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ALUTHGE TRANSFORMS OF $(\mathcal{C}_p, \alpha)$-HYPONORMAL OPERATORS

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ABSTRACT. Recently, the class of $(\mathcal{C}_p, \alpha)$-hyponromal operators is introduced and the Aluthge transforms of such operators is discussed by some researchers. This paper is to give a further development of the Aluthge transforms of $(\mathcal{C}_p, \alpha)$-hyponromal operators by using Loewner-Heinz inequality, Furuta inequality and Lauric's lemma. Especially, it is shown that, if $p \geq 1$, $\alpha \geq 1/2$ and T is $(\mathcal{C}_p, \alpha)$-hyponromal, then the Aluthge transform $T(1/2, 1/2)$ is $(\mathcal{C}_{4p\alpha/\beta}, \beta)-$ *hyponromal* where $0 < \beta \leq 1$ and $T(1/2, 1/2) = |T|^{1/2}U|T|^{1/2}$.

1. Introduction

Throughout this paper, an operator T means a bounded linear operator on a separate, infinite dimensional, complex Hilbert space $\mathcal{H}$. For $\alpha > 0$, $(T^*T)^\alpha - (TT^*)^\alpha$ is called the α-self-commutator of T and denote it by D_T^α. Let $\mathcal{K}$ be the ideal of all compact operators and $\mathcal{C}_p(\mathcal{H})$, $1 \leq p < \infty$, the ideal of operators in the Schatten p-class. For $0 < p < 1$, the usual definition of $\|\cdot\|_p$ does not satisfy the triangle inequality, nevertheless $(\mathcal{C}_p, \|\cdot\|_p)$ is closed and $\|TK\|_p \leq \|T\|\|K\|_p$ where T is an operator and $\mathcal{K} \in \mathcal{C}_p(\mathcal{H})$.

An operator T is called $(\mathcal{C}_p, \alpha)$-normal if $D_T^\alpha \in \mathcal{C}_p(\mathcal{H})$, and denote the class of $(\mathcal{C}_p, \alpha)$-normal operators by $\mathcal{N}_p^\alpha(\mathcal{H})$. Similarly, T is called $(\mathcal{C}_p, \alpha)$-hyponormal if $D_T^\alpha = P + K$, where P is a positive semidefinite operator (denote by $P \geq 0$) and $K \in \mathcal{C}_p(\mathcal{H})$. The class of $(\mathcal{C}_p, \alpha)$-hyponormal operators is denoted by $\mathcal{H}_p^\alpha(\mathcal{H})$. Especially, $T \in \mathcal{H}_1^1(\mathcal{H})$ is also called almost hyponormal. A α-hyponormal operator T can be regarded as a $(\mathcal{C}_0, \alpha)$-hyponormal operator, that is, $T \in \mathcal{H}_0^\alpha(\mathcal{H})$.

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It is known that, by Loewner-Heinz inequality (L-H), $\mathcal{H}_0^\beta(\mathcal{H}) \subseteq \mathcal{H}_0^\alpha(\mathcal{H})$ where $0 < \alpha \leq \beta$ (see [3, page 127]). However, the inclusion relations among $\mathcal{N}_p^\alpha(\mathcal{H})$ or $\mathcal{H}_p^\alpha(\mathcal{H})$ are less obvious. See [5] and [6].

Lauric [5, Theorem 13] gave a result on the case $s = t = 1/2$ of Aluthge transform $T(s,t)$ of $(\mathcal{C}_p, \alpha)$-hyponormal operators where $s > 0$, $t > 0$ and $T(s,t) = |T|^s U |T|^t$. Wang and Gao [6] showed a generalization of Lauric's result.

Theorem 1.1 ([6])**.** *Let $p \geq 1$ and $\alpha \geq \max\{s,t\}$. If T is $(\mathcal{C}_p, \alpha)$-hyponormal and $\alpha \leq \beta \leq 1$, then $T(s,t)$ is $(\mathcal{C}_{\frac{2p\alpha}{s\beta}}, \beta)$-hyponormal.*

Recall that $\mathcal{H}_0^\alpha(\mathcal{H})$ is regarded as the class of α-hyponormal operators. The case $p = 0$ of Theorem 1.1 follows by the result below easily.

