Ann. Funct. Anal. 1 (2010), no. 2, 92–102
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

ON A FORMULA OF LE MERDY FOR THE COMPLEX INTERPOLATION OF TENSOR PRODUCTS

CARSTEN MICHELS

Communicated by G. Androulakis

ABSTRACT. C. Le Merdy in [Proc. Amer. Math. Soc. 126 (1998), 715–719] proved the following complex interpolation formula for injective tensor products: $[\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2\tilde{\otimes}_\varepsilon\ell_\infty]_{\frac{1}{2}} = \mathcal{S}_4$. We investigate whether related formulas hold when considering arbitrary $0 < \theta < 1$ instead of $\frac{1}{2}$, and give a partially positive answer for $\theta < \frac{1}{2}$ and a negative answer for $\theta > \frac{1}{2}$. Furthermore, we briefly discuss the more general case when ℓ_2 is replaced by ℓ_q, $1 < q < 2$, and ℓ_1 and ℓ_∞ by ℓ_{p_0} and ℓ_{p_1}, respectively.

1. INTRODUCTION

Let (X_0, X_1) and (Y_0, Y_1) be regular Banach couples. In [8], O. Kouba investigated under which geometric assumptions on the spaces involved the complex interpolation formula

$$[X_0\tilde{\otimes}_\varepsilon Y_0, X_1\tilde{\otimes}_\varepsilon Y_1]_\theta = [X_0, X_1]_\theta\tilde{\otimes}_\varepsilon[Y_0, Y_1]_\theta$$

for injective tensor products holds for all $0 < \theta < 1$. For couples of ℓ_p-spaces with indices all less than or equal to 2, he gave an affirmative answer:

$$[\ell_{p_0}\tilde{\otimes}_\varepsilon\ell_{q_0}, \ell_{p_1}\tilde{\otimes}_\varepsilon\ell_{q_1}]_\theta = \ell_{p_\theta}\tilde{\otimes}_\varepsilon\ell_{q_\theta} \tag{1.1}$$

for all $0 < \theta < 1$, $1 \le p_0, p_1, q_0, q_1 \le 2$ and p_θ, q_θ such that $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Moreover, he showed that the formula above does not hold for $p_0 = p_1 = 2$ and $q_0 = 2, q_1 = \infty$. Further counterexamples were given recently in [13]. However, in the literature so far there seem to be only few counterexamples where actually a description of the resulting complex interpolation space is given.

Date: Received: 29 November 2010; Accepted: 25 December 2010.
2010 *Mathematics Subject Classification.* Primary 46B70; Secondary 46M35, 47B06, 47B10.
Key words and phrases. Complex interpolation of Banach spaces, injective tensor products, approximation numbers, quasi-Banach operator ideals.

G. Pisier in [17] gave one in terms of regular operators for $p_0 = q_1 = 1$ and $p_1 = q_0 = \infty$ and, more generally, for $q_0 = p_0'$ and $q_1 = p_1'$ in [18]. For $p_0 = p_1 = 2$, $q_0 = \ell_1$, $q_1 = \infty$ and $\theta = \frac{1}{2}$, C. Le Merdy in [10] proved the following in order to show that the Schatten class $\mathcal{S}_4$ (and subsequently $\mathcal{S}_p$ for all $2 \leq p \leq 4$) endowed with the Schur product is a so-called Q-algebra (that is, isomorphic to a quotient C/I, where C is a uniform algebra and I a closed ideal):

$$[\ell_2 \tilde{\otimes}_\varepsilon \ell_1, \ell_2 \tilde{\otimes}_\varepsilon \ell_\infty]_{\frac{1}{2}} = \mathcal{S}_4. \tag{1.2}$$

In this note, we discuss the question whether there are suitable generalizations of this formula for arbitrary $0 < \theta < 1$. Since $\mathcal{S}_4 = \mathcal{L}_4^{(a)}(\ell_2, \ell_2)$, the class of all operators T on ℓ_2 with sequence $(a_n(T))$ of approximation numbers contained in ℓ_4, it seems to be natural to look for connections to components of the operator ideals $\mathcal{L}_{r,w}^{(a)}$. We will show that for $\theta < \frac{1}{2}$ there is indeed a close connection, whereas the answer for $\theta > \frac{1}{2}$ is negative. A more general discussion follows.

For a given interpolation couple (X_0, X_1) of Banach spaces and $0 < \theta < 1$ we denote by $\overline{X}_\theta = [X_0, X_1]_\theta$ the complex interpolation space with respect to the given couple and theta. For all information needed on complex interpolation of Banach spaces, we refer to [1]. However, we need the following particular case of the reiteration theorem with no density assumptions stated in [4, Lemma 2]:

Lemma 1.1. *Let $\overline{X} = (X_0, X_1)$ be a Banach couple and $\theta_0, \theta_1, \eta \in [0, 1]$. Then $[\overline{X}_{\theta_0}, X_1]_\eta = \overline{X}_{(1-\eta)\theta_0+\eta}$ and $[X_0, \overline{X}_{\theta_1}]_\eta = \overline{X}_{\eta\theta_1}$.*

Also, we will frequently use the fact that the complex interpolation functor is of power type θ, that is, $\|x\|_{[X_0,X_1]_\theta} \leq \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}$ for all $x \in X_0 \cap X_1$. Moreover, for all $1 \leq p_0, p_1 \leq \infty$ it holds $[\ell_{p_0}, \ell_{p_1}]_\theta = \ell_{p_\theta}$ (isometrically), where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. This formula will be used without further reference throughout the paper, and if it is not mentioned in a given formulation, p_θ is always defined as in the above.

