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OSTROWSKI'S TYPE INEQUALITIES FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS ON HILBERT SPACES: A SURVEY OF RECENT RESULTS

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ABSTRACT. In this survey we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators defined on complex Hilbert spaces.

1. Introduction

Ostrowski's type inequalities provide sharp error estimates in approximating the value of a function by its integral mean. They can be utilized to obtain a priory error bounds for different quadrature rules in approximating the Riemann integral by different Riemann sums. They also shows, in general, that the mid-point rule provides the best approximation in the class of all Riemann sums sampled in the interior points of a given partition.

As revealed by a simple search in the data base *MathSciNet* of the *American Mathematical Society* with the key words "Ostrowski" and "inequality" in the title, an exponential evolution of research papers devoted to this result has been registered in the last decade. There are now at least 280 papers that can be found by performing the above search. Numerous extensions, generalizations in both the integral and discrete case have been discovered. More general versions for n-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well. Numerous applications in Numerical Analysis, Probability Theory and other fields have been also given.

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In the scalar case, comparison between functions and integral means are incorporated in Ostrowski type inequalities as mentioned below.

The first result in this direction is known in the literature as Ostrowski's inequality [46].

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \tag{1.1}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity..

The following Ostrowski type result for absolutely continuous functions holds (see [35] – [37]).

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty [a, b]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{x-a}{b-a} \right)^{p+1} + \left(\frac{b-x}{b-a} \right)^{p+1} \right]^{\frac{1}{p}} (b-a)^{\frac{1}{p}} \|f'\|_q & \text{if } f' \in L_q [a, b], \\ & \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 ; \end{cases} \tag{1.2}$$

where $\|\cdot\|_r$ $(r \in [1, \infty])$ are the usual Lebesgue norms on $L_r [a, b]$, i.e.,

$$\|g\|_\infty := ess \sup \lim_{t \in [a,b]} |g(t)|$$

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r dt \right)^{\frac{1}{r}},\ r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [40] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [33] and the references therein for earlier contributions):

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be of $r - H-$Hölder type, i.e.,*

$$|f(x) - f(y)| \leq H|x - y|^r, \text{ for all } x, y \in [a, b], \tag{1.3}$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then, for all $x \in [a, b]$, we have the inequality:

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\leq \frac{H}{r+1} \left[\left(\frac{b-x}{b-a} \right)^{r+1} + \left(\frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r. \end{aligned} \tag{1.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [26])

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) L. \tag{1.5}$$

Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [25]).

Theorem 1.4. *Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{1.6}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [14] (see also the monograph [34]).

Theorem 1.5. *Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:*

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \\ & \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b sgn(t-x) f(t)\, dt \right\} \\ & \leq \frac{1}{b-a} \{ (x-a)[f(x) - f(a)] + (b-x)[f(b) - f(x)] \} \\ & \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)] . \end{aligned} \tag{1.7}$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

For other scalar Ostrowski's type inequalities, see [1]-[5] and [27].

For some Ostrowski related results in abstract structures, see [2], [3], [8] and the references therein.

In this survey we present some recent results obtained by the author in extending Ostrowski inequality in various directions for continuous functions of selfadjoint operators in complex Hilbert spaces. As far as we know, the obtained results are new with no previous similar results ever obtained in the literature.

2. Continuous Functions of Selfadjoint Operators

Assume that A is a bounded selfadjoint operator on the Hilbert space H. If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A = \sup \{ |\varphi(\lambda)|, \lambda \in Sp(A) \} .$$

If φ is continuous, in particular if φ is a polynomial, then the supremum is actually assumed for some points in $Sp(A)$ which is compact. Therefore the supremum may then be written as a maximum and the above formula can be written in the form $\|\varphi(A)\| = \|\varphi\|_A$.

Consider $\mathcal{C}(\mathbb{R})$ the algebra of all continuous complex valued functions defined on $\mathbb{R}$. The following fundamental result for continuous functional calculus holds, see for instance [42, p. 232]:

Theorem 2.1. *If A is a bounded selfadjoint operator on the Hilbert space H and $\varphi \in \mathcal{C}(\mathbb{R})$, then there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ with the property that whenever $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{P}$ such that $lim_{n\to\infty} \|\varphi - \varphi_n\|_A = 0$, then $\varphi(A) = \lim_{n\to\infty} \varphi_n(A)$. The mapping $\varphi \to \varphi(A)$ is a homomorphism of the algebra $\mathcal{C}(\mathbb{R})$ into $\mathcal{B}(H)$ with the additional properties $[\varphi(A)]^* = \bar{\varphi}(A)$ and $\|\varphi(A)\| \leq 2\|\varphi\|_A$. Moreover, $\varphi(A)$ is a normal operator, i.e. $[\varphi(A)]^* \varphi(A) = \varphi(A)[\varphi(A)]^*$. If φ is real-valued, then $\varphi(A)$ is selfadjoint.*

As examples we notice that, if $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = e^{is}, s \in \mathbb{R}$ then

$$e^{iA} = \sum_{k=0}^{\infty} \frac{1}{k!} (iA)^k .$$

Moreover, e^{iA} is a unitary operator and its inverse is the operator

$$\left(e^{iA}\right)^* = e^{-iA} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-iA\right)^k .$$

Now, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A \in \mathcal{B}(H)$ is selfadjoint and $\varphi(s) = \frac{1}{s-\lambda} \in \mathcal{C}(\mathbb{R})$, then $\varphi(A) = (A - \lambda I)^{-1}$.

If the selfadjoint operator $A \in \mathcal{B}(H)$ and the functions $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ are given, then we obtain the commutativity property $\varphi(A)\psi(A) = \psi(A)\varphi(A)$. This property can be extended for another operator as follows, see for instance [42, p. 235]:

Theorem 2.2. *Assume that $A \in \mathcal{B}(H)$ and the function $\varphi \in \mathcal{C}(\mathbb{R})$ are given. If $B \in \mathcal{B}(H)$ is such that $AB = BA$, then $\varphi(A)B = B\varphi(A)$.*

The next result is well known in the case of continuous functions, see for instance [42, p. 235]:

Theorem 2.3. *If A is abounded selfadjoint operator on the Hilbert space H and φ is continuous, then $Sp(\varphi(A)) = \varphi(Sp(A))$.*

As a consequence of this result we have:

Corollary 2.4. *With the assumptions in Theorem 2.3 we have:*

a) *The operator $\varphi(A)$ is selfadjoint iff $\varphi(\lambda) \in \mathbb{R}$ for all $\lambda \in Sp(A)$;*
b) *The operator $\varphi(A)$ is unitary iff $|\varphi(\lambda)| = 1$ for all $\lambda \in Sp(A)$;*
c) *The operator $\varphi(A)$ is invertible iff $\varphi(\lambda) \neq 0$ for all $\lambda \in Sp(A)$;*
d) *If $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$.*

In order to develop inequalities for functions of selfadjoint operators we need the following result, see for instance [42, p. 240]:

Theorem 2.5. *Let A be a bounded selfadjoint operator on the Hilbert space H. The homomorphism $\varphi \to \varphi(A)$ of $\mathcal{C}(\mathbb{R})$ into $\mathcal{B}(H)$ is order preserving, meaning that, if $\varphi, \psi \in \mathcal{C}(\mathbb{R})$ are real valued on $Sp(A)$ and $\varphi(\lambda) \geq \psi(\lambda)$ for any $\lambda \in Sp(A)$, then*

$$\varphi(A) \geq \psi(A) \text{ in the operator order of } \mathcal{B}(H). \tag{P}$$

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [42, p. 240]:

Theorem 2.6. *If the operator $A \in \mathcal{B}(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$. If A is invertible, then so is B.*

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

Analogously to the familiar factorization of a complex number

$$\xi = |\xi| e^{i \arg \xi}$$

a bounded normal operator on H may be written as a commutative product of a positive selfadjoint operator, representing its absolute value, and a unitary operator, representing the factor of absolute value one.

In fact, the following more general result holds, see for instance [42, p. 241]:

Theorem 2.7. *For every bounded linear operator A on H, there exists a positive selfadjoint operator $B = |A| \in \mathcal{B}(H)$ and an isometric operator C with the domain $\mathcal{D}_C = \overline{B(H)}$ and range $\mathcal{R}_C = C(\mathcal{D}_C) = \overline{A(H)}$ such that $A = CB$.*

In particular, we have:

Corollary 2.8. *If the operator $A \in \mathcal{B}(H)$ is normal, then there exists a positive selfadjoint operator $B = |A| \in \mathcal{B}(H)$ and a unitary operator C such that $A = BC = CB$. Moreover, if A is invertible, then B and C are uniquely determined by these requirements.*

Remark 2.9. Now, suppose that $A = CB$ where $B \in \mathcal{B}(H)$ is a positive selfadjoint operator and C is an isometric operator. Then

a) $B = \sqrt{A^*A}$; consequently B is uniquely determined by the stated requirements;
b) C is uniquely determined by the stated requirements iff A is one-to-one.

3. The Spectral Representation Theorem

Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are summed up in the following fundamental result concerning the spectral decomposition of bounded selfadjoint operators in Hilbert spaces, see for instance [42, p. 256]

Theorem 3.1 (Spectral Representation Theorem)**.** *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\, \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\, \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda. \tag{3.2}$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \leq \varepsilon \tag{3.3}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases} \tag{3.4}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{3.5}$$

where the integral is of Riemann–Stieltjes type.

Corollary 3.2. *With the assumptions of Theorem 3.1 for A, E_λ and φ we have the representations*

$$\varphi(A)\, x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H \tag{3.6}$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{3.7}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H. \tag{3.8}$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H. \tag{3.9}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 3.1, see for instance [42, p. 258]:

Theorem 3.3. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{F_\lambda\}_{\lambda\in\mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 3.1, then $F_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$ where E_λ is defined by (3.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A. The spectral family also reflects in a direct way the properties of the operator A as follows, see [42, p. 263-p.266]

Theorem 3.4. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. If B is a bounded linear operator on H, then $AB = BA$ iff $E_\lambda B = BE_\lambda$ for all $\lambda \in \mathbb{R}$. In particular $E_\lambda A = AE_\lambda$ for all $\lambda \in \mathbb{R}$.*

Theorem 3.5. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A and $\mu \in \mathbb{R}$. Then*

a) *μ is a regular value of A,i.e., $A - \mu I$ is invertible iff there exists a $\theta > 0$ such that $E_{\mu-\theta} = E_{\mu+\theta}$;*

b) $\mu \in Sp(A)$ *iff* $E_{\mu-\theta} < E_{\mu+\theta}$ *for all* $\theta > 0$;
c) μ *is an eigenvalue of* A *iff* $E_{\mu-0} < E_{\mu}$.

The following result will play a key role in many results concerning inequalities for bounded selfadjoint operators in Hilbert spaces. Since we were not able to locate it in the literature, we will provide here a complete proof:

Theorem 3.6 (Total Variation Schwarz's Inequality)**.** *Let* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *be the spectral family of the bounded selfadjoint operator* A *and let* $m = \min Sp(A)$ *and* $M = \max Sp(A)$. *Then for any* $x, y \in H$ *the function* $\lambda \to \langle E_\lambda x, y\rangle$ *is of bounded variation on* $[m, M]$ *and we have the inequality*

$$\left|\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right| \le \|x\| \|y\|, \tag{TVSI}$$

where $\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle E_{(\cdot)}x, y\right\rangle$ *on* $[m, M]$.

Proof. If P is a nonnegative selfadjoint operator on H, i.e., $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle, \tag{3.10}$$

for any $x, y \in H$.

Now, if $d : m = t_0 < t_1 < ... < t_{n-1} < t_n = M$ is an arbitrary partition of the interval $[m, M]$, then we have by Schwarz's inequality for nonnegative operators (3.10) that

$$\begin{aligned} &\bigvee_m^M \left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \qquad (3.11)\\ &= \sup_d \left\{\sum_{i=0}^{n-1} \left|\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, y\right\rangle\right|\right\} \\ &\le \sup_d \left\{\sum_{i=0}^{n-1} \left[\left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle^{1/2} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle^{1/2}\right]\right\} := I. \end{aligned}$$

By the Cauchy-Buniakovski–Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned} I &\le \sup_d \left\{\left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle\right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle\right]^{1/2}\right\} \qquad (3.12)\\ &\le \sup_d \left\{\left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) x, x\right\rangle\right]^{1/2} \left[\sum_{i=0}^{n-1} \left\langle \left(E_{t_{i+1}} - E_{t_i}\right) y, y\right\rangle\right]^{1/2}\right\} \\ &= \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}x, x\right\rangle\right)\right]^{1/2} \left[\bigvee_m^M \left(\left\langle E_{(\cdot)}y, y\right\rangle\right)\right]^{1/2} = \|x\| \|y\| \end{aligned}$$

for any $x, y \in H$.

On making use of (3.11) and (3.12) we deduce the desired result (TVSI). $\square$

4. Ostrowski's type Inequalities for Hölder Continuous Functions

4.1. **Introduction.** Utilising the spectral representation theorem and the following Ostrowski's type inequality for the Riemann–Stieltjes integral obtained by the author in [30]:

$$\left|f(s)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right| \tag{4.1}$$
$$\leq L\left[\frac{1}{2}(b-a)+\left|s-\frac{a+b}{2}\right|\right]^r \bigvee_a^b (u)$$

for any $s \in [a,b]$, provided that f is of $r-L-$Hölder type on $[a,b]$ (see (4.2) below), u is of *bounded variation* on $[a,b]$ and $\bigvee_a^b(u)$ denotes the *total variation* of u on $[a,b]$, we obtained the following inequality of Ostrowski type for selfadjoint operators:

Theorem 4.1 (Dragomir, 2008, [31])**.** *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is of $r-L-$Hölder type, i.e., for a given $r \in (0,1]$ and $L > 0$ we have*

$$|f(s)-f(t)| \leq L|s-t|^r \text{ for any } s,t \in [m,M], \tag{4.2}$$

then we have the inequality:

$$|f(s)-\langle f(A)x,x\rangle| \leq L\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r, \tag{4.3}$$

for any $s \in [m,M]$ and any $x \in H$ with $\|x\|=1$.

Moreover, we have

$$|\langle f(B)y,y\rangle-\langle f(A)x,x\rangle| \tag{4.4}$$
$$\leq \langle|f(B)-\langle f(A)x,x\rangle\cdot 1_H|y,y\rangle$$
$$\leq L\left[\frac{1}{2}(M-m)+\left\langle\left|B-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle\right]^r,$$

for any $x, y \in H$ with $\|x\|=\|y\|=1$.

With the above assumptions for f, A and B we have the following particular inequalities of interest:

$$\left|f\left(\frac{m+M}{2}\right)-\langle f(A)x,x\rangle\right| \leq \frac{1}{2^r}L(M-m)^r \tag{4.5}$$

and

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \leq L\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^r, \tag{4.6}$$

for any $x \in H$ with $\|x\|=1$.

We also have the inequalities:

$$\begin{aligned} &\left|\left\langle f\left(A\right)y,y\right\rangle -\left\langle f\left(A\right)x,x\right\rangle \right| \\ &\leq \left\langle \left|f\left(A\right)-\left\langle f\left(A\right)x,x\right\rangle \cdot 1_H\right|y,y\right\rangle \\ &\leq L\left[\frac{1}{2}\left(M-m\right)+\left\langle \left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle \right]^r, \end{aligned} \tag{4.7}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$,

$$\begin{aligned} &\left|\left\langle \left[f\left(B\right)-f\left(A\right)\right]x,x\right\rangle \right| \\ &\leq \left\langle \left|f\left(B\right)-\left\langle f\left(A\right)x,x\right\rangle \cdot 1_H\right|x,x\right\rangle \\ &\leq L\left[\frac{1}{2}\left(M-m\right)+\left\langle \left|B-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle \right]^r \end{aligned} \tag{4.8}$$

and, more particularly,

$$\begin{aligned} &\left\langle \left|f\left(A\right)-\left\langle f\left(A\right)x,x\right\rangle \cdot 1_H\right|x,x\right\rangle \\ &\leq L\left[\frac{1}{2}\left(M-m\right)+\left\langle \left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle \right]^r, \end{aligned} \tag{4.9}$$

for any $x\in H$ with $\|x\|=1$.

We also have the norm inequality

$$\left\|f\left(B\right)-f\left(A\right)\right\| \leq L\left[\frac{1}{2}\left(M-m\right)+\left\|B-\frac{m+M}{2}\cdot 1_H\right\|\right]^r. \tag{4.10}$$

For various generalizations, extensions and related Ostrowski type inequalities for functions of one or several variables see the monograph [34] and the references therein.

4.2. **More Inequalities of Ostrowski's Type.** The following result holds:

Theorem 4.2 (Dragomir, 2010, [32])**.** *Let A be a selfadjoint operator with $Sp\left(A\right)\subseteq\left[m,M\right]$ for some real numbers $m<M$. If $f:\left[m,M\right]\longrightarrow\mathbb{R}$ is of $r-L-$Hölder type with $r\in\left(0,1\right]$, then we have the inequality:*

$$\begin{aligned} \left|f\left(s\right)-\left\langle f\left(A\right)x,x\right\rangle \right| &\leq L\left\langle \left|s\cdot 1_H-A\right|x,x\right\rangle ^r \\ &\leq L\left[\left(s-\left\langle Ax,x\right\rangle \right)^2+D^2\left(A;x\right)\right]^{r/2}, \end{aligned} \tag{4.11}$$

for any $s\in\left[m,M\right]$ and any $x\in H$ with $\|x\|=1$, where $D\left(A;x\right)$ is the variance of the selfadjoint operator A in x and is defined by

$$D\left(A;x\right):=\left(\|Ax\|^2-\left\langle Ax,x\right\rangle ^2\right)^{1/2},$$

where $x\in H$ with $\|x\|=1$.

