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# STABILITY OF A FUNCTIONAL EQUATION COMING FROM THE CHARACTERIZATION OF THE ABSOLUTE VALUE OF ADDITIVE FUNCTIONS

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ABSTRACT. In the present paper, we prove the stability of the functional equation

$$\max\{f((x \circ y) \circ y), f(x)\} = f(x \circ y) + f(y)$$

for real valued functions defined on a square-symmetric groupoid with a left unit element. As a consequence, we obtain the known result about the stability of the equation

$$\max\{f(x + y), f(x - y)\} = f(x) + f(y)$$

for real valued functions defined on an abelian group.

## 1. INTRODUCTION

The topic of this paper is related to the study of the functional equation

$$\max\{f(x + y), f(x - y)\} = f(x) + f(y). \tag{1.1}$$

The general solution of this equation has been determined in the class of real valued functions defined on an abelian group $G$ in a joint paper with Alice Simon [19]. It has been proved that a function $f$ of this type satisfies (1.1) if and only if it has the form

$$f(x) = |a(x)| \qquad (x \in G),$$

where $a : G \to \mathbb{R}$ is an additive function.
In the following, we will investigate the stability of a generalization of the equation above in the sense of Pólya–Szegő ([15]) and Hyers–Ulam ([20, 10]). Based on

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the simple observation that replacing $x-y$ by $\bar{x}$ in equation (1.1), it becomes

$$\max\{f((\bar{x}+y)+y), f(\bar{x})\} = f(\bar{x}+y)+f(y) \qquad (\bar{x}, y \in G),$$

we may consider it on more general structures than groups. We will prove the stability of this equation for real valued functions defined on a square-symmetric groupoid with a left unit element. As a consequence of our main theorem, we get the stability of equation (1.1) on abelian groups, which has been proved in a joint paper with Witold Jarczyk [8].

## 2. Notation and terminology

Throughout the paper, $(S, \circ)$ denotes a *groupoid*, that is, a nonempty set $S$ with a binary operation $\circ : S \times S \to S$. If $x \in S$, we define $x^1 = x$ and, for a nonnegative integer $n$, $x^{2^{n+1}} = x^{2^n} \circ x^{2^n}$.
A groupoid $(S, \circ)$ (or the operation $\circ$ on S) is called *square-symmetric* if

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y)$$

holds for all $x, y \in S$.
Obviously, a commutative and associative binary operation is also square-symmetric. However, neither commutativity nor associativity follow from square-symmetry. For example, the operation $x \circ y = y$ is associative, square-symmetric but not commutative on an arbitrary set $S$ with at least two elements (concerning the connection between commutativity and square-symmetry, we refer to [9]). Furthermore, the commutative operation $x \circ y = |x-y|$ on $S = [0, \infty[$ is also square-symmetric but not associative. Moreover, power-symmetry is a "weaker property" than bisymmetry: the commutative and square-symmetric operation $x \circ y = |x-y|$ above is not bisymmetric. (An operation on $S$ is called bisymmetric if $(x \circ \bar{x}) \circ (y \circ \bar{y}) = (x \circ y) \circ (\bar{x} \circ \bar{y})$ for $x, \bar{x}, y, \bar{y} \in S$; cf. [1]).
The stability of functional equations on square-symmetric groupoids was investigated by several authors. Among others, Karol Baron, Gian Luigi Forti, Zoltán Kaiser, R. Duncan Luce, Zenon Moszner, Zsolt Páles and Jürg Rätz proved such type of results in [2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 18, 21, 22, 23]. Let us mention recent results of Barbara Przebieracz ([16, 17]) on the stability of some equations connected with the topic of the present paper (cf. also the references in [8]).

## 3. Stability theorems

**Theorem 3.1.** *Let $(S, \circ)$ be a square-symmetric groupoid and let $\delta$ be a nonnegative real number. If a function $f : S \to \mathbb{R}$ satisfies the inequality*

$$|f(x \circ x) - 2f(x)| \le \delta \qquad (x \in S)$$

*then the function $g : S \to \mathbb{R}$ defined by*

$$g(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(x^{2^n}\right) \qquad (x \in S), \tag{3.1}$$

*solves the equation*

$$g(x \circ x) = 2g(x) \qquad (x \in S) \tag{3.2}$$

*and fulfils*

$$|f(x) - g(x)| \le \delta \qquad (x \in S). \tag{3.3}$$

*Moreover, there exists only one solution $g : S \to \mathbb{R}$ of (3.2) for which the left hand side in (3.3) is bounded.*

*Proof.* The statement is a simple consequence of the results presented in [23] and [7]. □

**Theorem 3.2.** *Let $(S, \circ)$ be a square-symmetric groupoid with a left unit element and let $f : S \to \mathbb{R}$ be a function. If there exists a nonnegative real number $\varepsilon$ such that*

$$|\max\{f((x \circ y) \circ y), f(x)\} - f(x \circ y) - f(y)| \le \varepsilon \qquad (x, y \in S) \tag{3.4}$$

*then there exists a solution $g : S \to \mathbb{R}$ of the functional equation*

$$\max\{g((x \circ y) \circ y), g(x)\} = g(x \circ y) + g(y) \qquad (x, y \in S) \tag{3.5}$$

*for which*

$$|f(x) - g(x)| \le 3\varepsilon \qquad (x \in S). \tag{3.6}$$

*Moreover, there exists only one solution $g : S \to \mathbb{R}$ of (3.5) for which the left hand side in (3.6) is bounded.*

*Proof.* Replacing $x$ and $y$ by the unit element $e$ of $S$ in (3.4), we get that

$$|f(e)| \le \varepsilon. \tag{3.7}$$

Writing $e$ instead of $x$ in (3.4), we obtain

$$|\max\{f(y \circ y), f(e)\} - 2f(y)| \le \varepsilon \qquad (y \in S). \tag{3.8}$$