For an s-scale s in the sense of [16, 2.2.1], an operator $T \in \mathcal{L}(X, Y)$ (the space of all linear and bounded operators between Banach spaces X and Y) is said to be of s-type $\ell_{r,w}$ if $(s_n(T))$ is contained in the Lorentz sequence space $\ell_{r,w}$. The set of these operators is denoted by $\mathcal{L}_{r,w}^{(s)}(X, Y)$. Together with the quasi-norm $\|T\|_{\mathcal{L}_{r,w}^{(s)}} := \|(s_n(T))\|_{\ell_{r,w}}$, this gives the quasi-Banach operator ideal $\mathcal{L}_{r,w}^{(s)}$ (see, e.g., [16, 2.2.5]). In this article, we will only consider the scale of approximation numbers

$$a_n(T) := \inf\{\|T - S\| : S \in \mathcal{L}(X, Y) \text{ with } \operatorname{rank}(S) < n\}.$$

All s-scales coincide for operators acting between Hilbert spaces (see, e.g., [16, 2.11.9]). Thus, $\mathcal{L}_{r,w}^{(s)}(\ell_2, \ell_2) = \mathcal{S}_{r,w}$, the quasi-Banach space of all compact operators on ℓ_2 with singular numbers contained in $\ell_{r,w}$, endowed with the natural norm, for any s-scale s.

Throughout the paper we use the following notation: Given two sequences (a_n) and (b_n) of nonnegative real numbers we write $a_n \prec b_n$, if there is a constant $c > 0$ such that $a_n \leq c\, b_n$ for all $n \in \mathbb{N}$, while $a_n \succ b_n$ stands for $b_n \prec a_n$ and $a_n \asymp b_n$

means that $a_n \prec b_n$ and $a_n \succ b_n$ holds. Also, for $1 \le p \le \infty$ we denote by p' its conjugate number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Inclusions into and from spaces of operators of approximation type

A crucial tool will be the notion of absolutely summing norms. If X and Y are Banach spaces and $1 \le s \le r \le \infty$, then for every operator $T : X \to Y$ and each $m \in \mathbb{N}$ we define

$$\pi_{r,s}^{(m)}(T) := \sup \Big\{ \Big\| \sum_{k=1}^{m} \|Tx_k\|_Y e_k \Big\|_{\ell_r} ; \sup_{\|x'\|_{X'} \le 1} \Big(\sum_{k=1}^{m} |x'(x_k)|^s \Big)^{1/s} \le 1 \Big\},$$

where e_k denotes the scalar sequence with 1 as its kth entry and 0 else. If $\pi_{r,s}(T) := \sup_{m\in\mathbb{N}} \pi_{r,s}^{(m)}(T) < \infty$, then T is called absolutely (r,s)-summing. In this case, we write $T \in \Pi_{r,s}$, and $T \in \Pi_r$ if $r = s$. It is easy to see (see, e.g., [7]) that

$$\pi_{r,s}^{(m)}(T) = \|\widehat{T} : \ell_s^m \otimes_\varepsilon X \to \ell_r^m(Y)\|, \qquad \widehat{T}(x_1,\dots,x_m) := (Tx_1,\dots,Tx_m).$$

Here, for two Banach spaces X and Y, we denote by $X \otimes_\varepsilon Y$ the algebraic tensor product of X and Y equipped with the injective norm and by $X \tilde{\otimes}_\varepsilon Y$ its completion (see, e.g., [5]). Also, $\ell_r^m(Y)$ stands for the vector space of m-tuples with entries from Y equipped with the Bochner r-norm.

The following interpolation result is a somewhat specialized and combined version of various known results on interpolation of summing norms, see e.g. [13, Lemma 2] for (i) and (ii). For $1 \le q_0, q_1, p_0, p_1 \le \infty$ and $0 < \theta < 1$ we define

$$d_\theta^{m,n}(q_0,q_1,p_0,p_1) := \|\mathrm{id} : \ell_{q_\theta}^m \otimes_\varepsilon \ell_{p_\theta}^n \to [\ell_{q_0}^m \otimes_\varepsilon \ell_{p_0}^n, \ell_{q_1}^m \otimes_\varepsilon \ell_{p_1}^n]_\theta\|.$$

To shorten the formulas, we denote the identity map $\mathrm{id} : \ell_p^n \to \ell_q^n$ by id_{pq}^n. Also, if $\|\cdot\|_n$ is any fixed norm on the vector space of all linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ (in what follows, the norm induced by $[\ell_{q_0}^n \otimes_\varepsilon \ell_{p_0}^n, \ell_{q_1}^n \otimes_\varepsilon \ell_{p_1}^n]_\theta$ will be crucial), then $\|\mathrm{id}\|_n$ stands for the respective norm of the identity map $\mathrm{id} : \mathbb{C}^n \to \mathbb{C}^n$.

Lemma 2.1. *Let $1 \le s_i \le r_i \le \infty$, $i = 0,1$, $1 \le p_0,p_1,q_0,q_1 \le \infty$, and $0 < \theta < 1$. Then for s_θ, r_θ and p_θ defined by $\frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$, $\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, respectively, we have the following:*

(i) $\pi_{r_\theta,s_\theta}^{(m)}(\mathrm{id}_{p_\theta q_\theta}^n) \le d_\theta^{m,n}(s_0,s_1,p_0,p_1)\, \pi_{r_0,s_0}^{(m)}(\mathrm{id}_{p_0q_0}^n)^{1-\theta} \pi_{r_1,s_1}^{(m)}(\mathrm{id}_{p_1q_1}^n)^{\theta}$;

(ii) $\pi_{r_\theta,s_\theta}(\mathrm{id}_{p_\theta q_\theta}^n) \le \sup_m d_\theta^{m,n}(s_0,s_1,p_0,p_1)\, \pi_{r_0,s_0}(\mathrm{id}_{p_0q_0}^n)^{1-\theta} \pi_{r_1,s_1}(\mathrm{id}_{p_1q_1}^n)^{\theta}$;

(iii) $[\Pi_{r_0,s}(X,\ell_{p_0}), \Pi_{r_1,s}(X,\ell_{p_1})]_\theta \subseteq \Pi_{r_\theta,s}(X,\ell_{p_\theta})$, *provided that $r_0, r_1 \ge s \ge 1$, for any Banach space X.*

Lemma 2.2. *Let $\frac{1}{2} < \theta < 1$. Then:*

(i) $\|\mathrm{id}\|_{[\ell_2^n \otimes_\varepsilon \ell_1^n, \ell_2^n \otimes_\varepsilon \ell_\infty^n]_\theta} \le n^{\frac{1-\theta}{2}}$;

(ii) $\|\mathrm{id} : \ell_2 \otimes_\varepsilon \ell_{\frac{1}{1-\theta}}^n \to [\ell_2 \otimes_\varepsilon \ell_1^n, \ell_2 \otimes_\varepsilon \ell_\infty^n]_\theta\| \succ n^{\theta(1-\theta)}$.