Proof. First of all, by the Jensen inequality for convex functions of selfadjoint operators (see for instance [41, p. 5]) applied for the modulus, we can state that

$$\left|\left\langle h\left(A\right)x,x\right\rangle \right| \leq \left\langle \left|h\left(A\right)\right|x,x\right\rangle \tag{M}$$

for any $x\in H$ with $\|x\|=1$, where h is a continuous function on $\left[m,M\right]$.

Utilising the property (M) we then get

$$|f(s)-\langle f(A)x,x\rangle| = |\langle f(s)\cdot 1_H - f(A)x,x\rangle| \le \langle |f(s)\cdot 1_H - f(A)|x,x\rangle \tag{4.12}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$.

Since f is of $r-L-$Hölder type, then for any $t, s \in [m, M]$ we have

$$|f(s)-f(t)| \le L|s-t|^r. \tag{4.13}$$

If we fix $s \in [m, M]$ and apply the property (P) for the inequality (4.13) and the operator A we get

$$\langle |f(s)\cdot 1_H - f(A)|x,x\rangle \le L\langle |s\cdot 1_H - A|^r x,x\rangle \le L\langle |s\cdot 1_H - A|x,x\rangle^r \tag{4.14}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$, where, for the last inequality we have used the fact that if P is a positive operator and $r \in (0,1)$ then, by the Hölder-McCarthy inequality [44],

$$\langle P^r x,x\rangle \le \langle Px,x\rangle^r \tag{HM}$$

for any $x \in H$ with $\|x\| = 1$. This proves the fist inequality in (4.11).

Now, observe that for any bounded linear operator T we have

$$\langle |T|x,x\rangle = \left\langle (T^*T)^{1/2}x,x\right\rangle \le \langle (T^*T)x,x\rangle^{1/2} = \|Tx\|$$

for any $x \in H$ with $\|x\| = 1$ which implies that

$$\begin{aligned}\langle |s\cdot 1_H - A|x,x\rangle^r &\le \|sx-Ax\|^r \\ &= \left(s^2 - 2s\langle Ax,x\rangle + \|Ax\|^2\right)^{r/2} \\ &= \left[(s-\langle Ax,x\rangle)^2 + \|Ax\|^2 - \langle Ax,x\rangle^2\right]^{r/2}\end{aligned} \tag{4.15}$$

for any $x \in H$ with $\|x\| = 1$ and any $s \in [m, M]$.

Finally, on making use of (4.12), (4.14) and (4.15) we deduce the desired result (4.11). □

Remark 4.3. If we choose in (4.11) $s = \frac{m+M}{2}$, then we get the sequence of inequalities

$$\begin{aligned}&\left|f\left(\frac{m+M}{2}\right) - \langle f(A)x,x\rangle\right| \\ &\le L\left\langle \left|\frac{m+M}{2}\cdot 1_H - A\right|x,x\right\rangle^r \\ &\le L\left[\left(\frac{m+M}{2} - \langle Ax,x\rangle\right)^2 + D^2(A;x)\right]^{r/2} \\ &\le L\left[\frac{1}{4}(M-m)^2 + D^2(A;x)\right]^{r/2} \le \frac{1}{2^r}L(M-m)^r\end{aligned} \tag{4.16}$$

for any $x \in H$ with $\|x\| = 1$, since, obviously,

$$\left(\frac{m+M}{2} - \langle Ax,x\rangle\right)^2 \le \frac{1}{4}(M-m)^2$$

and

$$D^2(A;x) \leq \frac{1}{4}(M-m)^2$$

for any $x \in H$ with $\|x\| = 1$.

We notice that the inequality (4.16) provides a refinement for the result (4.5) above.

The best inequality we can get from (4.11) is incorporated in the following:

Corollary 4.4 (Dragomir, 2010, [32])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality*

$$|f(\langle Ax, x\rangle) - \langle f(A)x, x\rangle| \leq L\langle|\langle Ax, x\rangle \cdot 1_H - A|\, x, x\rangle^r \leq LD^r(A;x), \quad (4.17)$$

for any $x \in H$ with $\|x\| = 1$.

The inequality (4.11) may be used to obtain other inequalities for two selfadjoint operators as follows:

Corollary 4.5 (Dragomir, 2010, [32])**.** *Let A and B be selfadjoint operators with $Sp(A), Sp(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality*

$$\begin{aligned} &|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle| \\ &\leq L\left[(\langle By, y\rangle - \langle Ax, x\rangle)^2 + D^2(A;x) + D^2(B;y)\right]^{r/2} \end{aligned} \quad (4.18)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Proof. If we apply the property (P) to the inequality (4.11) and for the operator B, then we get

$$\begin{aligned} &\langle|f(B) - \langle f(A)x, x\rangle \cdot 1_H|\, y, y\rangle \\ &\leq L\left\langle\left[(B - \langle Ax, x\rangle \cdot 1_H)^2 + D^2(A;x) \cdot 1_H\right]^{r/2} y, y\right\rangle \end{aligned} \quad (4.19)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Utilising the inequality (M) we also have that

$$|f(\langle By, y\rangle) - \langle f(A)x, x\rangle| \leq \langle|f(B) - \langle f(A)x, x\rangle \cdot 1_H|\, y, y\rangle \quad (4.20)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Now, by the Hölder-McCarthy inequality (HM) we also have

$$\begin{aligned} &\left\langle\left[(B - \langle Ax, x\rangle \cdot 1_H)^2 + D^2(A;x) \cdot 1_H\right]^{r/2} y, y\right\rangle \\ &\leq \left\langle\left[(B - \langle Ax, x\rangle \cdot 1_H)^2 + D^2(A;x) \cdot 1_H\right] y, y\right\rangle^{r/2} \\ &= \left((\langle By, y\rangle - \langle Ax, x\rangle)^2 + D^2(A;x) + D^2(B;y)\right)^{r/2} \end{aligned} \quad (4.21)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

On making use of (4.19)-(4.21) we deduce the desired result (4.18). □

Remark 4.6. Since

$$D^2(A;x) \le \frac{1}{4}(M-m)^2, \tag{4.22}$$

then we obtain from (4.18) the following vector inequalities

$$\begin{aligned} &|\langle f(A)y,y\rangle - \langle f(A)x,x\rangle| \qquad (4.23)\\ &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + D^2(A;x) + D^2(A;y)\right]^{r/2}\\ &\le L\left[(\langle Ay,y\rangle - \langle Ax,x\rangle)^2 + \frac{1}{2}(M-m)^2\right]^{r/2}, \end{aligned}$$

and

$$\begin{aligned} &|\langle [f(B)-f(A)]x,x\rangle| \qquad (4.24)\\ &\le L\left[\langle (B-A)x,x\rangle^2 + D^2(A;x) + D^2(B;x)\right]^{r/2}\\ &\le L\left[\langle (B-A)x,x\rangle^2 + \frac{1}{2}(M-m)^2\right]^{r/2}. \end{aligned}$$

In particular, we have the norm inequality

$$\|f(B)-f(A)\| \le L\left[\|B-A\|^2 + \frac{1}{2}(M-m)^2\right]^{r/2}. \tag{4.25}$$

The following result provides convenient examples for applications:

Corollary 4.7 (Dragomir, 2010, [32])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is absolutely continuous on $[m,M]$, then we have the inequality:*

$$\begin{aligned} &|f(s) - \langle f(A)x,x\rangle| \qquad (4.26)\\ &\le \begin{cases} \langle |s\cdot 1_H - A|x,x\rangle \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m,M],\\ \langle |s\cdot 1_H - A|x,x\rangle^{1/q} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m,M],\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases}\\ &\le \begin{cases} \left[(s-\langle Ax,x\rangle)^2 + D^2(A;x)\right]^{1/2} \|f'\|_{[m,M],\infty} & \text{if } f' \in L_\infty[m,M],\\ \left[(s-\langle Ax,x\rangle)^2 + D^2(A;x)\right]^{\frac{1}{2q}} \|f'\|_{[m,M],p} & \text{if } f' \in L_p[m,M],\ p>1, \frac{1}{p}+\frac{1}{q}=1, \end{cases} \end{aligned}$$

for any $s \in [m,M]$ and any $x \in H$ with $\|x\| = 1$, where $\|f'\|_{[m,M],\ell}$ are the Lebesgue norms, i.e.,

$$\|f'\|_{[m,M],\ell} := \begin{cases} ess\sup_{t\in[m,M]} |f'(t)| & \text{if } \ell = \infty\\ \left(\int_m^M |f'(t)|^p dt\right)^{1/p} & \text{if } \ell = p \ge 1. \end{cases}$$

Proof. Follows from Theorem 4.2 and on tacking into account that if $f:[m,M]\longrightarrow\mathbb{R}$ is absolutely continuous on $[m,M]$, then for any $s,t\in[m,M]$ we have

$$\begin{aligned}&|f(s)-f(t)|\\&=\left|\int_t^s f'(u)\,du\right|\\&\leq\begin{cases}|s-t|\,ess\sup_{t\in[m,M]}|f'(t)| & \text{if } f'\in L_\infty[m,M]\\ \\ |s-t|^{1/q}\left(\int_m^M|f'(t)|^p\,dt\right)^{1/p} & \text{if } f'\in L_p[m,M],\,p>1,\,\frac{1}{p}+\frac{1}{q}=1.\end{cases}\end{aligned}$$

□

Remark 4.8. It is clear that all the inequalities from Corollaries 4.4, 4.5 and Remark 4.6 may be stated for absolutely continuous functions. However, we mention here only one, namely

$$\begin{aligned}&|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \qquad (4.27)\\&\leq\begin{cases}\langle|\langle Ax,x\rangle\cdot 1_H-A|\,x,x\rangle\,\|f'\|_{[m,M],\infty} & \text{if } f'\in L_\infty[m,M]\\ \\ \langle|\langle Ax,x\rangle\cdot 1_H-A|\,x,x\rangle^{1/q}\,\|f'\|_{[m,M],p} & \begin{array}{l}\text{if } f'\in L_p[m,M],\\ p>1,\,\frac{1}{p}+\frac{1}{q}=1,\end{array}\end{cases}\\&\leq\begin{cases}D(A;x)\,\|f'\|_{[m,M],\infty} & \text{if } f'\in L_\infty[m,M]\\ \\ D^{1/q}(A;x)\,\|f'\|_{[m,M],p} & \begin{array}{l}\text{if } f'\in L_p[m,M],\\ p>1,\,\frac{1}{p}+\frac{1}{q}=1.\end{array}\end{cases}\end{aligned}$$

4.3. **The Case of (φ,Φ) −Lipschitzian Functions.** The following lemma may be stated.

Lemma 4.9. Let $u:[a,b]\to\mathbb{R}$ and $\varphi,\Phi\in\mathbb{R}$ be such that $\Phi>\varphi$. The following statements are equivalent:

(i) The function $u-\frac{\varphi+\Phi}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$, is $\frac{1}{2}(\Phi-\varphi)$ −Lipschitzian;
(ii) We have the inequality:

$$\varphi\leq\frac{u(t)-u(s)}{t-s}\leq\Phi \quad \text{for each} \quad t,s\in[a,b] \quad \text{with } t\neq s; \tag{4.28}$$

(iii) We have the inequality:

$$\varphi(t-s)\leq u(t)-u(s)\leq\Phi(t-s) \quad \text{for each} \quad t,s\in[a,b] \quad \text{with } t>s. \tag{4.29}$$

We can introduce the following class of functions, see also [43]:

Definition 4.10. The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (φ,Φ) −Lipschitzian on $[a,b]$.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides practical examples of (φ,Φ) −Lipschitzian functions.

Proposition 4.11. Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If

$$-\infty < \gamma := \inf_{t\in(a,b)} u'(t), \qquad \sup_{t\in(a,b)} u'(t) =: \Gamma < \infty \tag{4.30}$$

then u is (γ,Γ) −Lipschitzian on $[a,b]$.

The following result can be stated:

Proposition 4.12 (Dragomir, 2010, [32])**.** Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is (γ,Γ) −Lipschitzian on $[m,M]$, then we have the inequality

$$\begin{aligned} |f(\langle Ax,x\rangle) - \langle f(A)x,x\rangle| &\le \frac{1}{2}(\Gamma-\gamma)\langle |\langle Ax,x\rangle \cdot 1_H - A| x,x\rangle \\ &\le \frac{1}{2}(\Gamma-\gamma) D(A;x), \end{aligned} \tag{4.31}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Follows by Corollary 4.4 on taking into account that in this case we have $r = 1$ and $L = \frac{1}{2}(\Gamma-\gamma)$. □

We can use the result (4.31) for the particular case of convex functions to provide an interesting reverse inequality for the Jensen's type operator inequality due to Mond and Pečarić [45] (see also [41, p. 5]):

Theorem 4.13 (Mond-Pečarić, 1993, [45])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m,M]$, then*

$$f(\langle Ax,x\rangle) \le \langle f(A)x,x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

Corollary 4.14 (Dragomir, 2010, [32])**.** *With the assumptions of Theorem 4.13 we have the inequality*

$$\begin{aligned} &(0 \le) \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \\ &\le \frac{1}{2}\left(f'_-(M) - f'_+(m)\right)\langle |\langle Ax,x\rangle \cdot 1_H - A| x,x\rangle \\ &\le \frac{1}{2}\left(f'_-(M) - f'_+(m)\right) D(A;x) \le \frac{1}{4}\left(f'_-(M) - f'_+(m)\right)(M-m) \end{aligned} \tag{4.32}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Follows by Proposition 4.12 on taking into account that

$$f'_+(m)(t-s) \le f(t) - f(s) \le f'_-(M)(t-s)$$

for each s,t with the property that $M > t > s > m$. □

The following result may be stated as well:

Proposition 4.15 (Dragomir, 2010, [32])**.** Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is (γ, Γ) −Lipschitzian on $[m, M]$, then we have the inequality

$$\begin{aligned} & |f(\langle Ax, x\rangle) - \langle f(A) x, x\rangle| \\ & \leq \frac{1}{2}(\Gamma - \gamma)\left[\frac{1}{2}(M - m) + \left|\langle Ax, x\rangle - \frac{m + M}{2}\right|\right] \end{aligned} \tag{4.33}$$

for any $x \in H$ with $\|x\| = 1$.

The following particular case for convex functions holds:

Corollary 4.16 (Dragomir, 2010, [32])**.** *With the assumptions of Theorem 4.13 we have the inequality*

$$\begin{aligned} & (0 \leq) \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \\ & \leq \frac{1}{2}\left(f'_{-}(M) - f'_{+}(m)\right)\left[\frac{1}{2}(M - m) + \left|\langle Ax, x\rangle - \frac{m + M}{2}\right|\right] \end{aligned} \tag{4.34}$$

for each $x \in H$ with $\|x\| = 1$.

4.4. **Related Results.** In the previous sections we have compared amongst other the following quantities

$$f\left(\frac{m + M}{2}\right) \text{ and } f(\langle Ax, x\rangle)$$

with $\langle f(A) x, x\rangle$ for a selfadjoint operator A on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $f : [m, M] \longrightarrow \mathbb{R}$ a function of $r-$ $L-$Hölder type with $r \in (0, 1]$ and $x \in H$ with $\|x\| = 1$.

Since, obviously,

$$m \leq \frac{1}{M - m}\int_m^M f(t)\, dt \leq M,$$

then is also natural to compare $\frac{1}{M-m}\int_m^M f(t)\, dt$ with $\langle f(A) x, x\rangle$ under the same assumptions for f, A and x.

The following result holds:

Theorem 4.17 (Dragomir, 2010, [32])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \longrightarrow \mathbb{R}$ is of $r - L-$Hölder type with $r \in (0, 1]$, then we have the inequality:*

$$\begin{aligned} & \left|\frac{1}{M - m}\int_m^M f(s)\, dt - \langle f(A) x, x\rangle\right| \\ & \leq \frac{1}{r + 1} L (M - m)^r \\ & \times \left[\left\langle\left(\frac{M \cdot 1_H - A}{M - m}\right)^{r+1} x, x\right\rangle + \left\langle\left(\frac{A - m \cdot 1_H}{M - m}\right)^{r+1} x, x\right\rangle\right] \\ & \leq \frac{1}{r + 1} L (M - m)^r, \end{aligned} \tag{4.35}$$

for any $x \in H$ *with* $\|x\| = 1$.

In particular, if $f : [m, M] \longrightarrow \mathbb{R}$ *is Lipschitzian with a constant* K, *then*

$$\left| \frac{1}{M-m} \int_m^M f(s)\,dt - \langle f(A)x, x\rangle \right| \tag{4.36}$$
$$\leq K(M-m)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left(D^2(A;x) + \left(\langle Ax, x\rangle - \frac{m+M}{2}\right)^2\right)\right]$$
$$\leq \frac{1}{2}K(M-m)$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. We use the following Ostrowski's type result (see for instance [34, p. 3]) written for the function f that is of $r - L-$Hölder type on the interval $[m, M]$:

$$\left| \frac{1}{M-m} \int_m^M f(s)\,dt - f(t) \right| \tag{4.37}$$
$$\leq \frac{L}{r+1}(M-m)^r \left[\left(\frac{M-t}{M-m}\right)^{r+1} + \left(\frac{t-m}{M-m}\right)^{r+1}\right]$$

for any $t \in [m, M]$.

