Annals of Functional Analysis
Ann. Funct. Anal. 1 (2010), no. 2, 12–27
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF PSEUDO-DIFFERENTIAL OPERATORS

ZHIWEI WANG[1] AND LANZHE LIU[2*]

Communicated by J. Esterle

ABSTRACT. In this paper, we prove the boundedness for some multilinear commutators generated by the pseudo-differential operator and Lipschitz functions.

1. INTRODUCTION

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [4, 5, 6, 7, 8, 9, 10]). In [4, 5, 6, 7, 8, 9, 10], the authors prove that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [4, 11, 12, 13, 16], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on Triebel–Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [14], the weighted boundedness for the commutators generated by the singular integral operators and BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. The purpose of this paper is to prove the weighted boundedness on Lebesgue spaces for some multilinear operators associated to the pseudo-differential operators and the weighted Lipschitz functions. To do this, we first prove a sharp function estimate for the multilinear operators. Our results are new, even in the unweighted cases.

Date: Received: 25 July 2010; Accepted: 15 December 2010.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B25.
Key words and phrases. Multilinear commutator, pseudo-differential operator, Lipschitz space, Lebesgue spaces, Triebel–Lizorkin space.

The purpose of this paper is to prove the boundedness on Lebesgue spaces for some multilinear commutators generated by the pseudo-differential operator and Lipschitz functions.

2. Notations and Theorems

In order to state our results, we begin by introducing the relevant notions and definitions.

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes.

The A_1 weight is defined by $A_1 = \{0 < \omega \in L^1_{loc} : \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \omega(y)dy \leq c\omega(x), a.e.\}$(see [5]).

The Lipschitz space $Lip_\beta(R^n)$ with $0 < \beta < 1$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

For $0 < \beta < 1,\ 1 < p < \infty, m > 0$, the Triebel–Lizorkin space $\tilde{F}_p^{m\beta,\infty}(R^n)$ (see [14]) is the space of functions f such that

$$||f||_{\tilde{F}_p^{m\beta,\infty}} = \left|\left|\sup_{Q \ni \tilde{x}} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x) - f_Q|dx\right|\right|_{L^p} < \infty.$$

Given some function $b_j \in Lip_\beta(R^n), 1 \leq j \leq m, 0 < \beta < 1$, we denote by C_j^m the family of all finite subsets $\gamma = \{\gamma(1), \cdots, \gamma(j)\}$ of j different elements in $\{1, \cdots, m\}$, where $\gamma(i) < \gamma(j)$ when $i < j$.

For $\gamma \in C_j^m$, we denote $\gamma^c = \{1, \cdots, m\} \backslash \gamma$.

For $\vec{b} = \{b_1, \cdots, b_m\}$ and $\gamma = \{\gamma(1), \cdots, \gamma(j)\} \in C_j^m$, we denote $\vec{b_\gamma} = \{b_{\gamma(1)}, \cdots, b_{\gamma(j)}\}$, and $b_\gamma = b_{\gamma(1)} \cdots b_{\gamma(j)}$, and $||\vec{b_\gamma}||_{Lip_\beta} = ||b_{\gamma(1)}||_{Lip_\beta} \cdots ||b_{\gamma(j)}||_{Lip_\beta}$.

We say $\delta(x,\xi) \in S^m_{\epsilon,\sigma}$, if for $x, \xi \in R^n$,

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \delta(x,\xi)\right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m - \epsilon|\beta| + \sigma|\alpha|},$$

where α, β are multiindex consisting of n nonnegative integers.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ with symbols $\delta(x,\xi) \in S^m_{\epsilon,\sigma}$ is given by

$$T(f)(x) = \int_{R^n} e^{2\pi i(x,\xi)} \delta(x,\xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ also have another expression

$$T(f)(x) = \int_{R^n} K(x, x-y) f(y) dy,$$

where $K(x, x-y) = \int_{R^n} \delta(x,\xi) e^{2\pi i(x-y,\xi)} d\xi$.

Let $b_j, 1 \leq j \leq m$ be the fixed locally integrable functions on R^n. The multilinear commutator associated to the pseudo-differential operator is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} K(x, x-y) \prod_{j=1}^{m} (b_j(x) - b_j(y)) f(y) dy.$$

Now, we state the main results as follows.

Theorem 2.1. *Let T be a $\psi \cdot d \cdot o\cdot$ with symbol $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m, 0 < \beta < \frac{1}{m(1-a)}$, and $1 < p < \infty$. Then $T_{\vec{b}}$ is bounded from $L^p(R^n)$ to $\tilde{F}^{m\beta,\infty}_p(R^n)$.*

Theorem 2.2. *Let T be a $\psi \cdot d \cdot o\cdot$ with symbol $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $b_j \in Lip_\beta(R^n)$ for $1 \leq j \leq m, 0 < \beta < \frac{1}{m(1-a)}$, and $1 < p < \infty, 1/q = 1/p - \frac{m\beta}{n}$. Then $T_{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

3. Preliminary Lemmas

Lemma 3.1. *(see [14]) For $0 < \beta < 1, 1 \leq p \leq \infty$,*

$$||b||_{Lip_\beta} \approx \sup_Q |Q|^{-1-\frac{\beta}{n}} \int_Q |b(x) - b_Q| dx \approx \sup_Q |Q|^{-\frac{\beta}{n}} \left(|Q|^{-1} \int_Q |b(x) - b_Q|^p dx \right)^{1/p},$$

where $b_Q = |Q|^{-1} \int_Q b(x) dx$.