Proof. (i) Since $\|\mathrm{id}\|_{\ell_2^n \otimes_\varepsilon \ell_1^n} = n^{\frac{1}{2}}$ and $\|\mathrm{id}\|_{\ell_2^n \otimes_\varepsilon \ell_\infty^n} = 1$, this is a simple consequence of the fact that the complex interpolation functor is of power type θ. (ii) It is

$\pi_{\infty,2}(\mathrm{id}:\ell_1^n \to \ell_1^n) = \|\mathrm{id}:\ell_1^n \to \ell_1^n\| = n^0$ and $\pi_{2,2}(\mathrm{id}:\ell_\infty^n \to \ell_1^n) = n^1$. Hence, by Lemma 2.1 (ii) it follows

$$\pi_{\frac{2}{\theta},2}(\mathrm{id}:\ell^n_{\frac{1}{1-\theta}} \to \ell_1^n) \prec \|\mathrm{id}:\ell_2 \otimes_\varepsilon \ell^n_{\frac{1}{1-\theta}} \to [\ell_2 \otimes_\varepsilon \ell_1^n, \ell_2 \otimes_\varepsilon \ell_\infty^n]_\theta\|\, n^\theta,$$

whereas by [3] $\pi_{\frac{2}{\theta},2}(\mathrm{id}:\ell^n_{\frac{1}{1-\theta}} \to \ell_1^n) \asymp n^{\theta+\theta(1-\theta)}$, which gives the lower estimate. □

For the following let us introduce for $0 < r < \infty$ the so-called $(r,2,2)$-nuclear operators according to the characterization given in [15, 18.1.3] (rather than stating the original definition which we will not use here).

An operator $T \in \mathcal{L}(X,Y)$ between Banach spaces X and Y is said to be $(r,2,2)$-*nuclear* if $T = RD_\sigma S$ with $S \in \mathcal{L}(X,\ell_2)$, $R \in \mathcal{L}(\ell_2,Y)$ and $D_\sigma \in \mathcal{L}(\ell_2,\ell_2)$ a diagonal operator of the form $D_\sigma(\xi_i) = (\sigma_i \xi_i)$ with $(\sigma_i) \in \ell_r$. In this case, $\nu_{r,2,2}(T) := \inf \|R\|\,\|\sigma\|_{\ell_r}\,\|S\|$, where the infimum is taken over all possible factorizations, defines a quasi-norm, and with this quasi-norm, the collection of all $(r,2,2)$-nuclear operators, denoted by $\mathcal{N}_{r,2,2}$, becomes a quasi-Banach operator ideal (see, e.g., [15, 18.1]).

Lemma 2.3. *Let $0 < r < \infty$. Then*

$$\mathcal{L}^{(a)}_{r,\frac{r}{r+1}}(\ell_2,Y) \subseteq \mathcal{N}_{r,2,2}(\ell_2,Y) \subseteq \mathcal{L}^{(a)}_r(\ell_2,Y)$$

for any Banach space Y.

Proof. It is easily verified that $\|\mathrm{id}:\mathcal{L}(\ell_2^n,Y) \to \mathcal{N}_{r,2,2}(\ell_2^n,Y)\| \le n^{\frac{1}{r}}$ for any Banach space Y. Hence, using projections in the Hilbert space ℓ_2, it follows by factorization that

$$\nu_{r,2,2}(T) \le n^{\frac{1}{r}}\|T\| \tag{2.1}$$

for every operator $T:\ell_2 \to Y$ with $\mathrm{rank}(T) \le n$. Then the first inclusion follows exactly as in the proof of [16, 2.3.10], since $\mathcal{N}_{r,2,2}$ is an $\frac{r}{r+1}$-normed ideal (cf. [15, 18.1.2]). The second inclusion is true by [15, 18.6.2] (even for any Banach space X replacing ℓ_2). □

Now we are ready to prove the following counterpart of Le Merdy's result for $\theta = \frac{1}{2}$:

Theorem 2.4. *Let $0 < \theta < 1$.*

(i) If $\theta < \frac{1}{2}$, then

$$\mathcal{L}^{(a)}_{\frac{2}{\theta},\frac{2}{2+\theta}}(\ell_2,\ell_{\frac{1}{1-\theta}}) \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2\tilde{\otimes}_\varepsilon\ell_\infty]_\theta \subseteq \mathcal{L}^{(a)}_{\frac{2}{\theta}}(\ell_2,\ell_{\frac{1}{1-\theta}}).$$

(ii) If $\theta > \frac{1}{2}$ and $\mathcal{L}^{(a)}_{r_0,w_0}(\ell_2,\ell_{\frac{1}{1-\theta}}) \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2\tilde{\otimes}_\varepsilon\ell_\infty]_\theta \subseteq \mathcal{L}^{(a)}_{r_1,w_1}(\ell_2,\ell_{\frac{1}{1-\theta}})$ holds for some $1 \le r_0 \le r_1 < \infty$ and $0 < w_0, w_1 \le \infty$, then $r_0 < r_1$.

Proof. (i) We first prove the second inclusion. Since $\Pi_{\frac{2}{\theta},2}(\ell_2,\ell_{\frac{1}{1-\theta}}) = \mathcal{L}^{(a)}_{\frac{2}{\theta}}(\ell_2,\ell_{\frac{1}{1-\theta}})$, (see, e.g., [16, 2.7.6]), it is sufficient to consider the corresponding ideals of summing operators, which behave nicely with respect to complex interpolation in the range.