If we apply the properties (P) and (M) then we have successively

$$\left| \frac{1}{M-m} \int_m^M f(s)\,dt - \langle f(A)x, x\rangle \right| \tag{4.38}$$
$$\leq \left\langle \left| \frac{1}{M-m} \int_m^M f(s)\,dt - f(A) \right| x, x \right\rangle$$
$$\leq \frac{L}{r+1}(M-m)^r$$
$$\times \left[\left\langle \left(\frac{M\cdot 1_H - A}{M-m}\right)^{r+1} x, x\right\rangle + \left\langle \left(\frac{A - m\cdot 1_H}{M-m}\right)^{r+1} x, x\right\rangle\right]$$

which proves the first inequality in (4.35).

Utilising the Lah-Ribarić inequality version for selfadjoint operators A with $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and convex functions $g : [m, M] \to \mathbb{R}$, namely (see for instance [41, p. 57]):

$$\langle g(A)x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M-m} g(m) + \frac{\langle Ax, x\rangle - m}{M-m} g(M)$$

for any $x \in H$ with $\|x\| = 1$, then we get for the convex function $g(t) := \left(\frac{M-t}{M-m}\right)^{r+1}$,

$$\left\langle \left(\frac{M\cdot 1_H - A}{M-m}\right)^{r+1} x, x\right\rangle \leq \frac{M - \langle Ax, x\rangle}{M-m}$$

and for the convex function $g(t) := \left(\frac{t-m}{M-m}\right)^{r+1}$,

$$\left\langle \left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1} x,x\right\rangle \le \frac{\langle Ax,x\rangle - m}{M-m}$$

for any $x \in H$ with $\|x\| = 1$.

Now, on making use of the last two inequalities, we deduce the second part of (4.35).

Since

$$\begin{aligned}&\frac{1}{2}\left\langle \left(\frac{M\cdot 1_H - A}{M-m}\right)^{2} x,x\right\rangle + \left\langle \left(\frac{A-m\cdot 1_H}{M-m}\right)^{2} x,x\right\rangle \\ =\ &\frac{1}{4} + \frac{1}{(M-m)^2}\left(D^2(A;x) + \left(\langle Ax,x\rangle - \frac{m+M}{2}\right)^2\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$, then on choosing $r = 1$ in (4.35) we deduce the desired result (4.36). □

Remark 4.18. We should notice from the proof of the above theorem, we also have the following inequalities in the operator order of $B(H)$

$$\begin{aligned}&\left| f(A) - \left(\frac{1}{M-m}\int_m^M f(s)\,dt\right)\cdot 1_H\right| \\ &\le \frac{L}{r+1}(M-m)^r\left[\left(\frac{M\cdot 1_H - A}{M-m}\right)^{r+1} + \left(\frac{A-m\cdot 1_H}{M-m}\right)^{r+1}\right] \\ &\le \frac{1}{r+1}L(M-m)^r\cdot 1_H.\end{aligned} \tag{4.39}$$

The following particular case is of interest:

Corollary 4.19 (Dragomir, 2010, [32])**.** *Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $f : [m,M] \longrightarrow \mathbb{R}$ is $(\gamma,\Gamma)-$Lipschitzian on $[m,M]$, then we have the inequality*

$$\begin{aligned}&\left|\langle f(A)x,x\rangle - \frac{\Gamma+\gamma}{2} - \frac{1}{M-m}\int_m^M f(s)\,dt + \frac{\Gamma+\gamma}{2}\cdot\frac{m+M}{2}\right| \\ &\le \frac{1}{2}(\Gamma-\gamma)(M-m) \\ &\times\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left(D^2(A;x) + \left(\langle Ax,x\rangle - \frac{m+M}{2}\right)^2\right)\right] \\ &\le \frac{1}{4}(\Gamma-\gamma)(M-m).\end{aligned} \tag{4.40}$$

Proof. Follows by (4.36) applied for the $\frac{1}{2}(\Gamma-\gamma)$-Lipshitzian function $f - \frac{\Gamma+\gamma}{2}\cdot e$. □

5. Other Ostrowski Inequalities for Continuous Functions

5.1. **Inequalities for Absolutely Continuous Functions of Selfadjoint Operators.** We start with the following scalar inequality that is of interest in itself since it provides a generalization of the Ostrowski inequality when upper and lower bounds for the derivative are provided:

Lemma 5.1 (Dragomir, 2010, [29])**.** Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function whose derivative is bounded above and below on $[a,b]$, i.e., there exists the real constants γ and $\Gamma,\gamma<\Gamma$ with the property that $\gamma\le f'(s)\le\Gamma$ for almost every $s\in[a,b]$. Then we have the double inequality

$$\begin{aligned}&-\frac{1}{2}\cdot\frac{\Gamma-\gamma}{b-a}\left[\left(s-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2-\Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right]\\&\le f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\le\frac{1}{2}\cdot\frac{\Gamma-\gamma}{b-a}\left[\left(s-\frac{a\Gamma-b\gamma}{\Gamma-\gamma}\right)^2-\Gamma\gamma\left(\frac{b-a}{\Gamma-\gamma}\right)^2\right]\end{aligned}\tag{5.1}$$

for any $s\in[a,b]$. The inequalities are sharp.

Proof. We start with *Montgomery's identity*

$$\begin{aligned}&f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&=\frac{1}{b-a}\int_a^s(t-a)f'(t)\,dt+\frac{1}{b-a}\int_s^b(t-b)f'(t)\,dt\end{aligned}\tag{5.2}$$

that holds for any $s\in[a,b]$.

Since $\gamma\le f'(t)\le\Gamma$ for almost every $t\in[a,b]$, then

$$\frac{\gamma}{b-a}\int_a^s(t-a)\,dt\le\frac{1}{b-a}\int_a^s(t-a)f'(t)\,dt\le\frac{\Gamma}{b-a}\int_a^s(t-a)\,dt$$

and

$$\frac{\Gamma}{b-a}\int_s^b(b-t)\,dt\le\frac{1}{b-a}\int_s^b(b-t)f'(t)\,dt\le\frac{\Gamma}{b-a}\int_s^b(b-t)\,dt$$

for any $s\in[a,b]$.

Now, due to the fact that

$$\int_a^s(t-a)\,dt=\frac{1}{2}(s-a)^2\text{ and }\int_s^b(b-t)\,dt=\frac{1}{2}(b-s)^2$$

then by (5.2) we deduce the following inequality that is of interest in itself:

$$\begin{aligned} & -\frac{1}{2(b-a)}\left[\Gamma(b-s)^2-\gamma(s-a)^2\right] \\ & \leq f(s)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ & \leq \frac{1}{2(b-a)}\left[\Gamma(s-a)^2-\gamma(b-s)^2\right] \end{aligned} \tag{5.3}$$

for any $s\in[a,b]$.

Further on, if we denote by

$$A:=\gamma(s-a)^2-\Gamma(b-s)^2 \text{ and } B:=\Gamma(s-a)^2-\gamma(b-s)^2$$

then, after some elementary calculations, we derive that

$$A=-(\Gamma-\gamma)\left(s-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2+\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^2$$

and

$$B=(\Gamma-\gamma)\left(s-\frac{a\Gamma-b\gamma}{\Gamma-\gamma}\right)^2-\frac{\Gamma\gamma}{\Gamma-\gamma}(b-a)^2$$

which, together with (5.3), produces the desired result (5.1).

The sharpness of the inequalities follow from the sharpness of some particular cases outlined below. The details are omitted. □

Corollary 5.2. *With the assumptions of Lemma 5.1 we have the inequalities*

$$\frac{1}{2}\gamma(b-a)\leq\frac{1}{b-a}\int_a^b f(t)\,dt-f(a)\leq\frac{1}{2}\Gamma(b-a) \tag{5.4}$$

and

$$\frac{1}{2}\gamma(b-a)\leq f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{2}\Gamma(b-a) \tag{5.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}(\Gamma-\gamma)(b-a) \tag{5.6}$$

respectively. The constant $\frac{1}{8}$ *is best possible in (5.6).*

The proof is obvious from (5.1) on choosing $s=a$, $s=b$ and $s=\frac{a+b}{2}$, respectively.

Corollary 5.3 (Dragomir, 2010, [29])**.** *With the assumptions of Lemma 5.1 and if, in addition* $\gamma=-\alpha$ *and* $\Gamma=\beta$ *with* $\alpha,\beta>0$ *then*

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{b\beta+a\alpha}{\beta+\alpha}\right)\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right) \tag{5.7}$$

and

$$f\left(\frac{a\beta+b\alpha}{\beta+\alpha}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{1}{2}\cdot\alpha\beta\left(\frac{b-a}{\beta+\alpha}\right). \tag{5.8}$$

The proof follows from (5.1) on choosing $s = \frac{b\beta + a\alpha}{\beta + \alpha} \in [a, b]$ and $s = \frac{a\beta + b\alpha}{\beta + \alpha} \in [a, b]$, respectively.

Remark 5.4. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous and

$$\|f'\|_{\infty} := ess \sup_{t \in [a,b]} |f'(t)| < \infty,$$

then by choosing $\gamma = -\|f'\|_{\infty}$ and $\Gamma = \|f'\|_{\infty}$ in (5.1) we deduce the classical Ostrowski's inequality for absolutely continuous functions. The constant $\frac{1}{4}$ in Ostrowski's inequality is best possible.

We are able now to state the following result providing upper and lower bounds for absolutely convex functions of selfadjoint operators in Hilbert spaces whose derivatives are bounded below and above:

Theorem 5.5 (Dragomir, 2010, [29]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is an absolutely continuous function such that there exists the real constants γ and $\Gamma, \gamma < \Gamma$ with the property that $\gamma \leq f'(s) \leq \Gamma$ for almost every $s \in [m, M]$, then we have the following double inequality in the operator order of $B(H)$:*

$$-\frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m} \left[\left(A - \frac{M\Gamma - m\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right] \tag{5.9}$$

$$\leq f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H$$

$$\leq \frac{1}{2} \cdot \frac{\Gamma - \gamma}{M - m} \left[\left(A - \frac{m\Gamma - M\gamma}{\Gamma - \gamma} \cdot 1_H \right)^2 - \Gamma\gamma \left(\frac{M - m}{\Gamma - \gamma} \right)^2 \cdot 1_H \right].$$

The proof follows by the property (P) applied for the inequality (5.1) in Lemma 5.1.

Theorem 5.6 (Dragomir, 2010, [29]). *With the assumptions in Theorem 5.5 we have in the operator order the following inequalities*

$$\left| f(A) - \left(\frac{1}{M - m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \tag{5.10}$$

$$\leq \begin{cases} \left[\frac{1}{4} 1_H + \left(\frac{A - \frac{m+M}{2} 1_H}{M - m} \right)^2 \right] (M - m) \|f'\|_{\infty} & \text{if } f' \in L_{\infty}[m, M]; \\ \frac{1}{(p+1)^{\frac{1}{p}}} \left[\left(\frac{A - m 1_H}{M - m} \right)^{p+1} + \left(\frac{M 1_H - A}{M - m} \right)^{p+1} \right] (M - m)^{\frac{1}{q}} \|f'\|_q & \\ \qquad \text{if } f' \in L_p[m, M],\ \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; & \\ \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M - m} \right| \right] \|f'\|_1 . & \end{cases}$$

The proof is obvious by the scalar inequalities from Theorem 1.2 and the property (P).

The third inequality in (5.10) can be naturally generalized for functions of bounded variation as follows:

Theorem 5.7 (Dragomir, 2010, [29])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a continuous function of bounded variation on $[m, M]$, then we have the inequality*

$$\left| f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H \right| \tag{5.11}$$
$$\leq \left[\frac{1}{2} 1_H + \left| \frac{A - \frac{m+M}{2} 1_H}{M-m} \right| \right] \bigvee_m^M (f)$$

where $\bigvee_m^M (f)$ denotes the total variation of f on $[m, M]$. The constant $\frac{1}{2}$ is best possible in (5.11).

Proof. Follows from the scalar inequality obtained by the author in [25], namely

$$\left| f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \left| \frac{s - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{5.12}$$

for any $s \in [a, b]$, where f is a function of bounded variation on $[a, b]$. The constant $\frac{1}{2}$ is best possible in (5.12). □

5.2. **Inequalities for Convex Functions of Selfadjoint Operators.** The case of convex functions is important for applications.

We need the following lemma.

Lemma 5.8 (Dragomir, 2010, [29])**.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function such that the derivative f' is continuous on (a, b) and with the lateral derivative finite and $f'_-(b) \neq f'_+(a)$. Then we have the following double inequality

$$-\frac{1}{2} \cdot \frac{f'_-(b) - f'_+(a)}{b-a} \tag{5.13}$$
$$\times \left[\left(s - \frac{b f'_-(b) - a f'_+(a)}{f'_-(b) - f'_+(a)} \right)^2 - f'_-(b) f'_+(a) \left(\frac{b-a}{f'_-(b) - f'_+(a)} \right)^2 \right]$$
$$\leq f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \leq f'(s) \left(s - \frac{a+b}{2} \right)$$

for any $s \in [a, b]$.

Proof. Since f is convex, then by the fact that f' is monotonic nondecreasing, we have

$$\frac{f'_+(a)}{b-a} \int_a^s (t-a)\, dt \leq \frac{1}{b-a} \int_a^s (t-a) f'(t)\, dt \leq \frac{f'(s)}{b-a} \int_a^s (t-a)\, dt$$

and

$$\frac{f'(s)}{b-a} \int_s^b (b-t)\, dt \leq \frac{1}{b-a} \int_s^b (b-t) f'(t)\, dt \leq \frac{f'_-(b)}{b-a} \int_s^b (b-t)\, dt$$

for any $s \in [a, b]$, where $f'_+ (a)$ and $f'_- (b)$ are the lateral derivatives in a and b respectively.

Utilising the Montgomery identity (5.2) we then have

$$\begin{aligned} &\frac{f'_+ (a)}{b-a} \int_a^s (t-a)\, dt - \frac{f'_- (b)}{b-a} \int_s^b (b-t)\, dt \\ &\leq f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \\ &\leq \frac{f'(s)}{b-a} \int_a^s (t-a)\, dt - \frac{f'(s)}{b-a} \int_s^b (b-t)\, dt \end{aligned}$$

which is equivalent with the following inequality that is of interest in itself

$$\begin{aligned} &\frac{1}{2(b-a)} \left[f'_+ (a) (s-a)^2 - f'_- (b) (b-s)^2 \right] \\ &\leq f(s) - \frac{1}{b-a} \int_a^b f(t)\, dt \leq f'(s) \left(s - \frac{a+b}{2} \right) \end{aligned} \tag{5.14}$$

for any $s \in [a, b]$.

A simple calculation reveals now that the left side of (5.14) coincides with the same side of the desired inequality (5.13). □

We are able now to sate our result for convex functions of selfadjoint operators:

Theorem 5.9 (Dragomir, 2010, [29])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbb{R}$ is a differentiable convex function such that the derivative f' is continuous on (m, M) and with the lateral derivative finite and $f'_- (M) \neq f'_+ (m)$, then we have the double inequality in the operator order of $B(H)$*

$$\begin{aligned} &-\frac{1}{2} \cdot \frac{f'_- (M) - f'_+ (m)}{M-m} \\ &\left[\times \left(A - \frac{M f'_- (M) - m f'_+ (m)}{f'_- (M) - f'_+ (m)} \cdot 1_H \right)^2 \right. \\ &\left. - f'_- (M) f'_+ (m) \left(\frac{M-m}{f'_- (M) - f'_+ (m)} \right)^2 \cdot 1_H \right] \\ &\leq f(A) - \left(\frac{1}{M-m} \int_m^M f(t)\, dt \right) \cdot 1_H \leq \left(A - \frac{m+M}{2} \cdot 1_H \right) f'(A). \end{aligned} \tag{5.15}$$

The proof follows from the scalar case in Lemma 5.8.

Remark 5.10. We observe that one can drop the assumption of differentiability of the convex function and will still have the first inequality in (5.15). This follows from the fact that the class of differentiable convex functions is dense in the class of all convex functions defined on a given interval.

A different lower bound for the quantity

$$f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H$$

expressed only in terms of the operator A and not its second power as above, also holds:

Theorem 5.11 (Dragomir, 2010, [29])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\to\mathbb{R}$ is a convex function on $[m,M]$, then we have the following inequality in the operator order of $B(H)$*

$$\begin{aligned}&f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H \\ &\geq\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H \\ &-\frac{f(M)(M\cdot 1_H-A)+f(m)(A-m\cdot 1_H)}{M-m}.\end{aligned}\tag{5.16}$$

Proof. It suffices to prove for the case of differentiable convex functions defined on (m,M).

So, by the gradient inequality we have that

$$f(t)-f(s)\geq(t-s)f'(s)$$

for any $t,s\in(m,M)$.

Now, if we integrate this inequality over $s\in[m,M]$ we get

$$\begin{aligned}&(M-m)f(t)-\int_m^M f(s)\,ds \\ &\geq\int_m^M(t-s)f'(s)\,ds \\ &=\int_m^M f(s)\,ds-(M-t)f(M)-(t-m)f(m)\end{aligned}\tag{5.17}$$

for each $s\in[m,M]$.

Finally, if we apply to the inequality (5.17) the property (P), we deduce the desired result (5.16). □

Corollary 5.12 (Dragomir, 2010, [29])**.** *With the assumptions of Theorem 5.11 we have the following double inequality in the operator order*

$$\begin{aligned}&\frac{f(m)+f(M)}{2}\cdot 1_H \\ &\geq\frac{1}{2}\left[f(A)+\frac{f(M)(M\cdot 1_H-A)+f(m)(A-m\cdot 1_H)}{M-m}\right] \\ &\geq\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)\cdot 1_H.\end{aligned}\tag{5.18}$$

Proof. The second inequality is equivalent with (5.16).

