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ERDÖS PROBLEM AND QUADRATIC EQUATION

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ABSTRACT. We investigate an Erdös problem on almost quadratic functions on $\mathbb{R}$.

1. INTRODUCTION

Motivated by a result of Hartman [9], Erdös asked an interesting problem concerning almost functions as follows:

Erdös Problem [5]**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x + y) = f(x) + f(y)$ for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Dose there exist an additive function $F : \mathbb{R} \to \mathbb{R}$ such that $f(x) = F(x)$ for almost all $x \in \mathbb{R}$?

Recall that we say a property holds for 'almost all' if it holds except on a set of measure zero. Affirmative answers to this problem were given by Bruijin [3] and Jurkat [11]. Several mathematicians have studied different functional equations under the assumption of being hold almost everywhere, among them we could refer [2, 6, 7, 8, 10].

One of important functional equations is

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.1}$$

The real function $f(x) = \alpha x^2$ is a solution of (1.1), and so this functional equation is called the *quadratic functional equation*. In particular, every solution Q of the quadratic functional equation is said to be a *quadratic mapping*. It is well known that a mapping f between real vector space is quadratic if and only if there exists a unique symmetric bi-additive mapping B is given by $B(x, y) = \frac{1}{4}\left(f(x + y) - f(x - y)\right)$ (see [14]). Another rather related notion to our work is that of stability in which one deals with the following essential question "When is

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it true that the solution of an equation differing slightly from a given one, must be close to the solution of the given equation?" The interested reader is refereed to [1, 4, 12, 13] and references therein for more information on stability of quadratic functional equation.

In this note we use the notation and strategy of [3] to give an answer to the Erdös problem above in the case where the function f satisfies (1.1) for almost all pairs (x, y) of $\mathbb{R} \times \mathbb{R}$.

2. Main result

Throughout this short paper the Lebesgue measure is denoted by m. If $N \subseteq \mathbb{R} \times \mathbb{R}$ and $(x, y) \in \mathbb{R}$, then $(x, y) + N$ is the set of all $(x + n_1, y + n_2)$ with $(n_1, n_2) \in N$, and $-N$ denotes the set of all $(-n_1, -n_2)$ with $(n_1, n_2) \in N$.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfies (1.1) for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then there exists a quadratic function h such that $f(x) = h(x)$ for almost all $x \in \mathbb{R}$.*

Proof. Assume that (1.1) holds for all $(x, y) \notin N$ where $N \subseteq \mathbb{R} \times \mathbb{R}$ and $m(N) = 0$. A set of measure zero in x-y-plan has the property that almost every line parallel to the y-axis intersects it in a set of measure zero. In the other words, there exists a subset $M \subseteq \mathbb{R}$ with $m(M) = 0$ such that for all $x \notin M$ it is true that (1.1) holds for almost all y (see [3]). Let x be an arbitrary real number. Since $m(M) = m(x - M) = m(\frac{x - M}{2}) = 0$, we have $M \cup (x - M) \cup \frac{(x - M)}{2} \neq \mathbb{R}$, so there exists $x_1 \in \mathbb{R}$ such that $x_1 \notin M$, $x - 2x_1 \notin M$ and $x - x_1 \notin M$. Therefore,

$$f(x_1 + y) + f(x_1 - y) = 2f(x_1) + 2f(y) \tag{2.1}$$

for almost all y.

$$f(x - 2x_1 + y) + f(x - 2x_1 - y) = 2f(x - 2x_1) + 2f(y) \tag{2.2}$$

for almost all y, and

$$f(x - x_1 + z) + f(x - x_1 - z) = 2f(x - x_1) + 2f(z) \tag{2.3}$$

for almost all z. Putting $z = x_1 + y$ and $z = x_1 - y$, in (2.3) we obtain

$$f(x + y) + f(x - 2x_1 - y) = 2f(x - x_1) + 2f(x_1 + y) \tag{2.4}$$

for almost all y, and

$$f(x - y) + f(x - 2x_1 + y) = 2f(x - x_1) + 2f(x_1 - y) \tag{2.5}$$

for almost all y, respectively.
By (2.1), (2.2), (2.4) and (2.5) we get

$$\begin{aligned} f(x + y) + f(x - y) - 2f(y) &= 4f(x - x_1) + 4f(x_1) - 2f(x - 2x_1) \\ &= 2(2f(x - x_1) + 2f(x_1) - f(x - 2x_1)) \end{aligned}$$

for almost all y. Thus there exists a uniquely function h with the property that for every x,

$$f(x + y) + f(x - y) - 2f(y) = 2h(x) \tag{2.6}$$

for almost all y.
For every x, let K_x denote the set of all y for which (2.6) dose not hold, so that $m(K_x) = 0$. If $x \notin M$ we also have (1.1) for almost all y. Since $m(\mathbb{R}) = \infty$ it follows that $h(x) = f(x)$ $(x \notin M)$. Let $a \in \mathbb{R}$, $b \in \mathbb{R}$. We shall show the existence of w, z such that simultaneously

$$f(a+w) + f(a-w) - 2f(w) = 2h(a) \tag{2.7}$$

$$f(b+z) + f(b-z) - 2f(z) = 2h(b) \tag{2.8}$$

$$f(a+b+w+z) + f(a+b-w-z) - 2f(w+z) = 2h(a+b) \tag{2.9}$$

$$f(a-b+w-z) + f(a-b-w+z) - 2f(w-z) = 2h(a-b) \tag{2.10}$$

$$f(w+z) + f(w-z) = 2f(w) + 2f(z) \tag{2.11}$$

$$f(a+b+w+z) + f(a-b+w-z) = 2f(a+w) + 2f(b+z) \tag{2.12}$$

$$f(a+b-w-z) + f(a-b-w+z) = 2f(a-w) + 2f(b-z) \tag{2.13}$$

The exceptional sets are, respectively, for (2.7):$K_a \times \mathbb{R}$, for (2.8):$\mathbb{R} \times K_b$, for (2.9): the set of (w, z) with $w+z \in K_{a+b}$, for (2.10): the set (w, z) with $w-z \in K_{a-b}$, for (2.11): the set N, for (2.12): the set $(-a, -b) + N$, for (2.13): the set $(a, b) - N$. Since this sets have measure zero, therefore, the set of (w, z) for which (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13) hold simultaneously is non-empty. Thus (2.7), (2.8), (2.9) and (2.10) are compatible. It immediately follows that $h(a + b) + h(a - b) = 2h(a) + 2h(b)$. □

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