Let $\overline{A} := (\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2\tilde{\otimes}_\varepsilon\ell_\infty)$. Then for $\eta = 2\theta$, by Le Merdy's result (1.2), reiteration (Lemma 1.1) and Lemma 2.1 (iii),

$$\overline{A}_\theta = [\overline{A}_0, \overline{A}_{\frac{1}{2}}]_\eta \subseteq [\Pi_{\infty,2}(\ell_2,\ell_1), \Pi_{4,2}(\ell_2,\ell_2)]_\eta \subseteq \Pi_{\frac{2}{\theta},2}(\ell_2, \ell_{\frac{1}{1-\theta}}).$$

To prove the first inclusion, fix $R \in \mathcal{L}(\ell_2,\ell_2)$. Note that by Pitt's theorem (see, e.g., [11, 2.c.3]) $\overline{A}_0 = \ell_2\tilde{\otimes}_\varepsilon\ell_1 = \mathcal{L}(\ell_2,\ell_1)$. Then for $\Phi(\mu, U) := U\,D_\mu\,R$ we have that the operators

$$\Phi : \ell_\infty \times \mathcal{L}(\ell_2,\ell_1) \to \mathcal{L}(\ell_2,\ell_1) = \overline{A}_0$$

and

$$\Phi : \ell_4 \times \mathcal{L}(\ell_2,\ell_2) \to \mathcal{N}_{4,2,2}(\ell_2,\ell_2) = \mathcal{S}_4 = \overline{A}_{\frac{1}{2}}$$

have norm less than or equal to $\|R\|$. Hence, by complex interpolation with the parameter $\eta = 2\theta$ and reiteration we have that

$$\Phi : \ell_{\frac{2}{\theta}} \times [\mathcal{L}(\ell_2,\ell_1), \mathcal{L}(\ell_2,\ell_2)]_\eta \to [\overline{A}_0, \overline{A}_{\frac{1}{2}}]_\eta = \overline{A}_\theta$$

is defined and continuous. Once again by Pitt's theorem and by (1.1) it is

$$\mathcal{L}(\ell_2, \ell_{\frac{1}{1-\theta}}) = \ell_2\tilde{\otimes}_\varepsilon\ell_{\frac{1}{1-\theta}} \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2\tilde{\otimes}_\varepsilon\ell_2]_\eta \subseteq [\mathcal{L}(\ell_2,\ell_1), \mathcal{L}(\ell_2,\ell_2)]_\eta,$$

it follows that

$$\Phi : \ell_{\frac{2}{\theta}} \times \mathcal{L}(\ell_2, \ell_{\frac{1}{1-\theta}}) \to \overline{A}_\theta$$

is defined and continuous with norm less than or equal to $C\,\|R\|$, for some constant $C > 0$. Now since by definition every operator in $\mathcal{N}_{\frac{2}{\theta},2,2}$ is of the form $U\,D_\mu\,R$ for some $R \in \mathcal{L}(\ell_2,\ell_2)$, $\mu \in \ell_{\frac{2}{\theta}}$ and $U \in \mathcal{L}(\ell_2, \ell_{\frac{1}{1-\theta}})$, the claim follows by Lemma 2.3.

(ii) Assume that $\mathcal{L}^{(a)}_{r_0,w_0}(\ell_2, \ell_{\frac{1}{1-\theta}}) \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2, \tilde{\otimes}_\varepsilon\ell_\infty]_\theta$. Then for any $s < r_0$ it would hold $\mathcal{L}^{(a)}_s(\ell_2, \ell_{\frac{1}{1-\theta}}) \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2, \tilde{\otimes}_\varepsilon\ell_\infty]_\theta$, and subsequently, by the above lemma, $\mathcal{N}_{s,2,2}(\ell_2, \ell_{\frac{1}{1-\theta}}) \subseteq [\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2, \tilde{\otimes}_\varepsilon\ell_\infty]_\theta$. Hence, by (2.1) we would have

$$\begin{aligned}\|\mathrm{id} : \ell_2 \otimes_\varepsilon [\ell_1^n, \ell_\infty^n]_\theta &\to [\ell_2 \otimes_\varepsilon \ell_1^n, \ell_2 \otimes_\varepsilon \ell_\infty^n]_\theta\| \\ &\leq C\,\|\mathrm{id} : \mathcal{L}(\ell_2, \ell^n_{\frac{1}{1-\theta}}) \to \mathcal{N}_{s,2,2}(\ell_2, \ell^n_{\frac{1}{1-\theta}})\| \leq C\,n^{\frac{1}{s}}\end{aligned}$$

for some constant $C > 0$ not depending on n. Comparing with the results from Lemma 2.2 (ii), this implies $\frac{1}{s} \geq \theta(1-\theta)$, hence, since $s < r_0$ was arbitrary, $\frac{1}{r_0} \geq \theta(1-\theta)$.

Now assume that $[\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2, \tilde{\otimes}_\varepsilon\ell_\infty]_\theta \subseteq \mathcal{L}^{(a)}_{r_1,w_1}(\ell_2, \ell_{\frac{1}{1-\theta}})$. Then for all $s > \max(r_1, 2)$, it would follow that

$$[\ell_2\tilde{\otimes}_\varepsilon\ell_1, \ell_2, \tilde{\otimes}_\varepsilon\ell_\infty]_\theta \subseteq \mathcal{L}^{(a)}_s(\ell_2, \ell_{\frac{1}{1-\theta}}) = \Pi_{s,2}(\ell_2, \ell_{\frac{1}{1-\theta}}).$$

By [3] $\pi_{s,2}(\mathrm{id} : \ell_2^n \to \ell^n_{\frac{1}{1-\theta}}) \asymp n^{\frac{1}{s}}$, whereas by Lemma 2.2 (i) we have

$$\|\mathrm{id}\|_{[\ell_2^n\otimes_\varepsilon\ell_1^n, \ell_2^n\otimes_\varepsilon\ell_\infty^n]_\theta} \leq n^{\frac{1-\theta}{2}}.$$

This implies that $\frac{1}{s} \leq \frac{1-\theta}{2} < \frac{1}{2}$, hence, $r_1 > 2$ and $\frac{1}{r_1} \leq \frac{1-\theta}{2}$. Now (ii) follows since clearly $\theta(1-\theta) > \frac{1-\theta}{2}$ by the assumption on θ. $\square$

Remark 2.5. The theorem above is also valid for the scales c of Gelfand numbers and x of Weyl numbers, respectively, since these numbers coincide with the approximation numbers for operators starting from a Hilbert space. Moreover, (i) is also valid for the scales d of Kolmogorov numbers and e of (dyadic) entropy numbers, respectively. This follows from the fact that for $s \in \{a, c, d, e\}$ all $\mathcal{L}^{(s)}_{r,s}(X, Y)$ coincide whenever X and Y' are of type 2 (see [2, Proposition 5]).