For the first inequality, we observe, by the convexity of f we have that

$$\frac{f(M)(t-m)+f(m)(M-t)}{M-m}\geq f(t)$$

for any $t\in[m,M]$, which produces the operator inequality

$$\frac{f(M)(A-m\cdot 1_H)+f(m)(M\cdot 1_H-A)}{M-m}\geq f(A). \tag{5.19}$$

Now, if in both sides of (5.19) we add the same quantity

$$\frac{f(M)(M\cdot 1_H-A)+f(m)(A-m\cdot 1_H)}{M-m}$$

and perform the calculations, then we obtain the first part of (5.18) and the proof is complete. □

5.3. **Some Vector Inequalities.** The following result holds:

Theorem 5.13 (Dragomir, 2010, [29])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is an absolutely continuous function on $[m,M]$, then we have the inequalities*

$$\begin{aligned}
&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \qquad (5.20)\\
&\leq\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\[2ex] \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\, p>1,\\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases}\\
&\leq\|x\|\,\|y\|\\
&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\[2ex] \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\, p>1,\\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases}
\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$.

Proof. Since f is absolutely continuous, then we have

$$\begin{aligned}
&|f(s)-f(t)| \qquad (5.21)\\
&=\left|\int_s^t f'(u)\,du\right|\leq\left|\int_s^t|f'(u)|\,du\right|\\
&\leq\begin{cases}|t-s|\,\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\[2ex] |t-s|^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\,p>1,\,\frac{1}{p}+\frac{1}{q}=1,\end{cases}
\end{aligned}$$

for any $s,t\in[m,M]$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous functions and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Now, by the above property of the Riemann–Stieltjes integral we have from the representation (5.27) that

$$\begin{aligned}
&|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle| \qquad (5.22)\\
&=\left|\int_{m-0}^{M}[f(s)-f(t)]\,d(\langle E_t x,y\rangle)\right|\\
&\le \max_{t\in[m,M]}|f(s)-f(t)|\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\le \bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\times\begin{cases}\max_{t\in[m,M]}|t-s|\,\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \max_{t\in[m,M]}|t-s|^{1/q}\,\|f'\|_p & \text{if } f'\in L_p[m,M],\,p>1,\\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases}\\
&:=F
\end{aligned}$$

where $\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$ denotes the total variation of $\left\langle E_{(\cdot)}x,y\right\rangle$ and $x,y\in H$.

Since, obviously, we have $\max_{t\in[m,M]}|t-s|=\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|$, then

$$\begin{aligned}
F&=\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \qquad (5.23)\\
&\times\begin{cases}\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\,p>1,\\ & \frac{1}{p}+\frac{1}{q}=1,\end{cases}
\end{aligned}$$

for any $x,y\in H$.

The last part follows by the Total Variation Schwarz's inequality and the details are omitted. □

Corollary 5.14 (Dragomir, 2010, [29]). *With the assumptions of Theorem 5.13 we have the following inequalities*

$$\left|f\left(\frac{\langle Ax,x\rangle}{\|x\|^2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{5.24}$$
$$\leq \|x\|\|y\|$$
$$\times\begin{cases}\left[\frac{1}{2}(M-m)+\left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \left[\frac{1}{2}(M-m)+\left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

and

$$\left|f\left(\frac{m+M}{2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{5.25}$$
$$\leq \|x\|\|y\|$$
$$\times\begin{cases}\frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

for any $x,y\in H$.

Remark 5.15. In particular, we obtain from (5.8) the following inequalities

$$|f(\langle Ax,x\rangle)-\langle f(A)x,x\rangle| \tag{5.26}$$
$$\leq\begin{cases}\left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \left[\frac{1}{2}(M-m)+\left|\langle Ax,x\rangle-\frac{m+M}{2}\right|\right]^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

and

$$\left|f\left(\frac{m+M}{2}\right)-\langle f(A)x,x\rangle\right| \tag{5.27}$$
$$\leq\begin{cases}\frac{1}{2}(M-m)\|f'\|_\infty & \text{if } f'\in L_\infty[m,M]\\ \frac{1}{2^{1/q}}(M-m)^{1/q}\|f'\|_p & \text{if } f'\in L_p[m,M],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

for any $x\in H$ with $\|x\|=1$.

Theorem 5.16 (Dragomir, 2010, [29]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is $r-H$-Hölder continuous*

on $[m,M]$, *then we have the inequality*

$$\begin{aligned} &|f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \quad (5.28)\\ &\leq H\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r\\ &\leq H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r \end{aligned}$$

for any $x,y\in H$ *and* $s\in[m,M]$.

In particular, we have the inequalities

$$\begin{aligned} &\left|f\left(\frac{\langle Ax,x\rangle}{\|x\|^2}\right)\langle x,y\rangle - \langle f(A)x,y\rangle\right| \quad (5.29)\\ &\leq H\|x\|\|y\|\left[\frac{1}{2}(M-m)+\left|\frac{\langle Ax,x\rangle}{\|x\|^2}-\frac{m+M}{2}\right|\right]^r \end{aligned}$$

and

$$\left|f\left(\frac{m+M}{2}\right)\langle x,y\rangle - \langle f(A)x,y\rangle\right| \leq \frac{1}{2^r}H\|x\|\|y\|(M-m)^r \quad (5.30)$$

for any $x,y\in H$.

Proof. Utilising the inequality (5.22) and the fact that f is $r-H$-Hölder continuous we have successively

$$\begin{aligned} &|f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \quad (5.31)\\ &=\left|\int_{m-0}^M[f(s)-f(t)]\,d(\langle E_tx,y\rangle)\right|\\ &\leq \max_{t\in[m,M]}|f(s)-f(t)|\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\\ &\leq H\max_{t\in[m,M]}|s-t|^r\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right)\\ &=H\left[\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right]^r\bigvee_m^M\left(\langle E_{(\cdot)}x,y\rangle\right) \end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$.

The argument follows now as in the proof of Theorem 5.13 and the details are omitted. □

6. More Ostrowski's Type Inequalities

6.1. Some Vector Inequalities for Functions of Bounded Variation.

The following result holds:

Theorem 6.1 (Dragomir, 2010, [18])**.** *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A)\subseteq[m,M]$ *for some real numbers* $m<M$

and let $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f : [m, M] \to \mathbb{R}$ *is a continuous function of bounded variation on* $[m, M]$*, then we have the inequality*

$$\begin{aligned} &|f(s)\langle x, y\rangle - \langle f(A)x, y\rangle| \\ &\leq \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ &+ \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2} \bigvee_s^M (f) \\ &\leq \|x\| \|y\| \left(\frac{1}{2} \bigvee_m^M (f) + \frac{1}{2} \left| \bigvee_m^s (f) - \bigvee_s^M (f) \right| \right) \left(\leq \|x\| \|y\| \bigvee_m^M (f) \right) \end{aligned} \tag{6.1}$$

for any $x, y \in H$ *and for any* $s \in [m, M]$.

Proof. We use the following identity for the Riemann–Stieltjes integral established by the author in 2000 in [12] (see also [34, p. 452]):

$$\begin{aligned} &[u(b) - u(a)] f(s) - \int_a^b f(t)\, du(t) \\ &= \int_a^s [u(t) - u(a)]\, df(t) + \int_s^b [u(t) - u(b)]\, df(t), \end{aligned} \tag{6.2}$$

for any $s \in [a, b]$, provided the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$ exists.

A simple proof can be done by utilizing the integration by parts formula and starting from the right hand side of (6.2).

If we choose in (6.2) $a = m, b = M$ and $u(t) = \langle E_t x, y\rangle$, then we have the following identity of interest in itself

$$f(s)\langle x, y\rangle - \langle f(A)x, y\rangle = \int_{m-0}^s \langle E_t x, y\rangle\, df(t) + \int_s^M \langle (E_t - 1_H) x, y\rangle\, df(t) \tag{6.3}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function and $v : [a, b] \to \mathbb{C}$ is of bounded variation, then the Riemann–Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t)\, dv(t) \right| \leq \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v)$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Utilising this property we have from (6.3) that

$$\begin{aligned} & |f(s)\langle x,y\rangle - \langle f(A)x,y\rangle| \quad (6.4) \\ & \leq \left|\int_{m-0}^{s} \langle E_t x, y\rangle \, df(t)\right| + \left|\int_{s}^{M} \langle (E_t - 1_H)x, y\rangle \, df(t)\right| \\ & \leq \max_{t\in[m,s]} |\langle E_t x, y\rangle| \bigvee_m^s (f) + \max_{t\in[s,M]} |\langle (E_t - 1_H)x, y\rangle| \bigvee_s^M (f) := T \end{aligned}$$

for any $x, y \in H$ and for any $s \in [m, M]$.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle \quad (6.5)$$

for any $x, y \in H$.

On applying the inequality (6.5) we have

$$|\langle E_t x, y\rangle| \leq \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2}$$

and

$$|\langle (1_H - E_t)x, y\rangle| \leq \langle (1_H - E_t)x, x\rangle^{1/2} \langle (1_H - E_t)y, y\rangle^{1/2}$$

for any $x, y \in H$ and $t \in [m, M]$.

Therefore

$$\begin{aligned} T & \leq \max_{t\in[m,s]} \left[\langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2}\right] \bigvee_m^s (f) \quad (6.6) \\ & + \max_{t\in[s,M]} \left[\langle (1_H - E_t)x, x\rangle^{1/2} \langle (1_H - E_t)y, y\rangle^{1/2}\right] \bigvee_s^M (f) \\ & \leq \max_{t\in[m,s]} \langle E_t x, x\rangle^{1/2} \max_{t\in[m,s]} \langle E_t y, y\rangle^{1/2} \bigvee_m^s (f) \\ & + \max_{t\in[s,M]} \langle (1_H - E_t)x, x\rangle^{1/2} \max_{t\in[s,M]} \langle (1_H - E_t)y, y\rangle^{1/2} \bigvee_s^M (f) \\ & = \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} \bigvee_m^s (f) \\ & + \langle (1_H - E_s)x, x\rangle^{1/2} \langle (1_H - E_s)y, y\rangle^{1/2} \bigvee_s^M (f) \\ & := V \end{aligned}$$

for any $x, y \in H$ and for any $s \in [m, M]$, proving the first inequality in (6.1).

Now, observe that

$$V \le \max\left\{\bigvee_m^s (f), \bigvee_s^M (f)\right\}$$
$$\times \left[\langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} + \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2}\right].$$

Since

$$\max\left\{\bigvee_m^s (f), \bigvee_s^M (f)\right\} = \frac{1}{2}\bigvee_m^M (f) + \frac{1}{2}\left|\bigvee_m^s (f) - \bigvee_s^M (f)\right|$$

and by the Cauchy-Buniakovski–Schwarz inequality for positive real numbers a_1, b_1, a_2, b_2

$$a_1 b_1 + a_2 b_2 \le \left(a_1^2 + a_2^2\right)^{1/2} \left(b_1^2 + b_2^2\right)^{1/2} \tag{6.7}$$

we have

$$\langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} + \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2}$$
$$\le [\langle E_s x, x\rangle + \langle (1_H - E_s) x, x\rangle]^{1/2} [\langle E_s y, y\rangle + \langle (1_H - E_s) y, y\rangle]^{1/2}$$
$$= \|x\| \|y\|$$

for any $x, y \in H$ and $s \in [m, M]$, then the last part of (6.1) is proven as well. □

Remark 6.2. For the continuous function with bounded variation $f : [m, M] \to \mathbb{R}$ if $p \in [m, M]$ is a point with the property that

$$\bigvee_m^p (f) = \bigvee_p^M (f)$$

then from (6.1) we get the interesting inequality

$$|f(p) \langle x, y\rangle - \langle f(A) x, y\rangle| \le \frac{1}{2} \|x\| \|y\| \bigvee_m^M (f) \tag{6.8}$$

for any $x, y \in H$.

If the continuous function $f : [m, M] \to \mathbb{R}$ is monotonic nondecreasing and therefore of bounded variation, we get from (6.1) the following inequality as well

$$|f(s) \langle x, y\rangle - \langle f(A) x, y\rangle| \tag{6.9}$$
$$\le \langle E_s x, x\rangle^{1/2} \langle E_s y, y\rangle^{1/2} (f(s) - f(m))$$
$$+ \langle (1_H - E_s) x, x\rangle^{1/2} \langle (1_H - E_s) y, y\rangle^{1/2} (f(M) - f(s))$$
$$\le \|x\| \|y\| \left(\frac{1}{2}(f(M) - f(m)) + \left|f(s) - \frac{f(m) + f(M)}{2}\right|\right)$$
$$(\le \|x\| \|y\| f(M) - f(m))$$

for any $x, y \in H$ and $s \in [m, M]$.

Moreover, if the continuous function $f : [m, M] \to \mathbb{R}$ is nondecreasing on $[m, M]$, then the equation

$$f(s) = \frac{f(m) + f(M)}{2}$$

has got at least a solution in $[m, M]$. In his case we get from (6.9) the following trapezoidal type inequality

$$\left| \frac{f(m) + f(M)}{2} \langle x, y \rangle - \langle f(A) x, y \rangle \right| \leq \frac{1}{2} \|x\| \|y\| (f(M) - f(m)) \tag{6.10}$$

for any $x, y \in H$.

6.2. **Some Vector Inequalities for Lipshitzian Functions.** The following result that incorporates the case of Lipschitzian functions also holds

Theorem 6.3 (Dragomir, 2010, [18])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.,*

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } s, t \in [m, M],$$

then we have the inequality

$$\begin{aligned} &|f(s) \langle x, y \rangle - \langle f(A) x, y \rangle| \\ &\leq L \left[\left(\int_{m-0}^{s} \langle E_t x, x \rangle dt \right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y \rangle dt \right)^{1/2} \right. \\ &\left. + \left(\int_{s}^{M} \langle (1_H - E_t) x, x \rangle dt \right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y \rangle dt \right)^{1/2} \right] \\ &\leq L \langle |A - s1_H| x, x \rangle^{1/2} \langle |A - s1_H| y, y \rangle^{1/2} \\ &\leq L \left[D^2(A; x) + \left(s \|x\|^2 - \langle Ax, x \rangle \right)^2 \right]^{1/4} \\ &\times \left[D^2(A; y) + \left(s \|y\|^2 - \langle Ay, y \rangle \right)^2 \right]^{1/4} \end{aligned} \tag{6.11}$$

for any $x, y \in H$ and $s \in [m, M]$, where $D(A; x)$ is the variance of the selfadjoint operator A in x and is defined by

$$D(A; x) := \left(\|Ax\|^2 \|x\|^2 - \langle Ax, x \rangle^2 \right)^{1/2}.$$

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L |s - t| \text{ for any } t, s \in [a, b],$$

then the Riemann–Stieltjes integral $\int_a^b p(t) dv(t)$ exists and the following inequality holds

$$\left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| dt.$$

Now, on applying this property of the Riemann–Stieltjes integral, we have from the representation (6.3) that

$$\begin{aligned}
&\left|f(s)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \\
&\leq\left|\int_{m-0}^{s}\langle E_t x,y\rangle\,df(t)\right|+\left|\int_{s}^{M}\langle (E_t-1_H)x,y\rangle\,df(t)\right| \\
&\leq L\left[\int_{m-0}^{s}\left|\langle E_t x,y\rangle\right|dt+\int_{s}^{M}\left|\langle (E_t-1_H)x,y\rangle\right|dt\right]:=LW
\end{aligned}$$

for any $x,y\in H$ and $s\in[m,M]$.

By utilizing the generalized Schwarz inequality for nonnegative operators (6.5) and the Cauchy-Buniakovski–Schwarz inequality for the Riemann integral we have

$$\begin{aligned}
W&\leq\int_{m-0}^{s}\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}\,dt \\
&+\int_{s}^{M}\langle (1_H-E_t)x,x\rangle^{1/2}\langle (1_H-E_t)y,y\rangle^{1/2}\,dt \\
&\leq\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt\right)^{1/2} \\
&+\left(\int_{s}^{M}\langle (1_H-E_t)x,x\rangle\,dt\right)^{1/2}\left(\int_{s}^{M}\langle (1_H-E_t)y,y\rangle\,dt\right)^{1/2} \\
&:=Z
\end{aligned}\tag{6.12}$$

for any $x,y\in H$ and $s\in[m,M]$.

On the other hand, by making use of the elementary inequality (6.7) we also have

$$\begin{aligned}
Z&\leq\left(\int_{m-0}^{s}\langle E_t x,x\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)x,x\rangle\,dt\right)^{1/2} \\
&\times\left(\int_{m-0}^{s}\langle E_t y,y\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)y,y\rangle\,dt\right)^{1/2}
\end{aligned}\tag{6.13}$$

for any $x,y\in H$ and $s\in[m,M]$.

Now, observe that, by the use of the representation (6.3) for the continuous function $f:[m,M]\to\mathbb{R}$, $f(t)=|t-s|$ where s is fixed in $[m,M]$ we have the following identity that is of interest in itself

$$\langle |A-s\cdot 1_H|\,x,y\rangle=\int_{m-0}^{s}\langle E_t x,y\rangle\,dt+\int_{s}^{M}\langle (1_H-E_t)x,y\rangle\,dt \tag{6.14}$$

for any $x,y\in H$.

On utilizing (6.14) for x and then for y we deduce the second part of (6.11).