Lemma 3.2. *(see [14]) Let $Q_1 \subset Q_2$. Then $|b_{Q_1} - b_{Q_2}| \leq C||b||_{Lip_\beta} |Q_2|^{\beta/n}$.*

Lemma 3.3. *(see [1]) Let $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \leq \sigma < 1-a, 0 < a < 1$, and $K(x,w)$ denote the inverse Fourier transformations in the ξ-variable and in the distribution sense of $\delta(x,\xi)$, that is informally $K(x,w) = \int_{R^n} \delta(x,\xi) e^{2\pi i (w,\xi)} d\xi$. Then for $|x - x_0| \leq d \leq 1/2$ and $N \geq 0$,*

$$\begin{aligned} &\left(\int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy \right)^{1/2} \\ \leq\ & C|x - x_0|^{(1-a)(h-n/2)} (2^{N+1} d)^{-h(1-a)} \end{aligned}$$

where h is an integer such that $n/2 < h < n/2 + 1/(1-a)$.

Lemma 3.4. *(see [1]) Let $\delta(x,\xi) \in S^0_{\epsilon,\sigma}, 0 < \epsilon < 1$, and, as usual, $K(x,w) = \int_{R^n} \delta(x,\xi) e^{2\pi i (w,\xi)} d\xi$. Then for $|w| \geq 1/4$ and arbitrarily large $M, |K(x,w)| \leq C_M |w|^{-2M}$.*

Lemma 3.5. *(see [3]) For $0 < \beta < 1,\ 1 < p < \infty, m > 0$, we have*

$$\begin{aligned} ||f||_{\tilde{F}^{m\beta,\infty}_p} &\approx \left|\left| \sup_{Q \ni \tilde{x}} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x) - f_Q| dx \right|\right|_{L^p} \\ &\approx \left|\left| \sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |f(x) - c| dx \right|\right|_{L^p}, \end{aligned}$$

where $f_Q = |Q|^{-1} \int_Q f(x) dx$.

Lemma 3.6. *(see [15, Section 1.9 of Chapter 5]) $\chi_Q \in A_1$ for any cube Q.*

Lemma 3.7. *(see [1]) Let $\delta(x,\xi) \in S^{-na/2}_{1-a,\sigma}, 0 \le \sigma < 1-a, 0 < a < 1$, $T(f)(x) = \int_{R^n} e^{2\pi i(x,\xi)}\delta(x,\xi)\hat{f}(\xi)d\xi$, and $\omega \in A_{p/2}$ with $2 \le p < \infty$, $||T(f)||_{p,\omega}$ $= \left(\int_{R^n} |T(f)(x)|^p \omega(x)dx\right)^{\frac{1}{p}}$. Then $||T(f)||_{p,\omega} \le C_p ||f||_{p,\omega}$ for $f \in C_0^\infty(R^n)$.*

Lemma 3.8. *(see [15, Section 1.1 of Chapter 2]) Suppose $1 < p < \infty, 1 < r < \infty$, $M_r(f)(x) = sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{\frac{1}{r}}$. Then*

$$\left(\int_{R^n} M_r(f)(x)^p dx\right)^{1/p} \le C\left(\int_{R^n} |f(x)|^p dx\right)^{1/p}.$$

Lemma 3.9. *(see [2]) Let $M_{r,m\beta}(f)(x) = \sup_{x\in Q}\left(|Q|^{-1+\frac{rm\beta}{n}}\int_Q |f(y)|^r dy\right)^{1/r}$ for $1 < r < \infty, \beta > 0, m > 0$, and $r < p < m\beta/n, 1/q = 1/p - m\beta/n$. Then*

$$||M_{r,m\beta}(f)(x)||_{L^q} \le C||f||_{L^p}.$$

4. Proof of Theorem 1

Proof. Fix a cube $Q = Q(\tilde{x}, d)$ with the sidelength d. we first prove that there exists some constant C and $1 < r < \infty$ such that, for $f \in L^p(R^n)$,

$$\sup_{Q\ni\tilde{x}}\inf_{c\in C_Q}|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(x) - c|dx \le C||\vec{b}||_{Lip_\beta}\left(M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})\right).$$

We consider the case $m = 1$ and $d \le 1$.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_J(x)$ and $f_2(x) = f(x)\chi_{J^c}(x)$, and let $(b_1)_J = |J|^{-1}\int_J b_1(y)dy$, where J is a cube concentric with Q of side-length d^{1-a}.

$$T_{\vec{b_1}}(f)(x) = \int_{R^n} K(x, x-y)((b_1(x) - (b_1)_J) - (b_1(y) - (b_1)_J))f(y)dy$$
$$= (b_1(x) - (b_1)_J)\int_{R^n} K(x,x-y)f(y)dy - \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_J)f_1(y)dy$$
$$- \int_{R^n} K(x,x-y)(b_1(y) - (b_1)_J)f_2(y)dy$$
$$= (b_1(x) - (b_1)_J)T(f)(x) - T((b_1 - (b_1)_J)f_1)(x) - T((b_1 - (b_1)_J)f_2)(x),$$

thus

$$\begin{aligned} & |Q|^{-1-\frac{\beta}{n}}\int_Q |T_{\vec{b_1}}(f)(x) - T(((b_1)_J - b_1)f_2)(\tilde{x})|dx \\ \le\ & |Q|^{-1-\frac{\beta}{n}}\int_Q |b_1(x) - (b_1)_J||T(f)(x)|dx + |Q|^{-1-\frac{\beta}{n}}\int_Q |T((b_1 - (b_1)_J)f_1)(x)|dx \\ & + |Q|^{-1-\frac{\beta}{n}}\int_Q |T((b_1 - (b_1)_J)f_2)(x) - T((b_1 - (b_1)_J)f_2)(\tilde{x})|dx \\ =\ & A_{1,1} + A_{1,2} + A_{1,3}. \end{aligned}$$

