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COMPACT SUBSTITUTION OPERATORS ON WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Abstract. In this short note we present a characterization of compact substitution operators on some weighted spaces of continuous functions.

1. Introduction

Let S and T be two non-empty sets, and let $L(S)$ and $L(T)$ denote vector spaces of complex valued functions on S and T, respectively, with vector space operations defined pointwise. If $\varphi : S \to T$ is a mapping such that the composite function $f \circ \varphi \in L(S)$ for every $f \in L(T)$, then we define the mapping $C_\varphi : L(T) \to L(S)$ by $C_\varphi f = f \circ \varphi$. The mapping C_φ is a linear transformation from $L(T)$ to $L(S)$, and it is called a substitution transformation (or composition transformation) induced by φ. If $L(S)$ and $L(T)$ are linear topological spaces, and C_φ is continuous, then it is called a substitution operator (or composition operator) induced by φ. The mappings taking φ to C_φ behave like contravariant functors from some categories of spaces to some categories of function spaces on them. These operators have been the subject matter for a systematic study for the last four decades or so. For initial details, we refer to the survey lecture by Nordgren [5] and for more recent details to [6]. The following are the three main situations under which these operators have been studied:

(1) The underlying spaces are measure spaces, and the inducing maps are measurable transformations.

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(2) The underlying spaces are regions in $\mathbb{C}$ or $\mathbb{C}^n$, and the inducing operators are holomorphic functions.
(3) The underlying spaces are topological spaces, and the inducing maps are continuous functions.

In the first case above, the theory of substitution operators comes in contact with ergodic theory [1], Markov processes and measurable dynamical systems. In the second case, it makes its impact on differentiable dynamical systems [3] and analytic function theory. In the third case, it establishes its rapport with topological dynamics, transformation groups and spaces of continuous functions [6]. In this short note we plan to study the compactness of these operators on weighted spaces $CV_b(T)$ and $CV_0(T)$, which fall in the third case described above. For results on substitution operators on these spaces and their applications, we refer to [2, 6, 8].

2. PRELIMINARIES

Let T be a topological space, and let $\mathbb{R}^+$ denote the set of all non-negative real numbers. Then a function $v : T \to \mathbb{R}^+$ is called a weight if it is upper-semi-continuous. A family V of weights on T is called a system of weights if

(1) $\lambda v \in V$ for every $v \in V$ and $\lambda \geq 0$.
(2) For every $u, v \in V$, there exists $w \in V$ such that $u \leq w$ and $v \leq w$.
(3) For every $t \in T$, there exists $v_t \in V$ such that $v_t(t) > 0$.

If V is a system of weights on T, S is a topological space, and $\varphi : S \to T$ is a continuous map, then the family $V(\varphi)$ defined by $V(\varphi) = \{v \circ \varphi : v \in V\}$ is a system of weights on S. In the case $S = T, V(\varphi)$ is a system of weights on T. We denote the vector space of all continuous functions on T by $C(T)$. $CV_b(T)$ and $CV_0(T)$ are defined by

$$CV_b(T) = \{f \in C(T) : \sup_{t\in T}\{v(t)\,|f(t)|\} < \infty \text{ for each } v \in V\}$$

and

$$CV_0(T) = \{f \in C(T) : vf \text{ vanishes at } \infty \text{ for each } v \in V\}$$

A function f on T is said to vanish at infinity if for every $\epsilon > 0$ there exists a compact subset K of T such that $|f(t)| < \epsilon$ for $t \notin K$. For $v \in V$, define $p_v(f) = \sup_{t\in T}\{v(t)\,|f(t)|\}$. Then p_v is a seminorm on $CV_b(T)$. Let $P = \{p_v : v \in V\}$. Then with the family P of seminorms, $CV_b(T)$ and $CV_0(T)$ become topological vector spaces of continuous functions. For details we refer to [4, 8]. A net $\{f_a\}$ in $CV_b(T)$(or $CV_0(T)$) converges to f if and only if $\sup_{t\in T}\{v(t)\,|f_a(t) - f(t)|\} \to 0$ for each $v \in V$.