Finally, by the well known inequality for the modulus of a bounded linear operator

$$\langle |T|\,x,x\rangle\leq\|Tx\|\,\|x\|\,,x\in H$$

we have

$$
\begin{aligned}
\langle |A-s\cdot 1_H|\, x,x\rangle^{1/2} &\le \|Ax-sx\|^{1/2}\|x\|^{1/2} \\
&= \left(\|Ax\|^2-2s\langle Ax,x\rangle+s^2\|x\|^2\right)^{1/4}\|x\|^{1/2} \\
&= \left[\|Ax\|^2\|x\|^2-\langle Ax,x\rangle^2+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/4} \\
&= \left[D^2(A;x)+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/4}
\end{aligned}
$$

and a similar relation for y. The proof is thus complete. □

Remark 6.4. Since A is a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$, then

$$
\left|A-\frac{m+M}{2}\cdot 1_H\right| \le \frac{M-m}{2}1_H
$$

giving from (6.11) that

$$
\begin{aligned}
&\left|f\left(\frac{m+M}{2}\right)\langle x,y\rangle-\langle f(A)x,y\rangle\right| \qquad (6.15)\\
&\le L\left[\left(\int_{m-0}^{\frac{m+M}{2}}\langle E_t x,x\rangle\,dt\right)^{1/2}\left(\int_{m-0}^{\frac{m+M}{2}}\langle E_t y,y\rangle\,dt\right)^{1/2}\right.\\
&\left.+\left(\int_{\frac{m+M}{2}}^{M}\langle(1_H-E_t)x,x\rangle\,dt\right)^{1/2}\left(\int_{\frac{m+M}{2}}^{M}\langle(1_H-E_t)y,y\rangle\,dt\right)^{1/2}\right]\\
&\le L\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|x,x\right\rangle^{1/2}\left\langle\left|A-\frac{m+M}{2}\cdot 1_H\right|y,y\right\rangle^{1/2}\\
&\le \frac{1}{2}L(M-m)\|x\|\|y\|
\end{aligned}
$$

for any $x,y\in H$.

The particular case of equal vectors is of interest:

Corollary 6.5 (Dragomir, 2010, [18])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$. If $f:[m,M]\to\mathbb{R}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then we have the inequality*

$$
\begin{aligned}
\left|f(s)\|x\|^2-\langle f(A)x,x\rangle\right| &\le L\langle|A-s\cdot 1_H|\,x,x\rangle \qquad (6.16)\\
&\le L\left[D^2(A;x)+\left(s\|x\|^2-\langle Ax,x\rangle\right)^2\right]^{1/2}
\end{aligned}
$$

for any $x\in H$ and $s\in[m,M]$.

Remark 6.6. An important particular case that can be obtained from (6.16) is the one when $s = \frac{\langle Ax,x\rangle}{\|x\|^2}, x \neq 0$, giving the inequality

$$\left| f\left(\frac{\langle Ax,x\rangle}{\|x\|^2}\right)\|x\|^2 - \langle f(A)x,x\rangle\right| \leq L\left\langle\left|A - \frac{\langle Ax,x\rangle}{\|x\|^2}\cdot 1_H\right|x,x\right\rangle \tag{6.17}$$
$$\leq LD(A;x) \leq \frac{1}{2}L(M-m)\|x\|^2$$

for any $x \in H, x \neq 0$.

We are able now to provide the following corollary:

Corollary 6.7 (Dragomir, 2010, [18])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m,M] \to \mathbb{R}$ is a $(\varphi,\Phi)-$Lipschitzian functions on $[m,M]$ with $\Phi > \varphi$, then we have the inequality*

$$\left|\langle f(A)x,y\rangle - \frac{\Phi+\varphi}{2}\langle Ax,y\rangle + \frac{\Phi+\varphi}{2}s\langle x,y\rangle - f(s)\langle x,y\rangle\right| \tag{6.18}$$
$$\leq \frac{1}{2}(\Phi-\varphi)\left[\left(\int_{m-0}^{s}\langle E_t x,x\rangle dt\right)^{1/2}\left(\int_{m-0}^{s}\langle E_t y,y\rangle dt\right)^{1/2}\right.$$
$$\left.+\left(\int_{s}^{M}\langle(1_H - E_t)x,x\rangle dt\right)^{1/2}\left(\int_{s}^{M}\langle(1_H - E_t)y,y\rangle dt\right)^{1/2}\right]$$
$$\leq \frac{1}{2}(\Phi-\varphi)\langle|A - s1_H|x,x\rangle^{1/2}\langle|A - s1_H|y,y\rangle^{1/2}$$
$$\leq \frac{1}{2}(\Phi-\varphi)\left[D^2(A;x) + \left(s\|x\|^2 - \langle Ax,x\rangle\right)^2\right]^{1/4}$$
$$\times\left[D^2(A;y) + \left(s\|y\|^2 - \langle Ay,y\rangle\right)^2\right]^{1/4}$$

for any $x, y \in H$.

Remark 6.8. Various particular cases can be stated by utilizing the inequality (6.18), however the details are left to the interested reader.

7. Some Vector Inequalities for Monotonic Functions

The case of monotonic functions is of interest as well. The corresponding result is incorporated in the following

Theorem 7.1 (Dragomir, 2010, [18])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m,M] \to \mathbb{R}$ is a continuous monotonic* *nondecreasing function on* $[m, M]$, *then we have the inequality*

$$\begin{aligned} & \left|f(s)\langle x, y\rangle-\langle f(A) x, y\rangle\right| \\ & \leq\left(\int_{m-0}^{s}\langle E_{t} x, x\rangle d f(t)\right)^{1/2}\left(\int_{m-0}^{s}\langle E_{t} y, y\rangle d f(t)\right)^{1/2} \\ & +\left(\int_{s}^{M}\langle(1_{H}-E_{t}) x, x\rangle d f(t)\right)^{1/2}\left(\int_{s}^{M}\langle(1_{H}-E_{t}) y, y\rangle d f(t)\right)^{1/2} \\ & \leq\langle|f(A)-f(s) 1_{H}| x, x\rangle^{1/2}\langle|f(A)-f(s) 1_{H}| y, y\rangle^{1/2} \\ & \leq\left[D^{2}(f(A) ; x)+\left(f(s)\|x\|^{2}-\langle f(A) x, x\rangle\right)^{2}\right]^{1/4} \\ & \times\left[D^{2}(f(A) ; y)+\left(f(s)\|y\|^{2}-\langle f(A) y, y\rangle\right)^{2}\right]^{1/4} \end{aligned} \tag{7.1}$$

for any $x, y \in H$ *and* $s \in [m, M]$, *where, as above* $D(f(A); x)$ *is the variance of the selfadjoint operator* $f(A)$ *in* x.

Proof. From the theory of Riemann–Stieltjes integral is well known that if $p : [a, b] \to \mathbb{C}$ is of bounded variation and $v : [a, b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann–Stieltjes integrals $\int_a^b p(t)\, dv(t)$ and $\int_a^b |p(t)|\, dv(t)$ exist and

$$\left|\int_{a}^{b} p(t) d v(t)\right| \leq \int_{a}^{b}|p(t)| d v(t).$$

On utilizing this property and the representation (6.3) we have successively

$$\begin{aligned} & \left|f(s)\langle x, y\rangle-\langle f(A) x, y\rangle\right| \\ & \leq\left|\int_{m-0}^{s}\langle E_{t} x, y\rangle d f(t)\right|+\left|\int_{s}^{M}\langle(E_{t}-1_{H}) x, y\rangle d f(t)\right| \\ & \leq \int_{m-0}^{s}\left|\langle E_{t} x, y\rangle\right| d f(t)+\int_{s}^{M}\left|\langle(E_{t}-1_{H}) x, y\rangle\right| d f(t) \\ & \leq \int_{m-0}^{s}\langle E_{t} x, x\rangle^{1/2}\langle E_{t} y, y\rangle^{1/2} d f(t) \\ & +\int_{s}^{M}\langle(1_{H}-E_{t}) x, x\rangle^{1/2}\langle(1_{H}-E_{t}) y, y\rangle^{1/2} d f(t) \\ & := Y, \end{aligned} \tag{7.2}$$

for any $x, y \in H$ and $s \in [m, M]$.

We use now the following version of the Cauchy-Buniakovski–Schwarz inequality for the Riemann–Stieltjes integral with monotonic nondecreasing integrators

$$\left(\int_{a}^{b} p(t) q(t) d v(t)\right)^{2} \leq \int_{a}^{b} p^{2}(t) d v(t) \int_{a}^{b} q^{2}(t) d v(t)$$

to get that

$$\begin{aligned}&\int_{m-0}^{s} \langle E_t x, x\rangle^{1/2} \langle E_t y, y\rangle^{1/2} df(t)\\ &\le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t)\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\int_{s}^{M} \langle (1_H - E_t) x, x\rangle^{1/2} \langle (1_H - E_t) y, y\rangle^{1/2} df(t)\\ &\le \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$.

Therefore

$$\begin{aligned}Y &\le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t)\right)^{1/2} \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t)\right)^{1/2}\\ &\quad + \left(\int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2} \left(\int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\\ &\le \left(\int_{m-0}^{s} \langle E_t x, x\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) x, x\rangle df(t)\right)^{1/2}\\ &\quad \times \left(\int_{m-0}^{s} \langle E_t y, y\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) y, y\rangle df(t)\right)^{1/2}\end{aligned}$$

for any $x, y \in H$ and $s \in [m, M]$, where, to get the last inequality we have used the elementary inequality (6.7).

Now, since f is monotonic nondecreasing, on applying the representation (6.3) for the function $|f(\cdot) - f(s)|$ with s fixed in $[m, M]$ we deduce the following identity that is of interest in itself as well:

$$\langle |f(A) - f(s)| x, y\rangle = \int_{m-0}^{s} \langle E_t x, y\rangle df(t) + \int_{s}^{M} \langle (1_H - E_t) x, y\rangle df(t) \tag{7.3}$$

for any $x, y \in H$.

The second part of (7.1) follows then by writing (7.3) for x then by y and utilizing the relevant inequalities from above.

The last part is similar to the corresponding one from the proof of Theorem 6.3 and the details are omitted. □

The following corollary is of interest:

Corollary 7.2 (Dragomir, 2010, [18]). *With the assumption of Theorem 7.1 we have the inequalities*

$$\left|\frac{f(m)+f(M)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{7.4}$$
$$\leq\left\langle\left|f(A)-\frac{f(m)+f(M)}{2}\cdot 1_H\right|x,x\right\rangle^{1/2}$$
$$\times\left\langle\left|f(A)-\frac{f(m)+f(M)}{2}\cdot 1_H\right|y,y\right\rangle^{1/2}$$
$$\leq\frac{1}{2}(f(M)-f(m))\|x\|\|y\|,$$

for any $x,y\in H$.

Proof. Since f is monotonic nondecreasing, then $f(u)\in[f(m),f(M)]$ for any $u\in[m,M]$. By the continuity of f it follows that there exists at list one $s\in[m,M]$ such that

$$f(s)=\frac{f(m)+f(M)}{2}.$$

Now, on utilizing the inequality (7.1) for this s we deduce the first inequality in (7.4). The second part follows as above and the details are omitted. □

8. Ostrowski's Type Vector Inequalities

8.1. **Some Vector Inequalities.** The following result holds:

Theorem 8.1 (Dragomir, 2010, [28]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$, then we have the inequality*

$$\left|\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{8.1}$$
$$\leq\frac{1}{M-m}\bigvee_m^M(f)\max_{t\in[m,M]}\Big[(M-t)\langle E_tx,x\rangle^{1/2}\langle E_ty,y\rangle^{1/2}$$
$$+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2}\Big]$$
$$\leq\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

Proof. Assume that $f:[m,M]\to\mathbb{C}$ is a continuous function on $[m,M]$. Then under the assumptions of the theorem for A and $\{E_\lambda\}_\lambda$, we have the following representation

$$\langle x,y\rangle \frac{1}{M-m}\int_{m}^{M} f(s)\,ds-\langle f(A)x,y\rangle = \frac{1}{M-m}\int_{m-0}^{M}\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\,df(t) \tag{8.2}$$

for any $x,y\in H$.

Indeed, integrating by parts in the Riemann–Stieltjes integral and using the spectral representation theorem we have

$$\begin{aligned}
&\frac{1}{M-m}\int_{m-0}^{M}\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle\,df(t)\\
&=\int_{m-0}^{M}\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)df(t)\\
&=\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)f(t)\Big|_{m-0}^{M}\\
&-\int_{m-0}^{M}f(t)\,d\left(\langle E_t x,y\rangle-\frac{t-m}{M-m}\langle x,y\rangle\right)\\
&=-\int_{m-0}^{M}f(t)\,d\langle E_t x,y\rangle+\langle x,y\rangle\frac{1}{M-m}\int_{m}^{M}f(t)\,dt\\
&=\langle x,y\rangle\frac{1}{M-m}\int_{m}^{M}f(t)\,dt-\langle f(A)x,y\rangle
\end{aligned}$$

for any $x,y\in H$ and the equality (8.2) is proved.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\max_{t\in[a,b]}|p(t)|\bigvee_a^b(v)$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising this property we have from (8.2) that

$$\left|\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \le\frac{1}{M-m}\max_{t\in[m,M]}|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\bigvee_m^M(f) \tag{8.3}$$

for any $x,y\in H$.

Now observe that

$$\begin{aligned} & |\langle [(M-t)\,E_t + (t-m)\,(E_t - 1_H)]\,x, y\rangle| \qquad (8.4)\\ & = |(M-t)\,\langle E_t x, y\rangle + (t-m)\,\langle (E_t - 1_H)\,x, y\rangle| \\ & \le (M-t)\,|\langle E_t x, y\rangle| + (t-m)\,|\langle (E_t - 1_H)\,x, y\rangle| \end{aligned}$$

for any $x, y \in H$ and $t \in [m, M]$.

If P is a nonnegative operator on H, i.e., $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\,\langle Py, y\rangle \qquad (8.5)$$

for any $x, y \in H$.

On applying the inequality (8.5) we have

$$\begin{aligned} & (M-t)\,|\langle E_t x, y\rangle| + (t-m)\,|\langle (E_t - 1_H)\,x, y\rangle| \qquad (8.6)\\ & \le (M-t)\,\langle E_t x, x\rangle^{1/2}\,\langle E_t y, y\rangle^{1/2} \\ & + (t-m)\,\langle (1_H - E_t)\,x, x\rangle^{1/2}\,\langle (1_H - E_t)\,y, y\rangle^{1/2} \\ & \le \max\{M-t, t-m\} \\ & \times \left[\langle E_t x, x\rangle^{1/2}\,\langle E_t y, y\rangle^{1/2} + \langle (1_H - E_t)\,x, x\rangle^{1/2}\,\langle (1_H - E_t)\,y, y\rangle^{1/2}\right] \\ & \le \max\{M-t, t-m\} \\ & \times [\langle E_s x, x\rangle + \langle (1_H - E_s)\,x, x\rangle]^{1/2}\,[\langle E_s y, y\rangle + \langle (1_H - E_s)\,y, y\rangle]^{1/2} \\ & = \max\{M-t, t-m\}\,\|x\|\,\|y\|, \end{aligned}$$

where for the last inequality we used the elementary fact

$$a_1 b_1 + a_2 b_2 \le \left(a_1^2 + a_2^2\right)^{1/2}\left(b_1^2 + b_2^2\right)^{1/2} \qquad (8.7)$$

that holds for a_1, b_1, a_2, b_2 positive real numbers.

Utilising the inequalities (8.3), (8.4) and (8.6) we deduce the desired result (8.1). □

The case of Lipschitzian functions is embodied in the following result:

Theorem 8.2 (Dragomir, 2010, [28])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{C}$ is a Lipschitzian function with the constant $L > 0$ on $[m, M]$, then we have the inequality*

$$\begin{aligned} & \left|\langle x, y\rangle\,\frac{1}{M-m}\int_m^M f(s)\,ds - \langle f(A)\,x, y\rangle\right| \qquad (8.8)\\ & \le \frac{L}{M-m}\int_m^M \left[(M-t)\,\langle E_t x, x\rangle^{1/2}\,\langle E_t y, y\rangle^{1/2}\right. \\ & \left. + (t-m)\,\langle (1_H - E_t)\,x, x\rangle^{1/2}\,\langle (1_H - E_t)\,y, y\rangle^{1/2}\right] dt \\ & \le \frac{3}{4} L\,(M-m)\,\|x\|\,\|y\| \end{aligned}$$

for any $x, y \in H$.

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\le L|s-t| \text{ for any } t,s\in[a,b],$$

then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b |p(t)|\,dt.$$

Now, on applying this property of the Riemann–Stieltjes integral, we have from the representation (8.2) that

$$\begin{aligned}
&\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \\
&\le \frac{L}{M-m}\int_{m-0}^M |\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,dt.
\end{aligned} \tag{8.9}$$

Since, from the proof of Theorem 8.1, we have

$$\begin{aligned}
&|\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle| \\
&\le (M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2} \\
&+(t-m)\langle(1_H-E_t)x,x\rangle^{1/2}\langle(1_H-E_t)y,y\rangle^{1/2} \\
&\le \max\{M-t,t-m\}\|x\|\|y\| \\
&=\left[\frac{1}{2}(M-m)+\left|t-\frac{m+M}{2}\right|\right]\|x\|\|y\|
\end{aligned} \tag{8.10}$$

for any $x,y\in H$ and $t\in[m,M]$, then integrating (8.10) and taking into account that

$$\int_m^M \left|t-\frac{m+M}{2}\right|dt=\frac{1}{4}(M-m)^2$$

we deduce the desired result (8.8). □

Finally for the section, we provide here the case of monotonic nondecreasing functions as well:

Theorem 8.3 (Dragomir, 2010, [28])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{R}$ is a continuous monotonic*

nondecreasing function on $[m, M]$*, then we have the inequality*

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{8.11}$$
$$\leq \frac{1}{M-m}\int_m^M \Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle (1_H-E_t)x,x\rangle^{1/2}\langle (1_H-E_t)y,y\rangle^{1/2}\Big]\,df(t)$$
$$\leq \left[f(M)-f(m)-\frac{1}{M-m}\int_m^M sgn\left(t-\frac{m+M}{2}\right)f(t)\,dt\right]\|x\|\,\|y\|$$
$$\leq [f(M)-f(m)]\,\|x\|\,\|y\|$$

for any $x, y \in H$.