For $A_{1,1}$, by Hölder's inequality with exponent $1/r+1/r'=1$ and Lemma 1, we have

$$\begin{aligned} A_{1,1} &\leq C|Q|^{-1-\frac{\beta}{n}}|Q|\left(|Q|^{-1}\int_Q|b_1(x)-(b_1)_J|^{r'}dx\right)^{1/r'}\left(|Q|^{-1}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\ &\leq C|J|^{-\frac{\beta}{n}}\left(|J|^{-1}\int_J|b_1(x)-(b_1)_J|^{r'}dx\right)^{1/r'}\left(|Q|^{-1}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\ &\leq C||b_1||_{Lip_\beta}M_r(T(f))(\tilde{x}). \end{aligned}$$

For $A_{1,2}$, by Hölder's inequality with exponent $1/s+1/s'=1$ and $1/t+1/t'=1$, such that $st=r$, and Lemma 7, 1, we have

$$\begin{aligned} A_{1,2} &\leq C|Q|^{-1-\frac{\beta}{n}}\left(\int_{R^n}|T((b_1-(b_1)_J)f_1)(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\ &\leq C|Q|^{-1-\frac{\beta}{n}}\left(\int_{R^n}|b_1(x)-(b_1)_J|^s|f_1(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\ &= C|Q|^{-1-\frac{\beta}{n}}\left(\int_J|b_1(x)-(b_1)_J|^s|f(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\ &\leq C|Q|^{-1-\frac{\beta}{n}}|Q|^{1/s}\left(|J|^{-1}\int_J|b_1(x)-(b_1)_J|^{st'}dx\right)^{1/st'}\\ &\quad\times\left(|Q|^{-1}\int_Q|f(x)|^{st}dx\right)^{1/st}|Q|^{1/s'}\\ &\leq C|J|^{-\frac{\beta}{n}}\left(|J|^{-1}\int_J|b_1(x)-(b_1)_J|^{st'}dx\right)^{1/st'}\left(|Q|^{-1}\int_Q|f(x)|^{st}dx\right)^{1/st}\\ &\leq C||b_1||_{Lip_\beta}M_r(f)(\tilde{x}). \end{aligned}$$

For $A_{1,3}$, choose h such that $n/2<h<n/2+1/(1-a)$ and $n/2-h+\beta<0$. By Hölder's inequality with exponent $1/r+1/s+1/2=1$ and Lemma 2,3,1, we have

$$\begin{aligned} &|T((b_1-(b_1)_J)f_2)(x)-T((b_1-(b_1)_J)f_2)(\tilde{x})|\\ &=\left|\int_{R^n}(K(x,x-y)-K(\tilde{x},\tilde{x}-y))(b_1(y)-(b_1)_J)f_2(y)dy\right|\\ &\leq\int_{|y-x_0|>d^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)||b_1(y)-(b_1)_J||f(y)|dy\\ &=\sum_{N=0}^{\infty}\int_{(2^Nd)^{1-a}\leq|y-\tilde{x}|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)||b_1(y)-(b_1)_J||f(y)|dy \end{aligned}$$

$$= \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \le |y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |K(x,x-y) - K(\tilde{x},\tilde{x}-y)|$$
$$\times |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}||f(y)|dy$$
$$+ \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \le |y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |K(x,x-y) - K(\tilde{x},\tilde{x}-y)|$$
$$\times |(b_1)_{2^{(N+1)(1-a)}J} - (b_1)_J||f(y)|dy$$
$$\le C \sum_{N=0}^{\infty} \left(\int_{(2^N d)^{1-a} \le |y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |K(x,x-y) - K(\tilde{x},\tilde{x}-y)|^2 dy \right)^{1/2}$$
$$\times \left(\int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}|^s dy \right)^{1/s} \left(\int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r}$$
$$+ C \sum_{N=0}^{\infty} \left(\int_{(2^N d)^{1-a} \le |y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |K(x,x-y) - K(\tilde{x},\tilde{x}-y)|^2 dy \right)^{1/2}$$
$$\times ||\vec{b_1}||_{Lip_\beta} (2^{N+1}d)^{\beta(1-a)} (2^{N+1}d)^{n(1-a)/s} \left(\int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r}$$
$$\le C \sum_{N=0}^{\infty} |x-\tilde{x}|^{(1-a)(h-n/2)} (2^{N+1}d)^{-h(1-a)+n(1-a)/2} (2^{N+1}d)^{\beta(1-a)}$$
$$\times (2^{N+1}d)^{-\beta(1-a)} \left((2^{N+1}d)^{-n(1-a)} \int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |b_1(y) - (b_1)_{2^{(N+1)(1-a)}J}|^s dy \right)^{1/s}$$
$$\times \left((2^{N+1}d)^{-n(1-a)} \int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r}$$
$$+ C \sum_{N=0}^{\infty} |x-\tilde{x}|^{(1-a)(h-n/2)} (2^{N+1}d)^{-h(1-a)+n(1-a)/2}$$
$$\times ||b_1||_{Lip_\beta} (2^{N+1}d)^{\beta(1-a)} \left((2^{N+1}d)^{-n(1-a)} \int_{|y-\tilde{x}| \le (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{1/r}$$
$$\le C \sum_{N=0}^{\infty} d^{(1-a)(h-n/2)} (2^{N+1}d)^{(1-a)(n/2-h)} (2^{N+1}d)^{\beta(1-a)} ||b_1||_{Lip_\beta} M_r(f)(\tilde{x})$$
$$+ C \sum_{N=0}^{\infty} d^{(1-a)(h-n/2)} (2^{N+1}d)^{(1-a)(n/2-h)} (2^{N+1}d)^{\beta(1-a)} ||b_1||_{Lip_\beta} M_r(f)(\tilde{x})$$
$$\le C \sum_{N=0}^{\infty} 2^{(N+1)(1-a)(n/2-h+\beta)} d^{\beta(1-a)} ||b_1||_{Lip_\beta} M_r(f)(\tilde{x})$$
$$\le C|J|^{\frac{\beta}{n}} ||b_1||_{Lip_\beta} M_r(f)(\tilde{x})$$