Inequality (3.7) gives $f(e) \ge -\varepsilon$, thus, $\max\{f(y \circ y), f(e)\} \ge -\varepsilon$, which together with (3.8) implies

$$f(y) \ge -\varepsilon \qquad (y \in S). \tag{3.9}$$

A simple calculation based on inequalities (3.8) and (3.9) yields

$$|f(y \circ y) - 2f(y)| \le 3\varepsilon \qquad (y \in S). \tag{3.10}$$

In fact, if $\max\{f(y \circ y), f(e)\} = f(y \circ y)$ for a $y \in S$ then $|f(y \circ y) - 2f(y)| \le \varepsilon$, thus, (3.10) follows from (3.8). On the other hand, if $\max\{f(y \circ y), f(e)\} = f(e)$ for a $y \in S$ then, by (3.7), we have $f(y \circ y) \le \varepsilon$, (3.8) gives $f(y) \le \varepsilon$, (3.9) implies $f(y) \ge -\varepsilon$ and $f(y \circ y) \ge -\varepsilon$, and these properties yield (3.10).
Based on inequality (3.10) and applying Theorem 3.1 with $\delta = 3\varepsilon$, we obtain that the function $g : S \to \mathbb{R}$ defined by (3.1) satisfies properties (3.2) and (3.6).
Writing $x^{2^n}$ instead of $x$ and $y^{2^n}$ instead of $y$ in (3.4), we obtain

$$\left|\max\{f((x^{2^n} \circ y^{2^n}) \circ y^{2^n}), f(x^{2^n})\} - f(x^{2^n} \circ y^{2^n}) - f(y^{2^n})\right| \le \varepsilon \tag{3.11}$$

for all $x, y \in S$ and $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$. It can be shown by induction that the square-symmetry of $S$ implies

$$x^{2^n} \circ y^{2^n} = (x \circ y)^{2^n} \qquad (x, y \in S,\ n \in \mathbb{N} \cup \{0\}),$$

which yields

$$(x^{2^n} \circ y^{2^n}) \circ y^{2^n} = ((x \circ y) \circ y)^{2^n} \qquad (x, y \in S,\ n \in \mathbb{N} \cup \{0\}).$$

Thus, dividing inequality (3.11) by $2^n$ and letting $n$ approach infinity, we obtain that the function $g$ satisfies (3.5).
In order to prove uniqueness, suppose that $\bar{g} : S \to \mathbb{R}$ is a solution of (3.5) which fulfils

$$|f(x) - \bar{g}(x)| \leq K \qquad (x \in S)$$

with a nonnegative constant $K \in \mathbb{R}$. The triangle inequality, with (3.6) and the previous inequality, implies

$$|g(x) - \bar{g}(x)| \leq K + 3\varepsilon \qquad (x \in S).$$

Since $g$ and $\bar{g}$ are solutions of inequality (3.4) with $\varepsilon = 0$, inequality (3.10) gives

$$g(x \circ x) = 2g(x) \qquad (x \in S)$$

and

$$\bar{g}(x \circ x) = 2\bar{g}(x) \qquad (x \in S).$$

Based on the last three formulas, Theorem 3.1 yields $g(x) = \bar{g}(x)$, $(x \in S)$, that is, the uniqueness part of our theorem. □

As a corollary of Theorem 3.2, we obtain the stability result of [8].

**Theorem 3.3.** *Let $(G, +)$ be an abelian group and $f : G \to \mathbb{R}$ be a function. If there exists a nonnegative real number $\varepsilon$ such that*

$$|\max\{f(x+y), f(x-y)\} - f(x) - f(y)| \leq \varepsilon \qquad (x, y \in G) \tag{3.12}$$

*then there exists a solution $g : G \to \mathbb{R}$ of the functional equation*

$$\max\{g(x+y), g(x-y)\} = g(x) + g(y) \qquad (x, y \in G) \tag{3.13}$$

*for which*

$$|f(x) - g(x)| \leq 3\varepsilon \qquad (x \in G). \tag{3.14}$$

*Moreover, there exists only one solution $g : G \to \mathbb{R}$ of (3.13) for which the left hand side in (3.14) is bounded.*

*Proof.* Substituting $x - y$ by $\bar{x}$ in inequality (3.12), we obtain that

$$|\max\{f((\bar{x}+y)+y), f(\bar{x})\} - f(\bar{x}+y) - f(y)| \leq \varepsilon \qquad (\bar{x}, y \in G).$$

Since the abelian group $G$ is a square-symmetric groupoid with a left unit element, Theorem 3.2 implies the existence of a function $g : G \to \mathbb{R}$ which solves the equation

$$\max\{g((\bar{x}+y)+y), g(\bar{x})\} = g(\bar{x}+y) + g(y) \qquad (\bar{x}, y \in G)$$

and satisfies inequality (3.14). Writing $x = \bar{x} + y$ in the last equation, we get (3.13), which yields the existence part of our statement. The uniqueness can be proved based on Theorem 3.2 using the substitution $\bar{x} = x - y$ again. □

*Remark* 3.4. According to the argumentation in the proof above, equations (3.5) and (3.13) are equivalent on abelian groups (i.e., they have the same solutions there). However, as it is well-known (cf., e.g., [24]), the equivalence of two functional equations does not imply that they are stable or unstable at the same time. This is the reason why we formulated our statement concerning the stability of equation (3.13) in a separate theorem.

Finally, we note that, in the class of real valued functions defined on a square-symmetric groupoid with a left unit element, the general solution of equation (3.5) has not been determined yet.

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