Proof. From the theory of Riemann–Stieltjes integral is well known that if $p : [a,b] \to \mathbb{C}$ is of bounded variation and $v : [a,b] \to \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann–Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \leq \int_a^b |p(t)|\,dv(t).$$

Now, on applying this property of the Riemann–Stieltjes integral, we have from the representation (8.2) that

$$\left|\langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds-\langle f(A)x,y\rangle\right| \tag{8.12}$$
$$\leq \frac{1}{M-m}\int_{m-0}^M |\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,df(t).$$

Further on, by utilizing the inequality (8.10) we also have that

$$\int_{m-0}^M |\langle[(M-t)E_t+(t-m)(E_t-1_H)]x,y\rangle|\,df(t) \tag{8.13}$$
$$\leq \int_m^M \Big[(M-t)\langle E_t x,x\rangle^{1/2}\langle E_t y,y\rangle^{1/2}$$
$$+(t-m)\langle (1_H-E_t)x,x\rangle^{1/2}\langle (1_H-E_t)y,y\rangle^{1/2}\Big]\,df(t)$$
$$\leq \left[\frac{1}{2}(M-m)[f(M)-f(m)]+\int_m^M \left|t-\frac{m+M}{2}\right|df(t)\right]\|x\|\,\|y\|.$$

Now, integrating by parts in the Riemann–Stieltjes integral we have

$$\begin{aligned}
&\int_m^M \left|t-\frac{m+M}{2}\right| df(t) \\
&= \int_m^{\frac{M+m}{2}} \left(\frac{m+M}{2}-t\right) df(t) + \int_{\frac{m+M}{2}}^M \left(t-\frac{m+M}{2}\right) df(t) \\
&= \left(\frac{m+M}{2}-t\right) f(t)\Big|_m^{\frac{M+m}{2}} + \int_m^{\frac{M+m}{2}} f(t)\,dt \\
&+ \left(t-\frac{m+M}{2}\right) f(t)\Big|_{\frac{m+M}{2}}^M - \int_{\frac{m+M}{2}}^M f(t)\,dt \\
&= \frac{1}{2}(M-m)\left[f(M)-f(m)\right] - \int_m^M sgn\left(t-\frac{m+M}{2}\right) f(t)\,dt,
\end{aligned}$$

which together with (8.13) produces the second inequality in (8.11).

Since the functions $sgn\left(\cdot-\frac{m+M}{2}\right)$ and $f(\cdot)$ have the same monotonicity, then by the Čebyšev inequality we have

$$\begin{aligned}
&\int_m^M sgn\left(t-\frac{m+M}{2}\right) f(t)\,dt \\
&\geq \frac{1}{M-m}\int_m^M sgn\left(t-\frac{m+M}{2}\right) dt \int_m^M f(t)\,dt = 0
\end{aligned}$$

and the last part of (8.11) is proved. □

9. Bounds for the Difference Between Functions and Integral Means

9.1. **Vector Inequalities Via Ostrowski's Type Bounds.**

The following result holds:

Theorem 9.1 (Dragomir, 2010, [24])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f : [m, M] \to \mathbb{R}$ is a continuous function on $[m, M]$, then we have the inequality*

$$\begin{aligned}
&\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \qquad (9.1)\\
&\leq \max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \bigvee_m^M \left(\langle E_{(\cdot)}x, y\rangle\right) \\
&\leq \max_{t\in[m,M]}\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \|x\|\,\|y\|
\end{aligned}$$

for any $x, y \in H$.

Proof. Utilising the spectral representation theorem we have the following equality of interest

$$\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds \tag{9.2}$$
$$= \int_{m-0}^M \left[f(t) - \frac{1}{M-m}\int_m^M f(s)\,ds\right] d\left(\langle E_t x,y\rangle\right)$$

for any $x,y\in H$.

It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{9.3}$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a,b]$.

Utilising these two facts we get the first part of (9.1).

The last part follows by the Total Variation Schwarz's inequality and we omit the details. □

For particular classes of continuous functions $f:[m,M]\to\mathbb{C}$ we are able to provide simpler bounds as incorporated in the following corollary:

Corollary 9.2 (Dragomir, 2010, [24])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family and $f:[m,M]\to\mathbb{C}$ a continuous function on $[m,M]$.*

1. If f is of bounded variation on $[m,M]$, then

$$\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.4}$$
$$\le \bigvee_m^M(f)\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le \|x\|\,\|y\|\bigvee_m^M(f)$$

for any $x,y\in H$.

2. If $f:[m,M]\longrightarrow\mathbb{C}$ is of $r-H-$Hölder type, i.e., for a given $r\in(0,1]$ and $H>0$ we have

$$|f(s)-f(t)|\le H|s-t|^r \text{ for any } s,t\in[m,M], \tag{9.5}$$

then we have the inequality:

$$\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.6}$$
$$\le \frac{1}{r+1}H(M-m)^r\bigvee_m^M\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \le \frac{1}{r+1}H(M-m)^r\|x\|\,\|y\|$$

for any $x,y\in H$.

In particular, if $f:[m,M]\longrightarrow\mathbb{C}$ is Lipschitzian with the constant $L>0$, then

$$\begin{aligned}&\left|\left\langle f\left(A\right)x,y\right\rangle-\left\langle x,y\right\rangle\frac{1}{M-m}\int_{m}^{M}f\left(s\right)ds\right| \qquad (9.7)\\&\leq\frac{1}{2}L\left(M-m\right)\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\leq\frac{1}{2}L\left(M-m\right)\left\|x\right\|\left\|y\right\|\end{aligned}$$

for any $x,y\in H$.

3. If $f:[m,M]\longrightarrow\mathbb{C}$ is absolutely continuous, then

$$\begin{aligned}&\left|\left\langle f\left(A\right)x,y\right\rangle-\left\langle x,y\right\rangle\frac{1}{M-m}\int_{m}^{M}f\left(s\right)ds\right| \qquad (9.8)\\&\leq\bigvee_{m}^{M}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\&\times\begin{cases}\frac{1}{2}\left(M-m\right)\left\|f'\right\|_{\infty} & \text{if } f'\in L_{\infty}\left[m,M\right]\\ \frac{1}{(q+1)^{1/q}}\left(M-m\right)^{1/q}\left\|f'\right\|_{p} & \begin{array}{c}\text{if } f'\in L_{p}\left[m,M\right]\\ p>1,1/p+1/q=1;\end{array}\\ \left\|f'\right\|_{1}\end{cases}\\&\leq\left\|x\right\|\left\|y\right\|\\&\times\begin{cases}\frac{1}{2}\left(M-m\right)\left\|f'\right\|_{\infty} & \text{if } f'\in L_{\infty}\left[m,M\right]\\ \frac{1}{(q+1)^{1/q}}\left(M-m\right)^{1/q}\left\|f'\right\|_{p} & \begin{array}{c}\text{if } f'\in L_{p}\left[m,M\right]\\ p>1,1/p+1/q=1;\end{array}\\ \left\|f'\right\|_{1}\end{cases}\end{aligned}$$

for any $x,y\in H$, where $\left\|f'\right\|_{p}$ are the Lebesgue norms, i.e., we recall that

$$\left\|f'\right\|_{p}:=\begin{cases}ess\sup_{s\in[m,M]}\left|f'\left(s\right)\right| & \text{if } p=\infty;\\ \left(\int_{m}^{M}\left|f\left(s\right)\right|^{p}ds\right)^{1/p} & \text{if } p\geq1.\end{cases}$$

Proof. We use the Ostrowski type inequalities in order to provide upper bounds for the quantity

$$\max_{t\in[m,M]}\left|f\left(t\right)-\frac{1}{M-m}\int_{m}^{M}f\left(s\right)ds\right|$$

where $f:[m,M]\longrightarrow\mathbb{C}$ is a continuous function.

The following result may be stated (see [25]) for functions of bounded variation:

Lemma 9.3. Assume that $f:[m,M]\to\mathbb{C}$ is of bounded variation and denote by $\bigvee_m^M(f)$ its total variation. Then

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\le\left[\frac{1}{2}+\left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\bigvee_m^M(f) \tag{9.9}$$

for all $t\in[m,M]$. The constant $\frac{1}{2}$ is the best possible.

Now, taking the maximum over $x\in[m,M]$ in (9.9) we deduce (9.4).
If f is Hölder continuous, then one may state the result:

Lemma 9.4. Let $f:[m,M]\to\mathbb{C}$ be of $r-H-$Hölder type, where $r\in(0,1]$ and $H>0$ are fixed, then, for all $x\in[m,M]$, we have the inequality:

$$\begin{aligned}&\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \\ &\le\frac{H}{r+1}\left[\left(\frac{M-t}{M-m}\right)^{r+1}+\left(\frac{t-m}{M-m}\right)^{r+1}\right](M-m)^r.\end{aligned} \tag{9.10}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r=1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [19])

$$\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right|\le\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)L, \tag{9.11}$$

for any $x\in[m,M]$. Here the constant $\frac{1}{4}$ is also best.
Taking the maximum over $x\in[m,M]$ in (9.10) we deduce (9.6) and the second part of the corollary is proved.
The following Ostrowski type result for absolutely continuous functions holds.

Lemma 9.5. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $t\in[a,b]$, we have:

$$\begin{aligned}&\left|f(t)-\frac{1}{M-m}\int_m^M f(s)\,ds\right| \\ &\le\begin{cases}\left[\frac{1}{4}+\left(\frac{t-\frac{m+M}{2}}{M-m}\right)^2\right](M-m)\|f'\|_\infty & \text{if}\quad f'\in L_\infty[m,M];\\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{t-m}{M-m}\right)^{q+1}+\left(\frac{M-t}{M-m}\right)^{q+1}\right]^{\frac{1}{q}}(M-m)^{\frac{1}{q}}\|f'\|_p & \text{if}\quad f'\in L_p[m,M],\\ & \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1;\\ \left[\frac{1}{2}+\left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\|f'\|_1 .\end{cases}\end{aligned} \tag{9.12}$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented above.

The above inequalities can also be obtained from the Fink result in [40] on choosing $n = 1$ and performing some appropriate computations.

Taking the maximum in these inequalities we deduce (9.8). □

For other scalar Ostrowski's type inequalities, see [1] and [20].

9.2. **Other Vector Inequalities.** In [38], the authors have considered the following functional

$$D(f;u) := \int_a^b f(s)\,du(s) - [u(b) - u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\,dt, \tag{9.13}$$

provided that the Stieltjes integral $\int_a^b f(s)\,du(s)$ exists.

This functional plays an important role in approximating the Stieltjes integral $\int_a^b f(s)\,du(s)$ in terms of the Riemann integral $\int_a^b f(t)\,dt$ and the divided difference of the integrator u.

In [38], the following result in estimating the above functional $D(f;u)$ has been obtained:

$$|D(f;u)| \le \frac{1}{2} L(M-m)(b-a), \tag{9.14}$$

provided u is $L-$*Lipschitzian* and f is *Riemann integrable* and with the property that there exists the constants $m, M \in \mathbb{R}$ such that

$$m \le f(t) \le M \quad \text{for any} \quad t \in [a,b]. \tag{9.15}$$

The constant $\frac{1}{2}$ is best possible in (9.14) in the sense that it cannot be replaced by a smaller quantity.

If one assumes that u is of *bounded variation* and f is $K-$*Lipschitzian*, then $D(f,u)$ satisfies the inequality [39]

$$|D(f;u)| \le \frac{1}{2} K(b-a) \bigvee_a^b (u). \tag{9.16}$$

Here the constant $\frac{1}{2}$ is also best possible.

Now, for the function $u : [a,b] \to \mathbb{C}$, consider the following auxiliary mappings Φ, Γ and Δ [21]:

$$\Phi(t) := \frac{(t-a)\,u(b) + (b-t)\,u(a)}{b-a} - u(t), \qquad t \in [a,b],$$
$$\Gamma(t) := (t-a)[u(b) - u(t)] - (b-t)[u(t) - u(a)], \qquad t \in [a,b],$$
$$\Delta(t) := [u;b,t] - [u;t,a], \qquad t \in (a,b),$$

where $[u;\alpha,\beta]$ is the *divided difference* of u in α, β, i.e.,

$$[u;\alpha,\beta] := \frac{u(\alpha) - u(\beta)}{\alpha - \beta}.$$

The following representation of $D(f,u)$ may be stated, see [21] and [22]. Due to its importance in proving our new results we present here a short proof as well.

Lemma 9.6. Let $f, u : [a, b] \to \mathbb{C}$ be such that the Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integral $\int_a^b f(t)\, dt$ exist. Then

$$D(f, u) = \int_a^b \Phi(t)\, df(t) = \frac{1}{b-a} \int_a^b \Gamma(t)\, df(t) \tag{9.17}$$
$$= \frac{1}{b-a} \int_a^b (t-a)(b-t)\Delta(t)\, df(t).$$

Proof. Since $\int_a^b f(t)\, du(t)$ exists, hence $\int_a^b \Phi(t)\, df(t)$ also exists, and the integration by parts formula for Riemann–Stieltjes integrals gives that

$$\int_a^b \Phi(t)\, df(t) = \int_a^b \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t)\right] df(t)$$
$$= \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t)\right] f(t)\Big|_a^b$$
$$- \int_a^b f(t)\, d\left[\frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t)\right]$$
$$= -\int_a^b f(t)\left[\frac{u(b) - u(a)}{b-a} dt - du(t)\right] = D(f, u),$$

proving the required identity. □

For recent inequalities related to $D(f; u)$ for various pairs of functions (f, u), see [23].

The following representation for a continuous function of selfadjoint operator may be stated:

Lemma 9.7 (Dragomir, 2010, [24])**.** Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its *spectral family and* $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$. If $x, y \in H$, then we have the representation

$$\langle f(A)x, y\rangle = \langle x, y\rangle \frac{1}{M-m} \int_m^M f(s)\, ds \tag{9.18}$$
$$+ \frac{1}{M-m} \int_{m-0}^M \langle [(t-m)(1_H - E_t) - (M-t)E_t]\, x, y\rangle\, df(t).$$

Proof. Utilising Lemma 9.6 we have

$$\int_m^M f(t)\, du(t) = [u(M) - u(m)] \cdot \frac{1}{M-m} \int_m^M f(s)\, ds \tag{9.19}$$
$$+ \int_m^M \left[\frac{(t-m)u(M) + (M-t)u(m)}{M-m} - u(t)\right] df(t),$$

for any continuous function $f : [m, M] \to \mathbb{C}$ and any function of bounded variation $u : [m, M] \to \mathbb{C}$.

Now, if we write the equality (9.19) for $u(t) = \langle E_t x, y\rangle$ with $x, y \in H$, then we get

$$\int_{m-0}^{M} f(t)\, d\langle E_t x, y\rangle = \langle x, y\rangle \cdot \frac{1}{M-m}\int_m^M f(s)\, ds + \int_{m-0}^{M}\left[\frac{(t-m)\langle x, y\rangle}{M-m} - \langle E_t x, y\rangle\right] df(t), \tag{9.20}$$

which, by the spectral representation theorem, produces the desired result (9.18). □

The following result may be stated:

Theorem 9.8 (Dragomir, 2010, [24])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$ $\{E_\lambda\}_\lambda$ be its spectral family and $f : [m, M] \to \mathbb{C}$ a continuous function on $[m, M]$.*

1. If f is of bounded variation, then

$$\begin{aligned}&\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\, ds\right| \\ &\le \|y\| \bigvee_m^M (f) \\ &\times \max_{t\in[m,M]}\left[\left(\frac{t-m}{M-m}\right)^2 \|(1_H - E_t)x\|^2 + \left(\frac{M-t}{M-m}\right)^2 \|E_t x\|^2\right]^{1/2} \\ &\le \|x\|\,\|y\| \bigvee_m^M (f)\end{aligned} \tag{9.21}$$

for any $x, y \in H$.

2. If f is Lipschitzian with the constant $L > 0$, then

$$\begin{aligned}&\left|\langle f(A)x, y\rangle - \langle x, y\rangle \frac{1}{M-m}\int_m^M f(s)\, ds\right| \\ &\le \frac{L\|y\|}{M-m}\int_{m-0}^{M}\left[(t-m)^2\|(1_H - E_t)x\|^2 + (M-t)^2\|E_t x\|^2\right]^{1/2} dt \\ &\le \frac{1}{2}\left[1 + \frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right](M-m)L\|y\|\,\|x\|\end{aligned} \tag{9.22}$$

for any $x, y \in H$.