thus

$$A_{1,3} \le C||b_1||_{Lip_\beta} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m=1$ and $d\le 1$.

In case $m=1$ and $d>1$, we proceeds the case as follows.

Let $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f(x)\chi_{2J}(x)$ and $f_2(x)=f(x)\chi_{(2Q)^c}(x)$, and let $(b_1)_{2Q}=|2Q|^{-1}\int_{2Q}b_1(y)dy$. We have

$$\begin{aligned}
&T_{\vec{b_1}}(f)(x)=\int_{R^n}K(x,x-y)((b_1(x)-(b_1)_{2Q})-(b_1(y)-(b_1)_{2Q}))f(y)dy\\
=\ &(b_1(x)-(b_1)_{2Q})\int_{R^n}K(x,x-y)f(y)dy\\
&-\int_{R^n}K(x,x-y)(b_1(y)-(b_1)_{2Q})f_1(y)dy\\
&-\int_{R^n}K(x,x-y)(b_1(y)-(b_1)_{2Q})f_2(y)dy\\
=\ &(b_1(x)-(b_1)_{2Q})T(f)(x)-T((b_1-(b_1)_{2Q})f_1)(x)-T((b_1-(b_1)_{2Q})f_2)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&|Q|^{-1-\frac{\beta}{n}}\int_Q|T_{\vec{b_1}}(f)(x)|dx\\
\le\ &|Q|^{-1-\frac{\beta}{n}}\int_Q|b_1(x)-(b_1)_{2Q}||T(f)(x)|dx\\
+\ &|Q|^{-1-\frac{\beta}{n}}\int_Q|T((b_1-(b_1)_{2Q})f_1)(x)|dx\\
+\ &|Q|^{-1-\frac{\beta}{n}}\int_Q|T((b_1-(b_1)_{2Q})f_2)(x)|dx\\
=\ &A_{2,1}+A_{2,2}+A_{2,3}.
\end{aligned}$$

Similar to $A_{1,1}$, $A_{2,1}\le C||b_1||_{Lip_\beta}M_r(T(f))(\tilde{x})$.

Similar to $A_{1,2}$, $A_{2,1}\le C||b_1||_{Lip_\beta}M_r(f)(\tilde{x})$.

For $A_{2,3}$, by Hölder's inequality with exponent $1/r+1/r'=1$ and Lemma 4, 2, 1, we have

$$\begin{aligned}
&|T((b_1-(b_1)_{2Q})f_2)(x)|=\left|\int_{R^n}K(x,x-y)(b_1(y)-(b_1)_{2Q})f_2(y)dy\right|\\
&\le\int_{|y-\tilde{x}|>2d}|K(x,x-y)||b_1(y)-(b_1)_{2Q}||f(y)|dy\\
&\le C\int_{|y-\tilde{x}|>2d}|x-y|^{-2n}|b_1(y)-(b_1)_{2Q}||f(y)|dy
\end{aligned}$$

$$\le C\sum_{N=1}^{\infty}\int_{2^N d\le|y-\tilde{x}|\le 2^{N+1}d}|x-y|^{-2n}|b_1(y)-(b_1)_{2^{N+1}Q}||f(y)|dy$$
$$+C\sum_{N=1}^{\infty}\int_{2^N d\le|y-\tilde{x}|\le 2^{N+1}d}|x-y|^{-2n}|(b_1)_{2^{N+1}Q}-(b_1)_{2Q}||f(y)|dy$$
$$\le C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n+n}(2^{N+1}d)^{\beta}(2^{N+1}d)^{-\beta}\left((2^{N+1}d)^{-n}\int_{|y-\tilde{x}|\le 2^{N+1}d}|f(y)|^r dy\right)^{1/r}$$
$$\times\left((2^{N+1}d)^{-n}\int_{|y-\tilde{x}|\le 2^{N+1}d}|b_1(y)-(b_1)_{2^{N+1}Q}|^{r'}dy\right)^{1/r'}$$
$$+C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n+n}\left|(b_1)_{2^{N+1}Q}-(b_1)_{2Q}\right|\left((2^{N+1}d)^{-n}\int_{|y-\tilde{x}|\le 2^{N+1}d}|f(y)|^r dy\right)^{1/r}$$
$$\le C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-n}(2^{N+1}d)^{\beta}||b_1||_{Lip_\beta}M_r(f)(\tilde{x})$$
$$+C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-n}(2^{N+1}d)^{\beta}||b_1||_{Lip_\beta}M_r(f)(\tilde{x})$$
$$\le C\sum_{N=1}^{\infty}2^{(N+1)(-n+\beta)}d^{-n+\beta}||b_1||_{Lip_\beta}M_r(f)(\tilde{x})$$
$$\le C\sum_{N=1}^{\infty}2^{(N+1)(-n+\beta)}d^{\beta}||b_1||_{Lip_\beta}M_r(f)(\tilde{x})$$
$$\le C|Q|^{\frac{\beta}{n}}||b_1||_{Lip_\beta}M_r(f)(\tilde{x}),$$

thus

$$A_{2,3}\le C||b_1||_{Lip_\beta}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m=1$ and $d>1$.