Theorem 1.2 ([1, 4, 8])**.** *If T is a α-hyponormal operator and $\gamma = \min\{\alpha + s, \alpha + t, s + t\}$, then $T(s,t)$ is $\frac{\gamma}{s+t}$-hyponormal.*

Moreover, the outer exponent γ in the Theorem above is optimal [7]. In [9], it is proved that the complete form [10, Theorem 1.3] and original form of Furuta inequality [3, page 129] are equivalent to the order relations among Aluthge transforms of α-hyponormal operators.

Obviously, by (L-H) for $\alpha \leq \beta \leq 1$, Theorem 1.2 implies the case $p = 0$ of Theorem 1.1.

Inspired by Theorem 1.1-1.2, this paper is to provide a sharpening of Theorem 1.1 via (L-H), the original form of Furuta inequality and Lauric's lemma below.

Theorem 1.3 (Furuta inequality (F), [3])**.** *Let $r \geq 0$, $p > 0$, then $A \geq B \geq 0$ ensure*

$$\left(B^{r/2}A^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geq \left(B^{r/2}B^pB^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}},$$
$$\left(A^{r/2}A^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}} \geq \left(A^{r/2}B^pA^{r/2}\right)^{\frac{\min\{1,p\}+r}{p+r}}.$$

Tanahashi proved that the outer exponent $\min\{1,p\} + r$ above is optimal, see [2, 3] for related topics.

Lemma 1.4 ([5])**.** *Let $\alpha > 0$, $p \geq 1$, $A \geq 0$ and $B \geq 0$ such that $A - B \in \mathcal{C}_p(\mathcal{H})$. Then $A^\alpha - B^\alpha \in \mathcal{C}_{p\max\{1,1/\alpha\}}(\mathcal{H})$.*

It should be pointed out that, if $0 < \alpha < 1$, the condition $p \geq 1$ in Lemma 1.4 can be released to $p \geq \alpha$ [5, Lemma 10].

2. Results and Proofs

Denote $p(s,t) := \frac{\max\{2\alpha,s\}p(s+t)}{\min\{\alpha+s,\alpha+t,s+t\}s\beta}$ and $\alpha(s,t) := \frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}$.

Theorem 2.1 (Main result)**.** *Let $s > 0$, $t > 0$, $p \geq 1$ and $\alpha > 0$. If T is $(\mathcal{C}_p, \alpha)$-hyponormal and $0 < \beta \leq 1$, then $T(s,t)$ is $(\mathcal{C}_{p(s,t)}, \alpha(s,t))$-hyponormal.*

It is clear that Theorem 1.2 can be regarded as the case $p = 0$ and $\beta = 1$ of Theorem 2.1 which relates to Lauric's question closely [5, Question].

Proof. By assumption, let $D_T^\alpha = P + K$ where $P \geq 0$ and $K \in \mathcal{C}_p(\mathcal{H})$. Since $K = K^*$, K can be represented as $K = K_+ - K_-$ where K_+, K_- are positive part and negative part of K respectively, and K_+, K_- are in $\mathcal{C}_p(\mathcal{H})$. So assume that $D_T^\alpha = P - K$ where $P \geq 0$, $K \geq 0$ and $K \in \mathcal{C}_p(\mathcal{H})$ without loss of generality.

Hence $|T|^{2\alpha} + K = |T^*|^{2\alpha} + P \geq |T^*|^{2\alpha}$ where $T = U|T|$ is the polar decomposition of T, denote $A := |T|^{2\alpha} + K$ and $B := |T^*|^{2\alpha}$. By (F) and (L-H) for $0 < \beta \leq 1$,

$$\left(B^{\frac{t}{2\alpha}} A^{\frac{s}{\alpha}} B^{\frac{t}{2\alpha}}\right)^{\frac{\min\{\alpha,s\}+t}{s+t}\beta} \geq B^{\frac{\min\{\alpha,s\}+t}{\alpha}\beta}. \tag{2.1}$$