3. If $f:[m,M]\to\mathbb{R}$ is monotonic nondecreasing, then

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.23}$$
$$\leq\frac{\|y\|}{M-m}\int_{m-0}^{M}\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}df(t)$$
$$\leq\|y\|\,\|x\|\int_m^M\left[\left(\frac{t-m}{M-m}\right)^2+\left(\frac{M-t}{M-m}\right)^2\right]^{1/2}df(t)$$
$$\leq\|y\|\,\|x\|\,[f(M)-f(m)]^{1/2}$$
$$\times\left[f(M)-f(m)-\frac{4}{M-m}\int_m^M\left(t-\frac{m+M}{2}\right)f(t)\,dt\right]^{1/2}$$
$$\leq\|y\|\,\|x\|\,[f(M)-f(m)]$$

for any $x,y\in H$.

Proof. If we assume that f is of bounded variation, then on applying the property (9.3) to the representation (9.18) we get

$$\left|\langle f(A)x,y\rangle-\langle x,y\rangle\frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.24}$$
$$\leq\frac{1}{M-m}\max_{t\in[m,M]}\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right|\bigvee_m^M(f).$$

Now, on utilizing the Schwarz inequality and the fact that E_t is a projector for any $t\in[m,M]$, then we have

$$\left|\langle[(t-m)(1_H-E_t)-(M-t)E_t]x,y\rangle\right| \tag{9.25}$$
$$\leq\|[(t-m)(1_H-E_t)-(M-t)E_t]x\|\,\|y\|$$
$$=\left[(t-m)^2\|(1_H-E_t)x\|^2+(M-t)^2\|E_tx\|^2\right]^{1/2}\|y\|$$
$$\leq\left[(t-m)^2+(M-t)^2\right]^{1/2}\|x\|\,\|y\|$$

for any $x,y\in H$ and for any $t\in[m,M]$.

Taking the maximum in (9.25) we deduce the desired inequality (9.21).

It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, i.e.,

$$|f(s)-f(t)|\leq L|s-t|\ \text{ for any } t,s\in[a,b],$$

then the Riemann–Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\leq L\int_a^b|p(t)|\,dt.$$

Now, on applying this property of the Riemann–Stieltjes integral to the representation (9.18), we get

$$\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.26}$$
$$\le \frac{L}{M-m}\int_{m-0}^M \left|\langle [(t-m)(1_H - E_t) - (M-t)E_t]x,y\rangle\right| dt$$
$$\le \frac{L\|y\|}{M-m}\int_{m-0}^M \left[(t-m)^2\|(1_H-E_t)x\|^2 + (M-t)^2\|E_t x\|^2\right]^{1/2} dt$$
$$\le L\|y\|\|x\|\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right]^{1/2} dt,$$

for any $x,y\in H$.

Now, if we change the variable in the integral by choosing $u=\frac{t-m}{M-m}$ then we get

$$\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right]^{1/2} dt$$
$$= (M-m)\int_0^1 \left[u^2 + (1-u)^2\right]^{1/2} du$$
$$= \frac{1}{2}(M-m)\left[1+\frac{\sqrt{2}}{2}\ln\left(\sqrt{2}+1\right)\right],$$

which together with (9.26) produces the desired result (9.22).

From the theory of Riemann–Stieltjes integral is well known that if $p:[a,b]\to\mathbb{C}$ is of bounded variation and $v:[a,b]\to\mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann–Stieltjes integrals $\int_a^b p(t)\,dv(t)$ and $\int_a^b |p(t)|\,dv(t)$ exist and

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dv(t).$$

Now, on applying this property of the Riemann–Stieltjes integral, we have from the representation (9.18)

$$\left|\langle f(A)x,y\rangle - \langle x,y\rangle \frac{1}{M-m}\int_m^M f(s)\,ds\right| \tag{9.27}$$
$$\le \frac{1}{M-m}\int_{m-0}^M \left|\langle [(t-m)(1_H - E_t) - (M-t)E_t]x,y\rangle\right| df(t)$$
$$\le \frac{\|y\|}{M-m}\int_{m-0}^M \left[(t-m)^2\|(1_H-E_t)x\|^2 + (M-t)^2\|E_t x\|^2\right]^{1/2} df(t)$$
$$\le \|y\|\|x\|\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right]^{1/2} df(t),$$

for any $x, y \in H$ and the proof of the first and second inequality in (9.23) is completed.

For the last part we use the following Cauchy-Buniakowski–Schwarz integral inequality for the Riemann–Stieltjes integral with monotonic nondecreasing integrator v

$$\left|\int_a^b p(t)\,q(t)\,dv(t)\right| \le \left[\int_a^b |p(t)|^2\,dv(t)\right]^{1/2}\left[\int_a^b |q(t)|^2\,dv(t)\right]^{1/2}$$

where $p, q : [a, b] \to \mathbb{C}$ are continuous on $[a, b]$.

By applying this inequality we conclude that

$$\begin{aligned}&\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right]^{1/2} df(t) \qquad (9.28)\\ &\le \left[\int_m^M df(t)\right]^{1/2}\left[\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right] df(t)\right]^{1/2}.\end{aligned}$$

Further, integrating by parts in the Riemann–Stieltjes integral we also have that

$$\begin{aligned}&\int_m^M \left[\left(\frac{t-m}{M-m}\right)^2 + \left(\frac{M-t}{M-m}\right)^2\right] df(t) \qquad (9.29)\\ &= f(M) - f(m) - \frac{4}{M-m}\int_m^M \left(t - \frac{m+M}{2}\right) f(t)\,dt\\ &\le f(M) - f(m)\end{aligned}$$

where for the last part we used the fact that by the Čebyšev integral inequality for monotonic functions with the same monotonicity we have that

$$\begin{aligned}&\int_m^M \left(t - \frac{m+M}{2}\right) f(t)\,dt\\ &\ge \frac{1}{M-m}\int_m^M \left(t - \frac{m+M}{2}\right) dt \int_m^M f(t)\,dt = 0.\end{aligned}$$

□

10. Ostrowski's Type Inequalities for n-Time Differentiable Functions

10.1. **Some Identities.** In [7], the authors have pointed out the following integral identity:

Lemma 10.1 (Cerone-Dragomir-Roumeliotis, 1999, [7]). Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the $(n-1)$-derivative $f^{(n-1)}$ (where $n \ge 1$) is absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$, we have the identity:

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x,t)\, f^{(n)}(t)\,dt \tag{10.1}$$

where the kernel $K_n : [a,b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) := \begin{cases} \frac{(t-a)^n}{n!}, & a \le t \le x \le b \\ \frac{(t-b)^n}{n!}, & a \le x < t \le b. \end{cases} \tag{10.2}$$

The identity (10.2) can be written in the following equivalent form as:

$$\begin{aligned} f(z) = {} & \frac{1}{b-a}\int_a^b f(t)\,dt \\ & - \frac{1}{b-a}\sum_{k=1}^{n-1}\frac{1}{(k+1)!}\left[(b-z)^{k+1}+(-1)^k(z-a)^{k+1}\right] f^{(k)}(z) \\ & + \frac{(-1)^{n-1}}{(b-a)\,n!}\left[\int_a^z (t-a)^n f^{(n)}(t)\,dt + \int_z^b (t-b)^n f^{(n)}(t)\,dt\right] \end{aligned} \tag{10.3}$$

for all $z \in [a,b]$.

Note that for $n = 1$, the sum $\sum_{k=1}^{n-1}$ is empty and we obtain the well known *Montgomery's identity* (see for example [4])

$$\begin{aligned} f(z) = {} & \frac{1}{b-a}\int_a^b f(t)\,dt \\ & + \frac{1}{b-a}\left[\int_a^z (t-a) f^{(1)}(t)\,dt + \int_z^b (t-b) f^{(1)}(t)\,dt\right], \end{aligned} \tag{10.4}$$

for any $z \in [a,b]$.

In a slightly more general setting, by the use of the identity (10.3), we can state the following result as well:

Lemma 10.2 (Dragomir, 2010, [10])**.** Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the n-derivative $f^{(n)}$ (where $n \ge 1$) is of bounded variation on $[a,b]$. Then for all

$\lambda \in [a,b]$, we have the identity:

$$f(\lambda) = \frac{1}{b-a}\int_a^b f(t)\,dt \tag{10.5}$$
$$- \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(b-\lambda)^{k+1} + (-1)^k(\lambda-a)^{k+1}\right] f^{(k)}(\lambda)$$
$$+ \frac{(-1)^n}{(b-a)(n+1)!}$$
$$\times \left[\int_a^{\lambda}(t-a)^{n+1}\,d\left(f^{(n)}(t)\right) + \int_{\lambda}^{b}(t-b)^{n+1}\,d\left(f^{(n)}(t)\right)\right].$$

Now we can state the following representation result for functions of selfadjoint operators:

Theorem 10.3 (Dragomir, 2010, [10]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m,M] \subset \mathring{I}$ (the interior of I) and let n be an integer with $n \geq 1$. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m,M]$, then we have the representation*

$$f(A) = \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right) 1_H - \frac{1}{M-m} \tag{10.6}$$
$$\times \sum_{k=1}^{n}\frac{1}{(k+1)!}\left[(M1_H - A)^{k+1} + (-1)^k (A - m1_H)^{k+1}\right] f^{(k)}(A)$$
$$+ T_n(A,m,M)$$

where the remainder is given by

$$T_n(A,m,M) := \frac{(-1)^n}{(M-m)(n+1)!} \tag{10.7}$$
$$\times \left[\int_{m-0}^{M}\left(\int_m^{\lambda}(t-m)^{n+1}\,d\left(f^{(n)}(t)\right)\right) dE_\lambda\right.$$
$$\left. + \int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right) dE_\lambda\right].$$

In particular, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m,M]$, then the remainder can be represented as

$$T_n(A,m,M) \tag{10.8}$$
$$= \frac{(-1)^n}{(M-m)(n+1)!}$$
$$\times \int_{m-0}^{M}\left[(\lambda-m)^{n+1}(1_H - E_\lambda) + (\lambda-M)^{n+1}E_\lambda\right] f^{(n+1)}(\lambda)\,d\lambda.$$

Proof. By Lemma 10.2 we have

$$f(\lambda) = \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{1}{M-m} \times \sum_{k=1}^{n} \frac{1}{(k+1)!}\left[(M-\lambda)^{k+1} + (-1)^k(\lambda-m)^{k+1}\right] f^{(k)}(\lambda) + \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right)\right] \tag{10.9}$$

for any $\lambda \in [m, M]$.

Integrating the identity (10.9) in the Riemann–Stieltjes sense with the integrator E_λ we get

$$\int_m^M f(\lambda)\,dE_\lambda = \frac{1}{M-m}\int_m^M f(t)\,dt \int_m^M dE_\lambda - \frac{1}{M-m} \times \sum_{k=1}^{n}\frac{1}{(k+1)!}\int_m^M \left[(M-\lambda)^{k+1} + (-1)^k(\lambda-m)^{k+1}\right] f^{(k)}(\lambda)\,dE_\lambda + T_n(A,m,M). \tag{10.10}$$

Since, by the spectral representation theorem we have

$$\int_{m-0}^M f(\lambda)\,dE_\lambda = f(A),\ \int_{m-0}^M dE_\lambda = 1_H$$

and

$$\int_{m-0}^M \left[(M-\lambda)^{k+1} + (-1)^k(\lambda-m)^{k+1}\right] f^{(k)}(\lambda)\,dE_\lambda = \left[(M1_H-A)^{k+1} + (-1)^k(A-m1_H)^{k+1}\right] f^{(k)}(A),$$

then by (10.10) we deduce the representation (10.6).

Now, if the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then

$$\int_m^\lambda (t-m)^{n+1}\,d\left(f^{(n)}(t)\right) = \int_m^\lambda (t-m)^{n+1} f^{(n+1)}(t)\,dt$$

and

$$\int_\lambda^M (t-M)^{n+1}\,d\left(f^{(n)}(t)\right) = \int_\lambda^M (t-M)^{n+1} f^{(n+1)}(t)\,dt$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Utilising the integration by parts formula for the Riemann–Stieltjes integral and the differentiation rule for the Stieltjes integral we have successively

$$\begin{aligned}
&\int_{m-0}^{M}\left(\int_{m}^{\lambda}(t-m)^{n+1}f^{(n+1)}(t)\,dt\right)dE_{\lambda}\\
=&\left(\int_{m}^{\lambda}(t-m)^{n+1}f^{(n+1)}(t)\,dt\right)E_{\lambda}\Big|_{m-0}^{M}-\int_{m-0}^{M}(\lambda-m)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda\\
=&\left(\int_{m}^{M}(t-m)^{n+1}f^{(n+1)}(t)\,dt\right)1_{H}-\int_{m-0}^{M}(\lambda-m)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda\\
=&\int_{m-0}^{M}(\lambda-m)^{n+1}f^{(n+1)}(\lambda)\,(1_{H}-E_{\lambda})\,d\lambda
\end{aligned}$$

and

$$\begin{aligned}
&\int_{m-0}^{M}\left(\int_{\lambda}^{M}(t-M)^{n+1}f^{(n+1)}(t)\,dt\right)dE_{\lambda}\\
=&\left(\int_{\lambda}^{M}(t-M)^{n+1}f^{(n+1)}(t)\,dt\right)E_{\lambda}\Big|_{m-0}^{M}+\int_{m-0}^{M}(\lambda-M)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda\\
=&\int_{m-0}^{M}(\lambda-M)^{n+1}f^{(n+1)}(\lambda)\,E_{\lambda}d\lambda
\end{aligned}$$

and the representation (10.8) is thus obtained. □

10.2. Error Bounds for $f^{(n)}$ of Bounded Variation.

From the identity (10.6), we define for any $x, y \in H$

$$\begin{aligned}
&T_{n}(A,m,M;x,y) \qquad (10.11)\\
&:=\langle f(A)\,x,y\rangle+\frac{1}{M-m}\sum_{k=1}^{n}\frac{1}{(k+1)!}\\
&\times\left[\left\langle (M1_{H}-A)^{k+1}f^{(k)}(A)\,x,y\right\rangle+(-1)^{k}\left\langle (A-m1_{H})^{k+1}f^{(k)}(A)\,x,y\right\rangle\right]\\
&-\left(\frac{1}{M-m}\int_{m}^{M}f(t)\,dt\right)\langle x,y\rangle .
\end{aligned}$$

We have the following result concerning bounds for the absolute value of T_n when the n-th derivative $f^{(n)}$ is of bounded variation:

Theorem 10.4 (Dragomir, 2010, [10])**.** *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$.*

1. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
&|T_n(A, m, M; x, y)| \qquad (10.12)\\
&\le \frac{1}{(M-m)(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \\
&\times \max_{\lambda\in[m,M]} \left[(\lambda - m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right) \right] \\
&\le \frac{(M-m)^n}{(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \bigvee_m^M \left(f^{(n)}\right) \le \frac{(M-m)^n}{(n+1)!} \bigvee_m^M \left(f^{(n)}\right) \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$.

2. If $f : I \to \mathbb{C}$ is such that the n-th derivative $f^{(n)}$ is Lipschitzian with the constant $L_n > 0$ on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
|T_n(A, m, M; x, y)| &\le \frac{L_n (M-m)^{n+1}}{(n+2)!} \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \qquad (10.13)\\
&\le \frac{L_n (M-m)^{n+1}}{(n+2)!} \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$.

3. If $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is monotonic nondecreasing on the interval $[m, M]$, then we have the inequalities

$$
\begin{aligned}
&|T_n(A, m, M; x, y)| \qquad (10.14)\\
&\le \frac{1}{(M-m)(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \\
&\times \max_{\lambda\in[m,M]} \left[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M-\lambda)^{n+1} \right) \right. \\
&\left. + (n+1) \left[\int_\lambda^M (M-t)^n f^{(n)}(t)\, dt - \int_m^\lambda (t-m)^n f^{(n)}(t)\, dt \right] \right] \\
&\le \frac{1}{(M-m)(n+1)!} \max_{\lambda\in[m,M]} \left[(\lambda-m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] \right. \\
&\left. + (M-\lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right] \right] \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \\
&\le \frac{(M-m)^n}{(n+1)!} \bigvee_m^M (\langle E_{(\cdot)}x, y\rangle) \left[f^{(n)}(M) - f^{(n)}(m) \right] \\
&\le \frac{(M-m)^n}{(n+1)!} \left[f^{(n)}(M) - f^{(n)}(m) \right] \|x\| \|y\|
\end{aligned}
$$

for any $x, y \in H$.

Proof. 1. By the identity (10.7) we have for any $x, y \in H$ that

$$T_n(A, m, M; x, y) := \frac{(-1)^n}{(M-m)(n+1)!} \times \left[\int_{m-0}^{M} \left(\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right)\right) d\langle E_\lambda x, y\rangle + \int_{m-0}^{M} \left(\int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right) d\langle E_\lambda x, y\rangle\right]. \tag{10.15}$$

It is well known that if $p : [a, b] \to \mathbb{C}$ is a continuous function, $v : [a, b] \to \mathbb{C}$ is of bounded variation then the Riemann–Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \le \max_{t\in[a,b]} |p(t)| \bigvee_a^b (v), \tag{10.16}$$

where $\bigvee_a^b (v)$ denotes the total variation of v on $[a, b]$.

Taking the modulus in (10.15) and utilizing the property (10.16), we have successively that

$$|T_n(A, m, M; x, y)| = \frac{1}{(M-m)(n+1)!} \times \left|\int_{m-0}^{M} \left[\left(\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \left(\int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right)\right)\right] \times d\langle E_\lambda x, y\rangle\right| \le \frac{1}{(M-m)(n+1)!} \bigvee_m^M \left(\langle E_{(\cdot)} x, y\rangle\right) \times \max_{\lambda\in[m,M]} \left|\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \tag{10.17}$$

for any $x, y \in H$.