Now, we consider the case $m\ge 2$ and $d\le 1$.

Let $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f(x)\chi_J(x)$ and $f_2(x)=f(x)\chi_{J^c}(x)$, and let $(b_j)_J=|J|^{-1}\int_J b_j(y)dy$ for $1\le j\le m$, where J is a cube concentric with Q of side-length d^{1-a}.

$$\begin{aligned} T_{\vec{b}}(f)(x) &= \int_{R^n}K(x,x-y)\prod_{j=1}^{m}((b_j(x)-(b_j)_J)-(b_j(y)-(b_j)_J))f(y)dy \\ &= \sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(-1)^{m-j}(b(x)-b_J)_\gamma\int_{R^n}K(x,x-y)(b(y)-b_J)_{\gamma^c}f(y)dy \end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{m}(b_j(x)-(b_j)_J)\int_{R^n}K(x,x-y)f(y)dy \\
&\quad +\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(-1)^{m-j}(b(x)-b_J)_\gamma\int_{R^n}K(x,x-y)(b(y)-b_J)_{\gamma^c}f(y)dy \\
&\quad +(-1)^m\int_{R^n}K(x,x-y)\prod_{j=1}^{m}(b_j(y)-(b_j)_J)f_1(y)dy \\
&\quad +(-1)^m\int_{R^n}K(x,x-y)\prod_{j=1}^{m}(b_j(y)-(b_j)_J)f_2(y)dy \\
&= \prod_{j=1}^{m}(b_j(x)-(b_j)_J)T(f)(x) \\
&\quad +\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(b(x)-b_J)_\gamma T((b-b_J)_{\gamma^c}f)(x) \\
&\quad +(-1)^m T(\prod_{j=1}^{m}(b_j-(b_j)_J)f_1)(x) \\
&\quad +(-1)^m T(\prod_{j=1}^{m}(b_j-(b_j)_J)f_2)(x),
\end{aligned}
$$

thus

$$
\begin{aligned}
&|Q|^{-1-\frac{m\beta}{n}}\int_Q|T_{\vec{b}}(f)(x)-T(\prod_{j=1}^{m}((b_j)_J-b_j)f_2)(\tilde{x})|dx \\
\leq\ &|Q|^{-1-\frac{m\beta}{n}}\int_Q\prod_{j=1}^{m}|b_j(x)-(b_j)_J||T(f)(x)|dx \\
&+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1-\frac{m\beta}{n}}\int_Q|(b(x)-b_J)_\gamma||T((b-b_J)_{\gamma^c}f)(x)|dx \\
&+|Q|^{-1-\frac{m\beta}{n}}\int_Q|T(\prod_{j=1}^{m}(b_j-(b_j)_J)f_1)(x)|dx \\
&+|Q|^{-1-\frac{m\beta}{n}}\int_Q|T(\prod_{j=1}^{m}(b_j-(b_j)_J)f_2)(x)-T(\prod_{j=1}^{m}(b_j-(b_j)_J)f_2)(\tilde{x})|dx \\
=\ &A_{3,1}+A_{3,2}+A_{3,3}+A_{3,4}.
\end{aligned}
$$

For $A_{3,1}$, by Hölder's inequality with exponent $1/r_1+\cdots+1/r_m+1/r=1$ and Lemma 1, we have

$$
\begin{aligned}
A_{3,1} &\le C|Q|^{-\frac{m\beta}{n}}\prod_{j=1}^{m}\left(|J|^{-1}\int_J|b_j(x)-(b_j)_J|^{r_j}dx\right)^{1/r_j}\left(|Q|^{-1}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^{m}\left(|J|^{-\beta/n}(|J|^{-1}\int_J|b_j(x)-(b_j)_J|^{r_j}dx)^{1/r_j}\right)\left(|Q|^{-1}\int_Q|T(f)(x)|^r dx\right)^{1/r}\\
&\le C\prod_{j=1}^{m}||b_j||_{Lip_\beta}M_r(T(f))(\tilde{x})\\
&\le C||\vec{b}||_{Lip_\beta}M_r(T(f))(\tilde{x}).
\end{aligned}
$$