By Lemma 1.4, $A^{\frac{s}{\alpha}} = |T|^{2s} + K_1$ where $K_1 \in \mathcal{C}_{p_1}(\mathcal{H})$ and $p_1 = p\max\{1, \alpha/s\}$. Furthermore,

$$\begin{aligned}
&\left(B^{\frac{t}{2\alpha}} A^{\frac{s}{\alpha}} B^{\frac{t}{2\alpha}}\right)^{\frac{\min\{\alpha,s\}+t}{s+t}\beta} \\
=&\left(|T^*|^t|T|^{2s}|T^*|^t + K_2\right)^{\frac{\min\{\alpha,s\}+t}{s+t}\beta} \\
=&\left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{\min\{\alpha,s\}+t}{s+t}\beta} + K_3
\end{aligned}$$

where $K_i \in \mathcal{C}_{p_i}(\mathcal{H})$ for $i \in \{2,3\}$, $p_2 = p_1$ and

$$p_3 = p_2\frac{s+t}{\min\{\alpha+t, s+t\}\beta} = p\frac{(s+t)\max\{\alpha, s\}}{\min\{\alpha+t, s+t\}s\beta}.$$

Let $K_4 = U^*K_3U \in \mathcal{C}_{p_3}(\mathcal{H})$, by (2.1),

$$\begin{aligned}
&|T(s,t)|^{2\frac{\min\{\alpha+t,s+t\}\beta}{s+t}} + K_4 \\
=&\left(U^*|T^*|^t|T|^{2s}|T^*|^tU\right)^{\frac{\min\{\alpha+t,s+t\}\beta}{s+t}} + K_4 \\
=&U^*\left(B^{\frac{t}{2\alpha}} A^{\frac{s}{\alpha}} B^{\frac{t}{2\alpha}}\right)^{\frac{\min\{\alpha,s\}+t}{s+t}\beta}U \\
\geq&U^*B^{\frac{\min\{\alpha,s\}+t}{\alpha}\beta}U = |T|^{2(\min\{\alpha,s\}+t)\beta},
\end{aligned} \tag{2.2}$$

So that the following follows by Lemma 1.4,

$$\begin{aligned}
&\left(|T(s,t)|^{2\frac{\min\{\alpha+t,s+t\}\beta}{s+t}} + K_4\right)^{\frac{\min\{\alpha+s,\alpha+t,s+t\}}{\min\{\alpha+t,s+t\}}} \\
=&|T(s,t)|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} + K_5
\end{aligned} \tag{2.3}$$

where $K_5 \in \mathcal{C}_{p_5}(\mathcal{H})$ and $p_5 = p\frac{(s+t)\max\{\alpha,s\}}{\min\{\alpha+s,\alpha+t,s+t\}s\beta}$. (2.2) and (2.3) deduce that

$$|T(s,t)|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} + K_5 \geq |T|^{2\min\{\alpha+s,\alpha+t,s+t\}\beta}. \tag{2.4}$$

On the other hand,

$$A^{\frac{\min\{\alpha,t\}+s}{\alpha}\beta} \geq \left(A^{\frac{s}{2\alpha}} B^{\frac{t}{\alpha}} A^{\frac{s}{2\alpha}}\right)^{\frac{\min\{\alpha,t\}+s}{s+t}\beta}. \tag{2.5}$$

By Lemma 1.4, $A^{\frac{s}{2\alpha}} = |T|^s + K_6$ where $K_6 \in \mathcal{C}_{p_6}(\mathcal{H})$ and $p_6 = p\max\{1, 2\alpha/s\}$. Thus,

$$\begin{aligned}
&\big(A^{\frac{s}{2\alpha}}B^{\frac{t}{\alpha}}A^{\frac{s}{2\alpha}}\big)^{\frac{\min\{\alpha,t\}+s}{s+t}\beta} \\
=&\big(|T|^s|T^*|^{2t}|T|^s + K_7\big)^{\frac{\min\{\alpha,t\}+s}{s+t}\beta} \\
=&\big(|T|^s|T^*|^{2t}|T|^s\big)^{\frac{\min\{\alpha,t\}+s}{s+t}\beta} + K_8 \\
=&\big|\big(T(s,t)\big)^*\big|^{2\frac{\min\{\alpha,t\}+s}{s+t}\beta} + K_8
\end{aligned} \tag{2.6}$$

where $K_i \in \mathcal{C}_{p_i}(\mathcal{H})$ for $i \in \{7, 8\}$, $p_7 = p_6$ and

$$p_8 = p_7\frac{s+t}{\min\{\alpha+s, s+t\}\beta} = p\frac{(s+t)\max\{s, 2\alpha\}}{\min\{\alpha+s, s+t\}s\beta}.$$