By the same property (10.16) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^{\lambda} (t-m)^{n+1} d\left(f^{(n)}(t)\right)\right| \le \max_{t\in[m,\lambda]} (t-m)^{n+1} \bigvee_m^{\lambda} \left(f^{(n)}\right) = (\lambda - m)^{n+1} \bigvee_m^{\lambda} \left(f^{(n)}\right)$$

and

$$\left|\int_{\lambda}^{M} (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \le \max_{t\in[\lambda,M]} (M-t)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right) = (M-\lambda)^{n+1} \bigvee_{\lambda}^{M} \left(f^{(n)}\right)$$

which produce the inequality

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) + \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right)\right| \tag{10.18}$$
$$\leq (\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right).$$

Taking the maximum over $\lambda \in [m, M]$ in (10.18) and utilizing (10.17) we deduce the first inequality in (10.12).

Now observe that

$$(\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right)$$
$$\leq \max\left\{(\lambda-m)^{n+1}, (M-\lambda)^{n+1}\right\} \left[\bigvee_m^\lambda \left(f^{(n)}\right) + \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$
$$= \max\left\{(\lambda-m)^{n+1}, (M-\lambda)^{n+1}\right\} \bigvee_m^M \left(f^{(n)}\right)$$
$$= \left[\frac{1}{2}(M-m) + \left|\lambda - \frac{m+M}{2}\right|\right]^{n+1} \bigvee_m^M \left(f^{(n)}\right)$$

giving that

$$\max_{\lambda\in[m,M]} \left[(\lambda-m)^{n+1} \bigvee_m^\lambda \left(f^{(n)}\right) + (M-\lambda)^{n+1} \bigvee_\lambda^M \left(f^{(n)}\right)\right]$$
$$\leq (M-m)^{n+1} \bigvee_m^M \left(f^{(n)}\right)$$

and the second inequality in (10.12) is proved.

The last part of (10.12) follows by the Total Variation Schwarz's inequality and we omit the details.

2. Now, recall that if $p : [a, b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a, b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, i.e.,

$$|f(s) - f(t)| \leq L|s-t| \text{ for any } t, s \in [a, b],$$

then the Riemann–Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\, dv(t)\right| \leq L \int_a^b |p(t)|\, dt. \tag{10.19}$$

By the property (10.19) we have for $\lambda \in (m, M)$ that

$$\left|\int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right)\right| \leq L_n \int_m^\lambda (t-m)^{n+1} d(t) = \frac{L_n}{n+2}(\lambda-m)^{n+2}$$

and

$$\left|\int_{\lambda}^{M}(t-M)^{n+1}d\left(f^{(n)}(t)\right)\right|\leq L_{n}\int_{\lambda}^{M}(M-t)^{n+1}dt=\frac{L_{n}}{n+2}(M-\lambda)^{n+2}.$$

By the inequality (10.17) we then have

$$\begin{aligned}&|T_{n}(A,m,M;x,y)| \qquad (10.20)\\&\leq\frac{1}{(M-m)(n+1)!}\bigvee_{m}^{M}(\langle E_{(\cdot)}x,y\rangle)\\&\times\max_{\lambda\in[m,M]}\left[\frac{L_{n}}{n+2}(\lambda-m)^{n+2}+\frac{L_{n}}{n+2}(M-\lambda)^{n+2}\right]\\&=\frac{L_{n}(M-m)^{n+1}}{(n+2)!}\bigvee_{m}^{M}(\langle E_{(\cdot)}x,y\rangle)\leq\frac{L_{n}(M-m)^{n+1}}{(n+2)!}\|x\|\|y\|\end{aligned}$$

for any $x,y\in H$ and the inequality (10.13) is proved.

3. Further, from the theory of Riemann–Stieltjes integral it is also well known that if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, then the Riemann–Stieltjes integrals $\int_{a}^{b}p(t)\,dv(t)$ and $\int_{a}^{b}|p(t)|\,dv(t)$ exist and

$$\left|\int_{a}^{b}p(t)\,dv(t)\right|\leq\int_{a}^{b}|p(t)|\,dv(t)\leq\max_{t\in[a,b]}|p(t)|\,[v(b)-v(a)]. \qquad (10.21)$$

On making use of (10.21) we have

$$\begin{aligned}\left|\int_{m}^{\lambda}(t-m)^{n+1}d\left(f^{(n)}(t)\right)\right|&\leq\int_{m}^{\lambda}(t-m)^{n+1}d\left(f^{(n)}(t)\right) \qquad (10.22)\\&\leq(\lambda-m)^{n+1}\left[f^{(n)}(\lambda)-f^{(n)}(m)\right]\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\lambda}^{M}(t-M)^{n+1}d\left(f^{(n)}(t)\right)\right|&\leq\int_{\lambda}^{M}(M-t)^{n+1}d\left(f^{(n)}(t)\right) \qquad (10.23)\\&\leq(M-\lambda)^{n+1}\left[f^{(n)}(M)-f^{(n)}(\lambda)\right]\end{aligned}$$

for any $\lambda\in(m,M)$.

Integrating by parts in the Riemann–Stieltjes integral, we also have

$$\begin{aligned}&\int_{m}^{\lambda}(t-m)^{n+1}d\left(f^{(n)}(t)\right)\\&=(\lambda-m)^{n+1}f^{(n)}(\lambda)-(n+1)\int_{m}^{\lambda}(t-m)^{n}f^{(n)}(t)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_{\lambda}^{M}(M-t)^{n+1}d\left(f^{(n)}(t)\right)\\&=(n+1)\int_{\lambda}^{M}(M-t)^{n}f^{(n)}(t)\,dt-(M-\lambda)^{n+1}f^{(n)}(\lambda)\end{aligned}$$

for any $\lambda \in (m, M)$.

Therefore, by adding (10.22) with (10.23) we get

$$\left| \int_m^\lambda (t-m)^{n+1} d\left(f^{(n)}(t)\right) \right| + \left| \int_\lambda^M (t-M)^{n+1} d\left(f^{(n)}(t)\right) \right|$$
$$\leq \left[f^{(n)}(\lambda) \left((\lambda - m)^{n+1} - (M-\lambda)^{n+1} \right) \right]$$
$$+ (n+1) \left[\int_\lambda^M (M-t)^n f^{(n)}(t)\, dt - \int_m^\lambda (t-m)^n f^{(n)}(t)\, dt \right]$$
$$\leq (\lambda - m)^{n+1} \left[f^{(n)}(\lambda) - f^{(n)}(m) \right] + (M-\lambda)^{n+1} \left[f^{(n)}(M) - f^{(n)}(\lambda) \right]$$

for any $\lambda \in (m, M)$.

Now, on making use of the inequality (10.17) we deduce (10.14). □

10.3. Error Bounds for $f^{(n)}$ Absolutely Continuous.

We consider the Lebesgue norms defined by

$$\|g\|_{[a,b],\infty} := ess \sup_{t \in [a,b]} |g(t)| \text{ if } g \in L_\infty [a,b]$$

and

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p dt \right)^{1/p} \text{ if } g \in L_p [a,b], p \geq 1.$$

Theorem 10.5 (Dragomir, 2010, [10]). *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \mathring{I}$ and let n be an integer with $n \geq 1$. If the n-th derivative $f^{(n)}$ is absolutely continuous on $[m, M]$, then*

$$|T_n(A, m, M; x, y)| \leq \frac{1}{(M-m)(n+1)!}$$
$$\times \int_{m-0}^M \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right| \left| f^{(n+1)}(\lambda) \right| d\lambda.$$
$$\leq \frac{1}{(M-m)(n+1)!}$$
$$\times \begin{cases} B_{n,1}(A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],\infty} & \text{if } f^{(n)} \in L_\infty [m, M], \\ B_{n,p}(A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],q} & \text{if } f^{(n)} \in L_q [m, M], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ B_{n,\infty}(A, m, M; x, y) \left\| f^{(n)} \right\|_{[m,M],1}, \end{cases} \tag{10.24}$$

for any $x, y \in H$, where

$$B_{n,p}(A, m, M; x, y)$$
$$:= \left(\int_{m-0}^M \left| (\lambda - m)^{n+1} \langle (1_H - E_\lambda) x, y \rangle + (\lambda - M)^{n+1} \langle E_\lambda x, y \rangle \right|^p d\lambda \right)^{1/p}, p \geq 1$$

and

$$B_{n,\infty}(A,m,M;x,y) := \sup_{t\in[m,M]} \left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle + (\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right|.$$

Proof. Follows from the representation

$$T_n(A,m,M;x,y) = \frac{(-1)^n}{(M-m)(n+1)!} \times \int_{m-0}^{M}\left[(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle + (\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right] f^{(n+1)}(\lambda)\,d\lambda$$

for any $x,y\in H$, by taking the modulus and utilizing the Hölder integral inequality.

The details are omitted. □

The bounds provided by $B_{n,p}(A,m,M;x,y)$ are not useful for applications, therefore we will establish in the following some simpler, however coarser bounds.

Proposition 10.6 (Dragomir, 2010, [10])**.** With the above notations, we have

$$B_{n,\infty}(A,m,M;x,y) \le (M-m)^{n+1}\|x\|\|y\|, \tag{10.25}$$

$$B_{n,1}(A,m,M;x,y) \le \frac{(2^{n+2}-1)}{(n+2)2^{n+1}}(M-m)^{n+2}\|x\|\|y\| \tag{10.26}$$

and for $p>1$

$$B_{n,p}(A,m,M;x,y) \le \frac{\left(2^{(n+1)p+1}-1\right)^{1/p}}{2^{n+1}\left[(n+1)p+1\right]^{1/p}}(M-m)^{n+1+1/p}\|x\|\|y\| \tag{10.27}$$

for any $x,y\in H$.

Proof. Utilising the triangle inequality for the modulus we have

$$\begin{aligned} &\left|(\lambda-m)^{n+1}\langle(1_H-E_\lambda)x,y\rangle + (\lambda-M)^{n+1}\langle E_\lambda x,y\rangle\right| \\ &\le (\lambda-m)^{n+1}|\langle(1_H-E_\lambda)x,y\rangle| + (M-\lambda)^{n+1}|\langle E_\lambda x,y\rangle| \\ &\le \max\left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}\left[|\langle(1_H-E_\lambda)x,y\rangle| + |\langle E_\lambda x,y\rangle|\right] \end{aligned} \tag{10.28}$$

for any $x,y\in H$.

Utilising the generalization of Schwarz's inequality for nonnegative selfadjoint operators we have

$$|\langle(1_H-E_\lambda)x,y\rangle| \le \langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}$$

and

$$|\langle E_\lambda x,y\rangle| \le \langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$

for any $x,y\in H$ and $\lambda\in[m,M]$.

Further, by making use of the elementary inequality

$$ac+bd \le \left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2}, a,b,c,d\ge 0$$

we have

$$\begin{aligned} &\left|\left\langle\left(1_{H}-E_{\lambda}\right) x, y\right\rangle\right|+\left|\left\langle E_{\lambda} x, y\right\rangle\right| \quad (10.29)\\ &\leq\left\langle\left(1_{H}-E_{\lambda}\right) x, x\right\rangle^{1 / 2}\left\langle\left(1_{H}-E_{\lambda}\right) y, y\right\rangle^{1 / 2}+\left\langle E_{\lambda} x, x\right\rangle^{1 / 2}\left\langle E_{\lambda} y, y\right\rangle^{1 / 2} \\ &\leq\left(\left\langle\left(1_{H}-E_{\lambda}\right) x, x\right\rangle+\left\langle E_{\lambda} x, x\right\rangle\right)^{1 / 2}\left(\left\langle\left(1_{H}-E_{\lambda}\right) y, y\right\rangle+\left\langle E_{\lambda} y, y\right\rangle\right)^{1 / 2} \\ &=\|x\|\|y\| \end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Combining (10.28) with (10.29) we deduce that

$$\begin{aligned} &\left|(\lambda-m)^{n+1}\left\langle\left(1_{H}-E_{\lambda}\right) x, y\right\rangle+(\lambda-M)^{n+1}\left\langle E_{\lambda} x, y\right\rangle\right| \quad (10.30)\\ &\leq \max \left\{(\lambda-m)^{n+1},(M-\lambda)^{n+1}\right\}\|x\|\|y\| \end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

Taking the supremum over $\lambda \in [m, M]$ in (10.30) we deduce the inequality (10.25).

Now, if we take the power $r \geq 1$ in (10.30) and integrate, then we get

$$\begin{aligned} &\int_{m-0}^{M}\left|(\lambda-m)^{n+1}\left\langle\left(1_{H}-E_{\lambda}\right) x, y\right\rangle+(\lambda-M)^{n+1}\left\langle E_{\lambda} x, y\right\rangle\right|^{r} d \lambda \quad (10.31)\\ &\leq\|x\|^{r}\|y\|^{r} \int_{m}^{M} \max \left\{(\lambda-m)^{(n+1) r},(M-\lambda)^{(n+1) r}\right\} d \lambda \\ &=\|x\|^{r}\|y\|^{r}\left[\int_{m}^{\frac{M+m}{2}}(M-\lambda)^{(n+1) r} d \lambda+\int_{\frac{M+m}{2}}^{M}(\lambda-m)^{(n+1) r} d \lambda\right] \\ &=\frac{\left(2^{(n+1) r+1}-1\right)}{[(n+1) r+1] 2^{(n+1) r}}(M-m)^{(n+1) r+1}\|x\|^{r}\|y\|^{r} \end{aligned}$$

for any $x, y \in H$.

Utilizing (10.31) for $r = 1$ we deduce the bound (10.26). Also, by making $r = p$ and then taking the power $1/p$, we deduce the last inequality (10.27). □

The following result provides refinements of the inequalities in Proposition 10.6:

Proposition 10.7 (Dragomir, 2010, [10]). With the above notations, we have

$$\begin{aligned} &B_{n, \infty}(A, m, M ; x, y) \\ &\quad \leq\|y\| \max _{\lambda \in[m, M]}\left[(\lambda-m)^{2(n+1)}\left\langle\left(1_{H}-E_{\lambda}\right) x, x\right\rangle+(M-\lambda)^{2(n+1)}\left\langle E_{\lambda} x, x\right\rangle\right]^{1 / 2} \\ &\qquad \leq(M-m)^{n+1}\|x\|\|y\|, \quad (10.32) \end{aligned}$$

$$\begin{aligned} &B_{n, 1}(A, m, M ; x, y) \\ &\quad \leq\|y\| \int_{m-0}^{M}\left[(\lambda-m)^{2(n+1)}\left\langle\left(1_{H}-E_{\lambda}\right) x, x\right\rangle+(M-\lambda)^{2(n+1)}\left\langle E_{\lambda} x, x\right\rangle\right]^{1 / 2} d \lambda \\ &\qquad \leq \frac{\left(2^{n+2}-1\right)}{(n+2) 2^{n+1}}(M-m)^{n+2}\|x\|\|y\| \quad (10.33) \end{aligned}$$

and for $p > 1$

$$\begin{aligned}
&B_{n,p}(A, m, M; x, y) \\
&\leq \|y\| \left(\int_{m-0}^{M} \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{p/2} d\lambda \right)^{1/p} \\
&\qquad \leq \frac{\left(2^{(n+1)p+1} - 1\right)^{1/p}}{2^{n+1} [(n+1) p + 1]^{1/p}} (M - m)^{n+1+1/p} \|x\| \|y\| \quad (10.34)
\end{aligned}$$

for any $x, y \in H$.

Proof. Utilising the Schwarz inequality in H, we have

$$\begin{aligned}
&\left| \left\langle (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x, y \right\rangle \right| \quad (10.35) \\
&\leq \|y\| \left\| (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x \right\|
\end{aligned}$$

for any $x, y \in H$.

Since E_λ are projectors for each $\lambda \in [m, M]$, then we have

$$\begin{aligned}
&\left\| (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x \right\|^2 \quad (10.36) \\
&= (\lambda - m)^{2(n+1)} \|(1_H - E_\lambda) x\|^2 \\
&+ 2 (\lambda - m)^{n+1} (\lambda - M)^{n+1} \operatorname{Re} \langle (1_H - E_\lambda) x, E_\lambda x \rangle \\
&+ (M - \lambda)^{2(n+1)} \|E_\lambda x\|^2 \\
&= (\lambda - m)^{2(n+1)} \|(1_H - E_\lambda) x\|^2 + (M - \lambda)^{2(n+1)} \|E_\lambda x\|^2 \\
&= (\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \\
&\leq \|x\|^2 \max \left\{ (\lambda - m)^{2(n+1)}, (M - \lambda)^{2(n+1)} \right\}
\end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

On making use of (10.35) and (10.36) we obtain the following refinement of (10.30)

$$\begin{aligned}
&\left| \left\langle (\lambda - m)^{n+1} (1_H - E_\lambda) x + (\lambda - M)^{n+1} E_\lambda x, y \right\rangle \right| \quad (10.37) \\
&\leq \|y\| \left[(\lambda - m)^{2(n+1)} \langle (1_H - E_\lambda) x, x \rangle + (M - \lambda)^{2(n+1)} \langle E_\lambda x, x \rangle \right]^{1/2} \\
&\leq \max \left\{ (\lambda - m)^{n+1}, (M - \lambda)^{n+1} \right\} \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$ and $\lambda \in [m, M]$.

The proof now follows the lines of the proof from Proposition 10.6 and we omit the details. □

Remark 10.8. One can apply Theorem 10.5 and Proposition 10.6 for particular functions including the exponential and logarithmic function. However the details are left to the interested reader.

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