For $A_{3,2}$, by Hölder's inequality with exponent $1/s+1/s'=1$ and $1/t+1/t'=1$, such that $st=r$, and Lemma 1, 6, 7, we have

$$
\begin{aligned}
A_{3,2} \;&\le\; C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-\frac{m\beta}{n}}\left(|J|^{-1}\int_J|(b(x)-b_J)_\gamma|^{s'}dx\right)^{1/s'}\\
&\quad\times\left(|Q|^{-1}\int_{R^n}|T((b-b_J)_{\gamma^c}f)(x)|^s\chi_Q(x)dx\right)^{1/s}\\
&\le\; C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-\frac{m\beta}{n}}\left(|J|^{-1}\int_J|(b(x)-b_J)_\gamma|^{s'}dx\right)^{1/s'}\\
&\quad\times\left(|Q|^{-1}\int_{R^n}|(b(x)-b_J)_{\gamma^c}|^s|f(x)|^s\chi_Q(x)dx\right)^{1/s}\\
&\le\; C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-\frac{m\beta}{n}}\left(|J|^{-1}\int_J|(b(x)-b_J)_\gamma|^{s'}dx\right)^{1/s'}\\
&\quad\times\left(|Q|^{-1}\int_Q|(b(x)-b_J)_{\gamma^c}|^s|f(x)|^s dx\right)^{1/s}\\
&\le\; C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|J|^{-\frac{j\beta}{n}}\left(|J|^{-1}\int_J|(b(x)-b_J)_\gamma|^{s'}dx\right)^{1/s'}\\
&\quad\times|J|^{-\frac{(m-j)\beta}{n}}\left(|J|^{-1}\int_J|(b(x)-b_J)_{\gamma^c}|^{st'}dx\right)^{1/st'}\left(|Q|^{-1}\int_Q|f(x)|^{st}dx\right)^{1/st}\\
&\le\; C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||\vec{b_\gamma}||_{Lip_\beta}||\vec{b_{\gamma^c}}||_{Lip_\beta}M_r(f)(\tilde{x})\\
&\le\; C||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}).
\end{aligned}
$$

For $A_{3,3}$, by Hölder's inequality with exponent $1/s+1/s'=1$ and $1/t_1+\cdots+1/t_m+1/t=1$, such that $st=r$, and Lemma 1, 7, we have

$$\begin{aligned}
A_{3,3} &\leq C|Q|^{-1-\frac{m\beta}{n}}\left(\int_{R^n}|T(\prod_{j=1}^m(b_j-(b_j)_J)f_1)(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\
&\leq C|Q|^{-1-\frac{m\beta}{n}}\left(\int_{R^n}\prod_{j=1}^m|b_j(x)-(b_j)_J|^s|f_1(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\
&\leq C|Q|^{-1-\frac{m\beta}{n}}\left(\int_J\prod_{j=1}^m|b_j(x)-(b_j)_J|^s|f(x)|^s dx\right)^{1/s}|Q|^{1/s'}\\
&\leq C|Q|^{-1-\frac{m\beta}{n}}|Q|^{1/s}\prod_{j=1}^m\left(|J|^{-1}\int_J|b_j(x)-(b_j)_J|^{st_j}dx\right)^{1/st_j}\\
&\quad\times\left(|Q|^{-1}\int_Q|f(x)|^{st}dx\right)^{1/st}|Q|^{1/s'}\\
&\leq C\prod_{j=1}^m\left(|J|^{-\frac{\beta}{n}}(|J|^{-1}\int_J|b_j(x)-(b_j)_J|^{st_j}dx)^{1/st_j}\right)\\
&\quad\times\left(|Q|^{-1}\int_Q|f(x)|^{st}dx\right)^{1/st}\\
&\leq C\prod_{j=1}^m||b_j||_{Lip_\beta}M_r(f)(\tilde{x})\\
&\leq C||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}).
\end{aligned}$$

For $A_{3,4}$, choose h such that $n/2<h<n/2+1/(1-a)$ and $n/2-h+m\beta<0$. By Hölder's inequality with exponent $1/s+1/r+1/2=1$ and Lemma 1, 2, 3, we have

$$\begin{aligned}
&\left|T(\prod_{j=1}^m(b_j-(b_j)_J)f_2)(x)-T(\prod_{j=1}^m(b_j-(b_j)_J)f_2)(\tilde{x})\right|\\
=&\left|\int_{R^n}(K(x,x-y)-K(\tilde{x},\tilde{x}-y))\prod_{j=1}^m(b_j(y)-(b_j)_J)f_2(y)dy\right|\\
\leq&\int_{|y-\tilde{x}|>d^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)|\prod_{j=1}^m|b_j(y)-(b_j)_J||f(y)|dy\\
\leq&\sum_{N=0}^\infty\int_{(2^Nd)^{1-a}\leq|y-\tilde{x}|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)|\\
&\times\prod_{j=1}^m|b_j(y)-(b_j)_J||f(y)|dy
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{N=0}^{\infty}\int_{(2^N d)^{1-a}\le|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)|\\
&\quad\times\prod_{j=1}^{m}\left(|b_j(y)-(b_j)_{2^{(N+1)(1-a)}J}|+|(b_j)_{2^{(N+1)(1-a)}J}-(b_j)_J|\right)|f(y)|dy\\
&\le \sum_{N=0}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}|(b_{2^{(N+1)(1-a)}J}-b_J)_\gamma|\\
&\quad\times\int_{(2^N d)^{1-a}\le|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)|\\
&\quad\times|(b(y)-b_{2^{(N+1)(1-a)}J})_{\gamma^c}||f(y)|dy\\
&\le C\sum_{N=0}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}||\vec{b_\gamma}||_{Lip_\beta}(2^{N+1}d)^{j(1-a)\beta}\\
&\quad\times\left(\int_{(2^N d)^{1-a}\le|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|K(x,x-y)-K(\tilde{x},\tilde{x}-y)|^2dy\right)^{1/2}\\
&\quad\times\left(\int_{|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|(b(y)-b_{2^{(N+1)(1-a)}J})_{\gamma^c}|^sdy\right)^{1/s}\\
&\quad\times\left(\int_{|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|f(y)|^rdy\right)^{1/r}\\
&\le C\sum_{N=0}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}||\vec{b_\gamma}||_{Lip_\beta}(2^{N+1}d)^{j(1-a)\beta}\\
&\quad\times|x-\tilde{x}|^{(1-a)(h-n/2)}(2^{N+1}d)^{-h(1-a)+n(1-a)/2}(2^{N+1}d)^{(m-j)(1-a)\beta}\\
&\quad\times(2^{N+1}d)^{-(m-j)(1-a)\beta}\\
&\quad\times\left((2^{N+1}d)^{-n(1-a)}\int_{|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|(b(y)-b_{2^{(N+1)(1-a)}J})_{\gamma^c}|^sdy\right)^{1/s}\\
&\quad\times\left((2^{N+1}d)^{-n(1-a)}\int_{|y-\tilde{x}|\le(2^{N+1}d)^{1-a}}|f(y)|^rdy\right)^{1/r}\\
&\le C\sum_{N=0}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}||\vec{b_\gamma}||_{Lip_\beta}(2^{N+1}d)^{j(1-a)\beta}d^{(1-a)(h-n/2)}(2^{N+1}d)^{(1-a)(n/2-h)}\\
&\quad\times(2^{N+1}d)^{(m-j)(1-a)\beta}||\vec{b_{\gamma^c}}||_{Lip_\beta}M_r(f)(\tilde{x})\\
&\le C\sum_{N=0}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}2^{(N+1)(1-a)(n/2-h+m\beta)}d^{m(1-a)\beta}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x})\\
&\le C|J|^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$A_{3,4}\le C||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m \geq 2$ and $d \leq 1$.