We do not know whether the second inclusion in (i) actually is an equality (as it is for $\theta = \frac{1}{2}$). There is no hope that our present proof, which involves $(r, 2, 2)$-nuclear operators, can be used to prove such an equality. This follows from the strict inclusion $\mathcal{N}_{r,2,2}(\ell_2, \ell_p) \subsetneqq \mathcal{L}^{(a)}_r(\ell_2, \ell_p)$ for all $2 \le r < \infty$ and $1 \le p < 2$. Indeed, by [16, 2.9.10], a diagonal operator D_λ is contained in $\mathcal{L}^{(a)}_r(\ell_2, \ell_p)$ if and only if $\lambda \in \ell_{s,r}$, where $\frac{1}{s} = \frac{1}{r} - \frac{1}{2} + \frac{1}{p} > \frac{1}{r}$. On the other hand, $D_\lambda \in \mathcal{N}_{r,2,2}(\ell_2, \ell_p)$ if and only if $\lambda \in \ell_s$.

3. A more general discussion on necessities and impossibilities

In this concluding section, we generalize the idea used in the proof of Theorem 2.4 (ii) and provide a more general scheme to obtain necessary conditions for certain inclusions to happen. The main tool will be asymptotically optimal estimates of $d^{m,n}_\theta(q, q, p_0, p_1)$ for a fixed $1 \le q \le 2$.

For $1 \le q_0, q_1, p_0, p_1 \le \infty$ and $0 < \theta < 1$ recall the following definition from the beginning:

$$d^{m,n}_\theta(q_0, q_1, p_0, p_1) := \|\mathrm{id} : \ell^m_{q_\theta} \otimes_\varepsilon \ell^n_{p_\theta} \to [\ell^m_{q_0} \otimes_\varepsilon \ell^n_{p_0}, \ell^m_{q_1} \otimes_\varepsilon \ell^n_{p_1}]_\theta\|.$$

Now for $\alpha > 0$, we define

$$\lambda^d_{\theta,\alpha}(q_0, q_1, p_0, p_1) := \inf\{\lambda > 0 : \exists\, \rho > 0 : d^{[n^\alpha],n}_\theta(q_0, q_1, p_0, p_1) \le \rho\, n^\lambda\}.$$

Moreover, we set

$$c^n_\theta(q_0, q_1, p_0, p_1) := \|\mathrm{id}\|_{[\ell^n_{q_0} \otimes_\varepsilon \ell^n_{p_0}, \ell^n_{q_1} \otimes_\varepsilon \ell^n_{p_1}]_\theta}$$

and

$$\lambda^c_\theta(q_0, q_1, p_0, p_1) := \inf\{\lambda > 0 : \exists\, \rho > 0 : c^n_\theta(q_0, q_1, p_0, p_1) \le \rho\, n^\lambda\}.$$

All this is motivated by the definition of the limit order of a quasi-Banach operator ideal $(\mathcal{A}, A)$ (see, e.g., [15, 14.4]):

$$\lambda(\mathcal{A}, q, p) := \inf\{\lambda > 0 : \exists\, \rho > 0 : A(\mathrm{id} : \ell^n_q \to \ell^n_p) \le \rho\, n^\lambda\}.$$

For the definiton of a quasi-Banach operator ideal and the basic theory we refer to [15]. In the following we will use the (metric) ideal property of a quasi-Banach operator ideal $(\mathcal{A}, A)$, that is, $A(RST) \le \|R\| A(S) \|T\|$ whenever the composition RST is well-defined, R and T are bounded operators and $S \in \mathcal{A}$.

Lemma 3.1. *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$, $0 < \theta < 1$ and $(\mathcal{A}, A), (\mathcal{B}, B)$ be quasi-Banach operator ideals.*

(i) If $\mathcal{A}(\ell_{q'_\theta}, \ell_{p_\theta}) \subseteq [\ell_{q_0} \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_{q_1} \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta$, then $\lambda(\mathcal{A}, q'_\theta, p_\theta) \ge \lambda^c_\theta(q_0, q_1, p_0, p_1)$ and $\min(\alpha\, \lambda(\mathcal{A}, q'_\theta, q'_\theta), \lambda(\mathcal{A}, p_\theta, p_\theta)) \ge \lambda^d_{\theta,\alpha}(q_0, q_1, p_0, p_1)$ for all $\alpha > 0$.

(ii) If $[\ell_{q_0} \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_{q_1} \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta \subseteq \mathcal{B}(\ell_{q'_\theta}, \ell_{p_\theta})$, then $\lambda(\mathcal{B}, q'_\theta, p_\theta) \le \lambda^c_\theta(q_0, q_1, p_0, p_1)$.

Proof. (i) Since the k-th section ℓ_r^k of ℓ_r is always 1-complemented, the ideal property of quasi-Banach operator ideals and the mapping properties of the injective tensor product and the complex interpolation functor give that the assumption implies that

$$c := \sup_{m,n} \|\mathrm{id} : \mathcal{A}(\ell_{q'_\theta}^m, \ell_{p_\theta}^n) \to [\ell_{q_0}^m \otimes_\varepsilon \ell_{p_0}^n, \ell_{q_1}^m \otimes_\varepsilon \ell_{p_1}^n]_\theta\| < \infty.$$

Hence, $c_\theta^n(q_0, q_1, p_0, p_1) \leq c\, A(\mathrm{id} : \ell_{q'_\theta}^n \to \ell_{p_\theta}^n)$. Furthermore, by factorization through $\mathcal{A}(\ell_{q'_\theta}^m, \ell_{p_\theta}^n)$,

$$\begin{aligned} d_\theta^{m,n}(q_0, q_1, p_0, p_1) &\leq c\, \|\mathrm{id} : \mathcal{L}(\ell_{q'_\theta}^m, \ell_{p_\theta}^n) \to \mathcal{A}(\ell_{q'_\theta}^m, \ell_{p_\theta}^n)\| \\ &\leq c\, \min(A(\mathrm{id} : \ell_{q'_\theta}^m \to \ell_{q'_\theta}^m), A(\mathrm{id} : \ell_{p_\theta}^n \to \ell_{p_\theta}^n)), \end{aligned}$$

which, by the definition of the various limit orders, gives the conclusion in (i). Clearly, (ii) goes the same way. □

Motivated by the above, we start with estimating $d_\theta^{m,n}(q, q, p_0, p_1)$ for $1 \leq q \leq 2$. For $p_\theta > 2$, the proof is based on calculating summing norms of identity maps between n-dimensional spaces *with less than n vectors*.