Example 2.1. (i) If V is any system of weights on a compact space T, then $CV_b(T) = C_b(T) = C_0(T) = C(T)$.

(ii) If V is the system of weights on T consisting of all positive constant functions, then $CV_b(T) = C_b(T)$ and $CV_0(T) = C_0(T)$.

(iii) If $V = \{\lambda\chi_F : \lambda \geq 0 \text{ and } F \text{ a compact subset of } T\}$, then $CV_b(T) = C(T)$ with compact-open topology, where χ_F denotes the characterstic function of F.

(iv) If $v \in CV_b(T)$ such that $v(t) > 0$ for $t \in T$, and $V = \{\lambda v : \lambda \geq 0\}$, then $CV_b(T)$ is a Banach space.

In what follows we shall assume that for every $t \in T$ there exists an $f \in CV_0(T)$ such that $f(t) \neq 0$. This follows automatically if T is a locally compact topological space. If $\varphi : S \to T$ is a continuous map, and V is a weighted system on T, then it was proved in [8] that C_φ is a substitution operator on $CV_b(T)$ if and only if $V \leq V(\varphi)$ i.e. for every $v \in V$, there exists $u \in V$ such that $v \leq u \circ \varphi$. In the case of $CV_0(T)$, C_φ is a substitution operator if and only if $V \leq V(\varphi)$, and for each $v \in V$ and $\delta > 0$, there exists a compact subset K of T such that $\{t \in T : v(t) \geq \delta\} \cap \varphi^{-1}(K)$ is compact in T. If φ is a homeomorphism, then $V \leq V(\varphi)$ is a sufficient condition for φ to induce a substitution operator on $CV_0(T)$. The inducibility of substitution operators on weighted spaces is very much influenced by the system of weights. Even the nice functions like homeomorphisms and constant maps sometimes fail to induce substitution operators.

Example 2.2. Let $T = \mathbb{R}$. Let $v(t) = e^{-t}$ for $t \geq 0$ and $v(t) = e^{-t^2}$ for $t < 0$, and let $V = \{\lambda v : \lambda \geq 0\}$. Then positive translations induce substitution operators on $CV_b(\mathbb{R})$, while all negative translations fail to do so because the condition $V \leq V(\varphi)$ is not satisfied for negative translations.

In the following theorem we record without proof of the conditions under which constant maps induce substitution operators on weighted spaces.

Theorem 2.3. *Let V be a system of weights on T.*

(i) The following statements are equivalent:

(1) *Every $v \in V$ is bounded on T.*
(2) $1 \in CV_b(T)$, *where* 1 *denotes the constant function mapping every element of T to* 1.
(3) *Every constant function on T induces a substitution operator on $CV_b(T)$.*

(ii) The following statements are equivalent:

(1) *For every $v \in V$ and $\delta > 0$, the set $\{t \in T : v(t) \geq \delta\}$ is compact in T.*
(2) $1 \in CV_0(T)$.
(3) *Every constant function on T induces a substitution operator on $CV_0(T)$.*

3. COMPACT SUBSTITUTION OPERATORS

A linear transformation A from a topological vector space X to itself is said to be compact if the image $A(E)$ of every bounded set E of X is relatively compact in X. Recall that a subset E of $CV_b(T)$(or $CV_0(T)$) is bounded if for each $v \in V$, there exists $k_v > 0$ such that

$$\sup_{t \in T}\{v(t)\,|f(t)|\} \leq k_v \text{ for every } f \in E$$

If A is compact on $CV_b(T)$(or $CV_0(T)$) and $\{f_n\}$ is a bounded sequence in $CV_b(T)$, then there exists a subsequence $\{f_{n_k}\}$ and a $g \in CV_b(T)$(or $CV_0(T)$) such that $\{f_{n_k}\}$ converges to g.