In case $m \geq 2$ and $d > 1$, we proceeds the case as follows.

Let $f(x) = f_1(x) + f_2(x)$, with $f_1(x) = f(x)\chi_{2Q}(x)$ and $f_2(x) = f(x)\chi_{(2Q)^c}(x)$, and let $(b_j)_{2Q} = |2Q|^{-1}\int_{2Q} b_j(y)dy$ for $1 \leq j \leq m$.

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= \prod_{j=1}^{m}(b_j(x) - (b_j)_{2Q})T(f)(x) \\
&\quad + \sum_{j=1}^{m-1}\sum_{\gamma \in C_j^m}(b(x) - b_{2Q})_\gamma T((b - b_{2Q})_{\gamma^c} f)(x) \\
&\quad + (-1)^m T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x) \\
&\quad + (-1)^m T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x),
\end{aligned}$$

thus

$$\begin{aligned}
&|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x)|dx \\
\leq\ &|Q|^{-1-\frac{m\beta}{n}}\int_Q \prod_{j=1}^{m}|b_j(x) - (b_j)_{2Q}||T(f)(x)|dx \\
&+ \sum_{j=1}^{m-1}\sum_{\gamma \in C_j^m}|Q|^{-1-\frac{m\beta}{n}}\int_Q |(b(x) - b_{2Q})_\gamma||T((b - b_{2Q})_{\gamma^c} f)(x)|dx \\
&+ |Q|^{-1-\frac{m\beta}{n}}\int_Q |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_1)(x)|dx \\
&+ |Q|^{-1-\frac{m\beta}{n}}\int_Q |T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x)|dx \\
=\ &A_{4,1} + A_{4,2} + A_{4,3} + A_{4,4}.
\end{aligned}$$

Similar to $A_{3,1}$, $A_{4,1} \leq C||\vec{b}||_{Lip_\beta} M_r(T(f))(\tilde{x})$.
Similar to $A_{3,2}$, $A_{4,2} \leq C||\vec{b}||_{Lip_\beta} M_r(f)(\tilde{x})$.
Similar to $A_{3,3}$, $A_{4,3} \leq C||\vec{b}||_{Lip_\beta} M_r(f)(\tilde{x})$.

For $A_{4,4}$, by Hölder's inequality with exponent $1/r + 1/r' = 1$ and Lemma 1, 2, 4, we have

$$\left|T(\prod_{j=1}^{m}(b_j - (b_j)_{2Q})f_2)(x)\right|$$

$$
\begin{aligned}
&= \left|\int_{R^n} K(x, x-y)\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})f_2(y)dy\right| \\
&\le \int_{|y-\tilde{x}|>2d} |K(x,x-y)|\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\le C\int_{|y-\tilde{x}|>2d} |x-y|^{-2n}\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\le C\sum_{N=1}^{\infty}\int_{2^N d\le|y-\tilde{x}|\le 2^{N+1}d} |x-y|^{-2n}\prod_{j=1}^{m}|b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\le C\sum_{N=1}^{\infty}(2^{N+1}d)^{-2n} \\
&\quad \times\int_{2^N d\le|y-\tilde{x}|\le 2^{N+1}d}\prod_{j=1}^{m}(|b_j(y)-(b_j)_{2^{N+1}Q}|+|(b_j)_{2^{N+1}Q}-(b_j)_{2Q}|)|f(y)|dy \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-2n}|(b_{2^{N+1}Q}-b_{2Q})_{\gamma}| \\
&\quad \times\int_{|y-\tilde{x}|\le 2^{N+1}d}|(b(y)-b_{2^{N+1}Q})_{\gamma^c}||f(y)|dy \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-2n+n}||\vec{b_\gamma}||_{Lip_\beta}(2^{N+1}d)^{j\beta}(2^{N+1}d)^{(m-j)\beta} \\
&\quad \times(2^{N+1}d)^{-(m-j)\beta}\left((2^{N+1}d)^{-n}\int_{|y-\tilde{x}|\le 2^{N+1}d}|(b(y)-b_{2^{N+1}Q})_{\gamma^c}|^{r'}dy\right)^{1/r'} \\
&\quad \times\left((2^{N+1}d)^{-n}\int_{|y-\tilde{x}|\le 2^{N+1}d}|f(x)|^{r}dy\right)^{1/r} \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-n}||\vec{b_\gamma}||_{Lip_\beta}(2^{N+1}d)^{m\beta}||\vec{b_{\gamma^c}}||_{Lip_\beta}M_r(f)(\tilde{x}) \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}2^{(N+1)(-n+m\beta)}d^{-n+m\beta}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}) \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}2^{(N+1)(-n+m\beta)}d^{m\beta}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}) \\
&\le C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}2^{(N+1)(-n+m\beta)}|Q|^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}) \\
&\le C|Q|^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}),
\end{aligned}
$$