Proposition 3.2. *Let $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q \leq 2$, $0 < \theta < 1$, and $\alpha_{p_\theta} := \min(\frac{2}{p_\theta}, 1)$. Then for all $m \geq n^{\alpha_{p_\theta}}$*

$$d_\theta^{m,n}(q, q, p_0, p_1) \asymp \begin{cases} 1 & \text{if } 1 \leq p_0, p_1 \leq 2, \\ n^{\theta(\frac{1}{2} - \frac{1}{p_1})} & \text{if } p_\theta \leq 2 < p_1, \\ n^{\theta(1-\theta)(\frac{1}{p_0} - \frac{1}{p_1})} & \text{if } p_\theta > 2. \end{cases}$$

Proof. The case $1 \leq p_0, p_1 \leq 2$ is clear by the results of [8]. In the case $p_\theta \leq 2 < p_1$, the upper estimate can be shown by a careful analysis of [6] (roughly speaking: in Lemma 5, one may replace ℓ_2 by $\ell_{q'}$, see also [13]), whereas the lower estimate is more or less already stated within the proof of [13, Example 3 (i)]—a slight modification similar to the one made for the upcoming remaining case has to be made, we leave this to the reader. If $p_\theta > 2$, the upper estimate is a consequence of factorization through $\ell_{p_\theta}^n$ and again the fact that the complex interpolation functor $[\cdot, \cdot]_\theta$ is of power type θ:

$$\begin{aligned} \|T\|_{[\ell_q^m \otimes_\varepsilon \ell_{p_0}^n, \ell_q^m \otimes_\varepsilon \ell_{p_1}^n]_\theta} &\leq \|T : \ell_{q'}^m \to \ell_{p_0}^n\|^{1-\theta} \|T : \ell_{q'}^m \to \ell_{p_1}^n\|^\theta \\ &\leq n^{(1-\theta)(\frac{1}{p_0} - \frac{1}{p_\theta})} \|T : \ell_{q'}^m \to \ell_{p_\theta}^n\|. \end{aligned}$$

The lower estimate is somehow a copy of the proof of [13, Example 3 (ii)], but we have to modify it slightly in order to get the estimate for $m \geq n^{\frac{2}{p_\theta}}$. By [15] we know that

$$\pi_{q,q}^{(m)}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) \leq \pi_{q,q}(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) = \pi_q(\mathrm{id} : \ell_{p_1}^n \hookrightarrow \ell_1^n) \asymp n^1,$$

and that for r defined by $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$

$$\pi_{r,q}^{(m)}(\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n) \leq \pi_{r,q}(\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n) \asymp \|\mathrm{id} : \ell_{p_0}^n \hookrightarrow \ell_1^n\| \asymp n^{1 - \frac{1}{p_0}}.$$

Thus, Lemma 2.1 (i) with r_θ defined by $\frac{1}{r_\theta} = \frac{1-\theta}{r} + \frac{\theta}{q}$ gives

$$\pi^{(m)}_{r_\theta,q}(\mathrm{id} : \ell^n_{p_\theta} \hookrightarrow \ell^n_1) \prec d^{m,n}_\theta(q,q,p_0,p_1)\, n^{1-\frac{1-\theta}{p_0}}.$$

However, the proof of [3, Satz 3] and some elementary calculations show that

$$\pi^{(m)}_{r_\theta,q}(\mathrm{id} : \ell^n_{p_\theta} \hookrightarrow \ell^n_1) \asymp n^{1-\frac{1-\theta}{p_0}+\theta(1-\theta)(\frac{1}{p_0}-\frac{1}{p_1})},$$

as long as $m \geq n^{\frac{2}{p_\theta}}$, which gives $d^{m,n}_\theta(q,q,p_0,p_1) \succ n^{\theta(1-\theta)(\frac{1}{p_0}-\frac{1}{p_1})}$. □

As a by-product, the preceding proof gives the following quantitative information about the deviation of the upper estimate in the usual interpolation theorem from the norm of the operator acting between the interpolation spaces:

Corollary 3.3. *Let $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q \leq 2$ and $0 < \theta < 1$ such that $p_\theta \geq 2$. Then for all $m \geq n^{\frac{2}{p_\theta}}$*

$$\max_{T\neq 0} \frac{\|T : \ell^m_{q'} \to \ell^n_{p_0}\|^{1-\theta} \|T : \ell^m_{q'} \to \ell^n_{p_1}\|^{\theta}}{\|T : \ell^m_{q'} \to \ell^n_{p_\theta}\|} \asymp n^{\theta(1-\theta)(\frac{1}{p_0}-\frac{1}{p_1})}.$$

We continue with determining the asymptotic values of $c^n_\theta(q,q,p_0,p_1)$:

Proposition 3.4. *Let $1 \leq p_0, p_1 \leq \infty$, $1 \leq q \leq 2$ and $0 < \theta < 1$. Then*

$$c^n_\theta(q,q,p_0,p_1) \asymp \begin{cases} n^{\frac{1}{p_\theta}-\frac{1}{q'}} & \text{if } 1 \leq p_0, p_1 \leq q' \\ n^{(1-\theta)(\frac{1}{p_0}-\frac{1}{q'})} & \text{if } p_0 < q' < p_1, \\ 1 & \text{if } p_0, p_1 \geq q'. \end{cases}$$

Proof. The upper estimates follow again from the fact that the complex interpolation functor $[\cdot,\cdot]_\theta$ is of power type θ. The lower ones are not essential for the theory within this section and we omit the proofs. □

After these preparations, we start with a quite general result which is much in contrast to the case $q = 2$:

Theorem 3.5. *Let $1 < q < 2$, $1 \leq p_0 < q' < p_1 \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{q'} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Let $\mathcal{A}_0, \mathcal{A}_1$ be two quasi-Banach operator ideals, and assume that $\mathcal{A}_0(\ell_{q'}, \ell_{q'}) \subseteq [\ell_q \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_q \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta \subseteq \mathcal{A}_1(\ell_{q'}, \ell_{q'})$. Then $\lambda(\mathcal{A}_0, q', q') > \lambda(\mathcal{A}_1, q', q')$. In particular, $[\ell_q \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_q \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta$ is not a component $\mathcal{A}(\ell_{q'}, \ell_{q'})$ of any quasi-Banach operator ideal $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}_0$ and $\mathcal{A}_1$ satisfy a chain of inclusions as in the above. Lemma 3.1 (i) and Proposition 3.2 imply that

$$\lambda(\mathcal{A}_0, q', q') \geq \frac{q'}{2}\theta(\frac{1}{p_\theta} - \frac{1}{p_1}) = \frac{q'}{2}(1-\theta)(\frac{1}{p_0} - \frac{1}{q'}).$$

whenever $p_\theta \geq 2$. On the other hand, Lemma 3.1 (ii) and Proposition 3.4 imply that

$$\lambda(\mathcal{A}_1, q', q') \leq (1-\theta)(\frac{1}{p_0} - \frac{1}{q'}).$$

Hence, $\lambda(\mathcal{A}_0, q', q') > \lambda(\mathcal{A}_1, q', q')$. □

Note that a slight modification of the above proof even gives a whole interval around q' so that for all θ with p_θ lying in this interval a similar result holds—we leave the details to the interested reader. For the special combination $p_0 = q'$ and $p_1 = \infty$, it even turns out that the whole scale of resulting interpolation spaces is far from any component of a quasi-Banach operator ideal:

Theorem 3.6. *Let $1 < q < 2$ and $0 < \theta < 1$. Let $\mathcal{A}_0, \mathcal{A}_1$ be two quasi-Banach operator ideals such that $\mathcal{A}_0(\ell_{q'}, \ell_{\frac{q'}{1-\theta}}) \subseteq [\ell_q \tilde{\otimes}_\varepsilon \ell_{q'}, \ell_q \tilde{\otimes}_\varepsilon \ell_\infty]_\theta \subseteq \mathcal{A}_1(\ell_{q'}, \ell_{\frac{q'}{1-\theta}})$. Then $\lambda(\mathcal{A}_0, q', q') > \lambda(\mathcal{A}_1, q', q')$ and $\lambda(\mathcal{A}_0, q', \frac{q'}{1-\theta}) > \lambda(\mathcal{A}_1, q', \frac{q'}{1-\theta})$. In particular, $[\ell_q \tilde{\otimes}_\varepsilon \ell_{q'}, \ell_q \tilde{\otimes}_\varepsilon \ell_\infty]_\theta$ is not a component $\mathcal{A}(\ell_{q'}, \ell_{\frac{q'}{1-\theta}})$ of any quasi-Banach operator ideal $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}_0$ and $\mathcal{A}_1$ satisfy a chain of inclusions as in the above. Lemma 3.1 (i) and Proposition 3.2 imply that $\lambda(\mathcal{A}_0, q', q') \geq \frac{\theta}{2}$. By factorization through $\ell^n_{\frac{q'}{1-\theta}}$, Lemma 3.1 (ii) and Proposition 3.4 imply that $\lambda(\mathcal{A}_1, q', q') \leq \frac{\theta}{q'}$. Hence, $\lambda(\mathcal{A}_0, q', q') > \lambda(\mathcal{A}_1, q', q')$. The proof that $\lambda(\mathcal{A}_0, q', \frac{q'}{1-\theta}) > \lambda(\mathcal{A}_1, q', \frac{q'}{1-\theta})$ is similar. $\square$

As a more concrete example, we will show that formulas of Le Merdy type for the ideals generated by approximation numbers definitely fail if ℓ_2 is replaced by ℓ_q, $1 < q < 2$. To see this, we will first list what is know about the respective limit orders of these ideals (see, e.g., [15, 14.4.9]).

Lemma 3.7. *Let $1 \leq q \leq 2$, $1 \leq p \leq \infty$, $2 \leq r < \infty$ and $0 < w \leq \infty$. Then*

$$\lambda(\mathcal{L}^{(a)}_{r,w}, q', p) = \frac{1}{r} + \max(0, \frac{1}{p} - \frac{1}{q'}).$$

Theorem 3.8. *Let $1 < q < 2$, $1 \leq p_0 < p_1 \leq \infty$, $0 < \theta < 1$. If the chain of inclusions $\mathcal{L}^{(a)}_{r_0,w_0}(\ell_{q'}, \ell_{p_\theta}) \subseteq [\ell_q \tilde{\otimes}_\varepsilon \ell_{p_0}, \ell_q \tilde{\otimes}_\varepsilon \ell_{p_1}]_\theta \subseteq \mathcal{L}^{(a)}_{r_1,w_1}(\ell_{q'}, \ell_{p_\theta})$ holds for some $1 \leq r_0 \leq r_1 < \infty$ and $0 < w_0, w_1 \leq \infty$, then it follows $r_0 < r_1$.*

Proof. Since by the above lemma and Proposition 3.4

$$\lambda(\mathcal{L}^{(a)}_{2,w}, q', p_\theta) \geq \frac{1}{p_\theta} > \lambda^c_\theta(q, q, p_0, p_1),$$

it follows by Lemma 3.1 (ii) that $r_1 > 2$ provided that the second inclusion in the above holds. Then for $r_0 \leq 2$ the above statement would be clear, whence we are only left to deal with $r_0, r_1 > 2$.

If $1 \leq p_0, p_1 \leq q'$, then we have

$$\lambda(\mathcal{L}^{(a)}_{r_1,w_1}, q', p_\theta) = \frac{1}{r_1} + \frac{1}{p_\theta} - \frac{1}{q'} > \frac{1}{p_\theta} - \frac{1}{q'} = \lambda^c_\theta(q, q, p_0, p_1),$$

thus, by Lemma 3.1 (ii), the second inclusion in the above can never hold in this case.

