generated by $\mathcal{B}$ is the linear span of $\mathcal{AB}$ (resp. $\mathcal{BA}$). The closed left (resp. right) ideal of $\mathcal{A}$ generated by $\mathcal{B}$ is defined to be the closure of the linear span of $\mathcal{AB}$ (resp. $\mathcal{BA}$). Clearly, if $\mathcal{A}$ is commutative then the two sided ideal generated by $\mathcal{B}$ is the linear span of $\mathcal{AB}$.

Theorem 3.5. *Let d be a (δ, ε)-double derivation on a Banach algebra $\mathcal{A}$. If d and δ are continuous then $\mathcal{S}(\varepsilon)\delta(\mathcal{A}) = \delta(\mathcal{A})\mathcal{S}(\varepsilon) = \{0\}$.*

Proof. Let $a \in \mathcal{A},\ b \in \mathcal{S}(\varepsilon)$. There is a sequence $\{b_n\}$ in $\mathcal{A}$ converging to 0 with $\lim\limits_{n\to\infty} \varepsilon(b_n) = b$. We have

$$d(ab_n) = ad(b_n) + d(a)b_n + \delta(a)\varepsilon(b_n) + \varepsilon(a)\delta(b_n).$$

Continuity of d and δ implies that $\delta(a)b = 0$. Similarly $b\delta(a) = 0$. □

Corollary 3.6. *Let d be a (δ, ε)-double derivation on a commutative unital prime Banach algebra $\mathcal{A}$. If d and δ are continuous and δ is nonzero, then ε is also continuous.*

Proof. We have $\delta(\mathcal{A})\mathcal{S}(\varepsilon) = \{0\}$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the ideals generated by $\delta(\mathcal{A})$ and $\mathcal{S}(\varepsilon)$, respectively. Then $\mathcal{I}_1\mathcal{I}_2 = \{0\}$. Since $\mathcal{I}_1 \neq \{0\}$, $\mathcal{I}_2$ and hence $\mathcal{S}(\varepsilon)$ is zero. □

Finally, we give some results concerning continuity of δ-double derivations.

Theorem 3.7. *If d is a continuous δ-double derivation on a Banach algebra $\mathcal{A}$ then $\mathcal{S}(\delta)\delta(\mathcal{A}) = \delta(\mathcal{A})\mathcal{S}(\delta) = \{0\}$. Moreover, for each $a \in \mathcal{S}(\delta)$, $a^2 = 0$.*

Proof. The same argument as in Theorem 3.5 gives that $\mathcal{S}(\delta)\delta(\mathcal{A}) = \delta(\mathcal{A})\mathcal{S}(\delta) = \{0\}$. Now let $a_n \to 0$ and $\delta(a_n) \to a$. Then

$$0 = \lim_{n\to\infty} d({a_n}^2) = \lim_{n\to\infty} a_n d(a_n) + d(a_n)a_n + 2\delta(a_n)^2,$$

which implies that $a^2 = 0$. □

Corollary 3.8. *If d is a continuous δ-double derivation on a commutative unital semi-prime Banach algebra $\mathcal{A}$, then δ is continuous.*

Proof. Consider $\mathcal{I}$ to be the closed ideal generated by $\mathcal{S}(\delta)$ in $\mathcal{A}$. Note that $\mathcal{I}$ contains $\mathcal{S}(\delta)$ since $\mathcal{A}$ is unital. Commutativity of $\mathcal{A}$ and Theorem 3.7 imply that $\mathcal{I}$ is a closed nil and hence nilpotent ideal, see [3]. Since $\mathcal{A}$ is semi-prime, $\mathcal{I} = \{0\}$. It follows that $\mathcal{S}(\delta) = \{0\}$. □

Corollary 3.9. *If D is a derivation on a Banach algebra $\mathcal{A}$ such that D^2 is continuous, then $\mathcal{S}(D)$ is nilpotent.*

Proof. When D is a derivation D^2 is a D-double derivation and $\mathcal{S}(D)$ is a closed nil and hence nilpotent ideal. □

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