thus

$$A_{4,4} \leq C||\vec{b}||_{Lip_\beta} M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m \geq 2$ and $d > 1$.

So, for any cube Q and $c \in C_Q$ and $1 < r < \infty$ and $f \in L^p$,

$$|Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(x) - c|dx \quad \leq \quad C||\vec{b}||_{Lip_\beta} \left(M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})\right),$$

thus

$$\sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(x) - c|dx \quad \leq \quad C||\vec{b}||_{Lip_\beta} \left(M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})\right).$$

By Minkowski's inequality and Lemma 7, 8, we have

$$\begin{aligned} ||T_{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}} &\approx \left|\left|\sup_{Q \ni \tilde{x}} \inf_{c \in C_Q} |Q|^{-1-\frac{m\beta}{n}} \int_Q |T_{\vec{b}}(f)(x) - c|dx\right|\right|_{L^p} \\ &\leq C||\vec{b}||_{Lip_\beta} \left(||M_r(T(f))(\tilde{x})||_{L^p} + C||M_r(f)(\tilde{x})||_{L^p}\right) \\ &\leq C||\vec{b}||_{Lip_\beta} \left(||T(f)||_{L^p} + ||f||_{L^p}\right) \\ &\leq C||\vec{b}||_{Lip_\beta} ||f||_{L^p}. \end{aligned}$$

This completes the proof of Theorem 1. □

5. Proof of Theorem 2

Proof. Similar to Theorem 1, we can prove that there exists some constant c and $1 < r < \infty$, such that for $f \in L^p$,

$$|Q|^{-1} \int_Q |T_{\vec{b}}(x) - c|dx \leq C||\vec{b}||_{Lip_\beta} \left(M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})\right).$$

Further, we have

$$(T_{\vec{b}}(f))^{\#}(\tilde{x}) \leq C||\vec{b}||_{Lip_\beta}(M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

By Minkowski's inequality and Lemma 7, 9, we have

$$\begin{aligned} ||T_{\vec{b}}(f)||_{L^q} &\leq C||M(T_{\vec{b}}(f))(\tilde{x})||_{L^q} \\ &\leq C||(T_{\vec{b}}(f))^{\#}(\tilde{x})||_{L^q} \\ &\leq C||\vec{b}||_{Lip_\beta} \left(||M_{r,m\beta}(T(f))(\tilde{x})||_{L^q} + ||M_{r,m\beta}(f)(\tilde{x})||_{L^q}\right) \\ &\leq C||\vec{b}||_{Lip_\beta} \left(||T(f)||_{L^p} + ||f||_{L^p}\right) \\ &\leq C||\vec{b}||_{Lip_\beta} ||f||_{L^p}. \end{aligned}$$

This completes the proof of Theorem 2. □

References

1. S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.
2. S. Chanillo and A. Torchinsky, *Sharp function and weighted L^p estimates for a class of pseudo-differential operators*, Ark. Math. **24** (1986), 1–25.
3. W.G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica **16** (2000), 613–626.
4. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.
5. J. Garcia-Cuerva and M.J.L. Herrero, *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc. **69** (1994), 605–628.
6. S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math. **16** (1978), 263–270.
7. L.Z. Liu, *Boundedness of multilinear operator on Triebel–Lizorkin spaces*, Inter J. Math. Math. Sci. **5** (2004), 259–271.
8. L.Z. Liu, *The continuity of commutators on Triebel–Lizorkin spaces*, Integral Equations Operator Theory **49** (2004), 65–76.
9. L.Z. Liu, *Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces*, Extracta Math. **19** (2004), 243–255.
10. S.Z. Lu, *Four lectures on real H^p spaces*, World Scientific, River Edge, NI, 1995.
11. S.Z. Lu, Q. Wu and D.C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. China Ser. A **45** (2002), 984–997.
12. S.Z. Lu and D.C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. China Ser. A **38** (1995), 147–158.
13. S.Z. Lu and D.C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. China Ser. A **38** (1995), 662–673.
14. M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochbeg and Weiss*, Indiana Univ. Math. J. **44** (1995), 1–17.
15. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ Press, Princeton, 1993.
16. A. Torchinsky and S. Wang, *A note on the Marcinkiewicz integral*, Colloq. Math. **60/61**(1990), 235–243.

[1,2] College of Mathematics, Changsha University of Science and Technology, Changsha, 410077, P. R. of China.

E-mail address: wangzhiwei402@163.com

E-mail address: lanzheliu@163.com