If $2 < q' \leq p_0, p_1 \leq \infty$ and the second inclusion holds, then by Lemma 3.1 (ii) and Proposition 3.4 we would have $\lambda(\mathcal{L}^{(a)}_{r_1,w_1}, q', p_\theta) = \lambda^c_\theta(q, q, p_0, p_1) = 0$, whereas $\lambda(\mathcal{L}^{(a)}_{r_1,w_1}, q', p_\theta) = \frac{1}{r_1} > 0$, a contradiction.

Now let $1 \le p_0 < q' < p_1 \le \infty$, and assume that a chain of inclusions as in the above holds. Consider three cases:
(i) If $p_\theta \le 2$, then by Lemma 3.1 (ii) it would follow that

$$\frac{1}{r_1} + \frac{1}{p_\theta} - \frac{1}{q'} = \lambda(\mathcal{L}^{(a)}_{r_1,w_1}, q', p_\theta) \le \lambda^c_\theta(q,q,p_0,p_1) = (1-\theta)(\frac{1}{p_0} - \frac{1}{q'}),$$

thus $\frac{1}{r_1} \le \theta(\frac{1}{q'} - \frac{1}{p_1})$. Lemma 3.1 (i) would imply

$$\frac{1}{r_0} = \lambda(\mathcal{L}^{(a)}_{r_0,w_0}, q', q') \ge \lambda^d_{\theta,1}(q,q,p_0,p_1) = \theta(\frac{1}{2} - \frac{1}{p_1}).$$

Since $q' > 2$, it follows $r_0 < r_1$.
(ii) If $2 \le p_\theta < q'$, then it would again follow that $\frac{1}{r_1} \le \theta(\frac{1}{q'} - \frac{1}{p_1})$, whereas

$$\frac{1}{r_0} = \lambda(\mathcal{L}^{(a)}_{r_0,w_0}, q', q') \ge \lambda^d_{\theta,1}(q,q,p_0,p_1) = \theta(\frac{1}{p_\theta} - \frac{1}{p_1}).$$

Since $q' > p_\theta$, it follows $r_0 < r_1$.
(iii) If $q' \le p_\theta$, then it would follow that

$$\frac{1}{r_1} = \lambda(\mathcal{L}^{(a)}_{r_1,w_1}, q', p_\theta) \le \lambda^c_\theta(q,q,p_0,p_1) = (1-\theta)(\frac{1}{p_0} - \frac{1}{q'}).$$

On the other hand,

$$\frac{1}{r_0} = \lambda(\mathcal{L}^{(a)}_{r_0,w_0}, q', q') \ge \frac{p_\theta}{2}\lambda^d_{\theta,\frac{2}{p_\theta}}(q,q,p_0,p_1) = \frac{p_\theta}{2}(1-\theta)(\frac{1}{p_0} - \frac{1}{p_\theta})$$
$$> (1-\theta)(\frac{1}{p_0} - \frac{1}{q'}),$$

since $2 < q' \le p_\theta$. This implies $r_0 < r_1$. □

Remark 3.9. In this article, we did not treat the case of different spaces on each side of the tensor product, a more general case which we did not want to pursue here for reasons of clarity and comprehensibility. In a forthcoming paper [14] we show by using Kouba's formulas and a result of Kwapień [9] that for $1 \le p,q \le 2$, $1 < s < \infty$ and $0 < \theta < 1$

$$\sup_n \|\mathrm{id} : [\mathcal{L}(\ell^n_q, \ell^n_p), \mathcal{L}(\ell^n_1, \ell^n_s)]_\theta \to \Pi_{\frac{\max(s,s')}{\theta},2}(\ell^n_u, \ell^n_v)\| < \infty,$$

where $\frac{1}{u} = \frac{1-\theta}{q} + \frac{\theta}{1}$ and $\frac{1}{v} = \frac{1-\theta}{p} + \frac{\theta}{s}$. Choosing θ so that $u = v$ (which is possible if one additionally assumes $p < q$), this yields interesting estimates for eigenvalues of matrices.

References

1. J. Bergh and J. Löfström, *Interpolation spaces* (Springer-Verlag, Berlin, 1978).
2. B. Carl and T. Kühn, *Local entropy moduli and eigenvalues of operators in Banach spaces*, Rev. Mat. Iberoam. **1** (1985), 127–148.
3. B. Carl, B. Maurey and J. Puhl, *Grenzordnungen von absolut-(r,p)-summierenden Operatoren*, Math. Nachr. **82** (1978), 205–218.
4. M. Cwikel and N.J. Kalton, *Interpolation of compact operators by the methods of Calderón and Gustavsson-Peetre*, Proc. Edinb. Math. Soc. II. Ser. **38** (1995), 261–276.
5. A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland 1993.

6. A. Defant and C. Michels, *A complex interpolation formula for tensor products of vector-valued Banach function spaces*, Arch. Math. **74** (2000), 441–451.
7. J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics 43, 1995.
8. O. Kouba, *On the interpolation of injective or projective tensor products of Banach spaces*, J. Funct. Anal. **96** (1991), 38–61.
9. S. Kwapień, *Some remarks on* (p, q)*-summing operators in* ℓ_p*-spaces*, Studia Math. **28** (1968), 327–337.
10. C. Le Merdy, *The Schatten space* S_4 *is a* Q*-algebra*, Proc. Amer. Math. Soc. **126** (1998), 715–719.
11. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: Sequence spaces*, Springer-Verlag, 1977.
12. C. Michels, *Absolutely* (r, p, q)*-summing inclusions*, Studia Math. **178** (2007), 19–45.
13. C. Michels, *One-sided interpolation of injective tensor products of Banach spaces*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), 531–538.
14. C. Michels, *Estimating eigenvalues of matrices by induced norms*, Oper. Matrices, to appear.
15. A. Pietsch, *Operator Ideals* (North-Holland, 1980).
16. A. Pietsch, *Eigenvalues and s-Numbers*, Cambridge University Press, 1987.
17. G. Pisier, *Complex interpolation and regular operators between Banach lattices*, Arch. Math. **62** (1994), 261–269.
18. G. Pisier, *Complex Interpolation between Hilbert, Banach and Operator spaces*, Mem. Amer. Math. Soc. **208** (2010), no. 978.

Department of Mathematics, University of Oldenburg, D-26111 Oldenburg, Germany.

E-mail address: michels@mathematik.uni-oldenburg.de