The main aim of this section is to study compact substitution operators on the weighted spaces. In case V consists of constant weights and T is a compact

Hausdroff space, $CV_b(T) = CV_0(T) = C(T)$. In this case compact substitution operators on $C(T)$ have been characterized by Kamowitz in [2]. In [7], Singh and Summers generalized the result of Kamowitz to spaces of vector valued continuous functions on completely regular spaces. To characterize compact substitution operators on the weighted spaces, we shall make use of the following lemma.

Lemma 3.1. *Let V be a system of weights on T, and let $t_0 \in T$. Then there exists an open set containing t_0 on which each $v \in V$ is bounded.*

Proof. Let $t_0 \in T$, let $f_{t_0} \in CV_b(T)$ such that $f_{t_0}(t_0) \neq 0$. Let $G = \{t \in T : |f_{t_0}(t)| > \frac{1}{2}|f_{t_0}(t_0)|\}$. Then G is an open set containing t_0. Let $v \in V$, and let $k_v = \sup\limits_{t\in T}\{v(t)\,|f_{t_0}(t)|\}$. Then $0 < k_v < \infty$, and for every $t \in G, v(t)\,|f_{t_0}(t)| \leq k_v$. Thus

$$v(t) \leq \frac{k_v}{|f_{t_0}(t)|} \leq \frac{2k_v}{|f_{t_0}(t_0)|} < \infty \text{ for } t \in G.$$

This shows that v is bounded on G. $\square$

The following theorem shows that there are not many compact substitution operators on the weighted spaces.

Theorem 3.2. *Let T be connected completely regular Hausdroff space, and let V be a system of weights on T. Then a substitution operator C_φ is compact on $CV_b(T)$(or $CV_0(T)$) implies that the inducing function φ is constant.*

Proof. Suppose that C_φ is a compact substitution operator on $CV_b(T)$(or $CV_0(T)$). We wish to show that the inducing map φ is constant. Let $t_1, t_2 \in T$, and let $s_1 = \varphi(t_1)$ and $s_2 = \varphi(t_2)$, and assume that $s_1 \neq s_2$. By lemma 3.1, there exists an open set G containing s_1 on which each $v \in V$ is bounded and for which $s_2 \notin G$. Using the complete regularity of T, there is a continuous function $g : T \to [0, 1]$ such that $g(s_1) = 1$ and $g(t) = 0$ for each $t \notin G$. In particular, $g(s_2) = 0$. Now let $g_n = g^n$ for $n \in \mathbb{N}$, and let $E = \{g_n : n \in \mathbb{N}\}$. Since each $v \in V$ is bounded on G, and each g_n vanishes off G, it follows that E is a bounded subset of $CV_b(T)$(or $CV_0(T)$). Since C_φ is compact, there exists a subsequence $\{g_{n_k}\}$ of g_n and $f \in CV_b(T)$(or $CV_0(T)$) such that $C_\varphi g_{n_k} \to f$. This implies that

$$v(t)\,|g_{n_k}(\varphi(t)) - f(t)| \to 0 \text{ for every } t \in T \text{ and for every } v \in V.$$

From this we conclude that

$$g_{n_k}(\varphi(t)) \to f(t) \text{ for every } t \in T.$$

Thus f is a characteristic function with $f(t_1) = 1$ and $f(t_2) = 0$. This is a contradiction since f is continuous and T is connected. Hence $\varphi(t_1) = \varphi(t_2)$ for every $t_1, t_2 \in T$. Thus φ is a constant map, which is what we needed to show. $\square$

Note: The conditions in theorem 2.3 indicate the types of systems of weights for which the corresponding weighted spaces admit compact substitution operators. If the weighted spaces do not contain the constant functions, then they fail to admit compact composition operators. If T is not necessarily connected, then for some systems of weights, it would not be difficult to conclude that C_φ is compact on $CV_b(T)$ implies that the range of φ is finite [7].

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