Again by Lemma 1.4,

$$\begin{aligned}
&\Big(\big|\big(T(s,t)\big)^*\big|^{2\frac{\min\{\alpha,t\}+s}{s+t}\beta} + K_8\Big)^{\frac{\min\{\alpha+s,\alpha+t,s+t\}}{\min\{\alpha+s,s+t\}}} \\
=&\big|\big(T(s,t)\big)^*\big|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} + K_9
\end{aligned} \tag{2.7}$$

where $K_9 \in \mathcal{C}_{p_9}(\mathcal{H})$ and $p_9 = p\frac{(s+t)\max\{2\alpha,s\}}{\min\{\alpha+s,\alpha+t,s+t\}s\beta}$, and

$$A^{\frac{\min\{\alpha+s,\alpha+t,s+t\}}{\alpha}\beta} = |T|^{2\min\{\alpha+s,\alpha+t,s+t\}\beta} + K_{10} \tag{2.8}$$

where $K_{10} \in \mathcal{C}_{p_{10}}(\mathcal{H})$ and $p_{10} = p\frac{\max\{\alpha,\min\{\alpha+s,\alpha+t,s+t\}\beta\}}{\min\{\alpha+s,\alpha+t,s+t\}\beta}$. (2.5)-(2.8) imply that

$$|T|^{2\min\{\alpha+s,\alpha+t,s+t\}\beta} + K_{10} \geq \big|\big(T(s,t)\big)^*\big|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} + K_9. \tag{2.9}$$

Lastly, (2.4) together with (2.9) imply that

$$|T(s,t)|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} - \big|\big(T(s,t)\big)^*\big|^{2\frac{\min\{\alpha+s,\alpha+t,s+t\}\beta}{s+t}} \geq K_{11}$$

where $K_{11} = K_9 - K_{10} - K_5 \in \mathcal{C}_{p_{11}}(\mathcal{H})$ and $p_{11} = \max\{p_5, p_9, p_{10}\} = p_9 = p(s,t)$ by

$$\max\{2\alpha, s\} \geq \max\{\frac{s}{s+t}\alpha, s\} \geq \frac{s}{s+t}\max\{\alpha, \min\{\alpha+s, \alpha+t, s+t\}\beta\}.$$

Therefore $T(s,t)$ is $(\mathcal{C}_{p(s,t)}, \alpha(s,t))$-hyponormal. □

Corollary 2.2. *Let $p \geq 1$ and $\alpha \geq \max\{s,t\}$. If T is $(\mathcal{C}_p, \alpha)$-hyponormal and $0 < \beta \leq 1$, then $T(s,t)$ is $(\mathcal{C}_{\frac{2p\alpha}{s\beta}}, \beta)$-hyponormal.*

Theorem 1.1 is the special case $\alpha \leq \beta \leq 1$ of Corollary 2.2.

Corollary 2.3. *Let $p \geq 1$ and $0 < \alpha \leq \min\{s,t\}$. If T is $(\mathcal{C}_p, \alpha)$-hyponormal and $0 < \beta' \leq \frac{\min\{\alpha+s,\alpha+t\}}{s+t}$, then $T(s,t)$ is $(\mathcal{C}_{\frac{p\max\{2\alpha,s\}}{s\beta'}}, \beta')$-hyponormal.*

The special case $\alpha \leq \beta' \leq \frac{2\alpha}{s+t}$ of Corollary 2.3 is just [6, Theorem 3.3].

Proof. Denote $\beta := \frac{(s+t)\beta'}{\min\{\alpha+s,\alpha+t,s+t\}}$. Since $0 < \alpha \leq \min\{s,t\}$, we have $0 < \beta \leq 1$,

$$p(s,t) = \frac{\max\{2\alpha, s\}p(s+t)}{\min\{\alpha+s, \alpha+t\}s\beta} = \frac{p\max\{2\alpha, s\}}{s\beta'},$$

$$\alpha(s,t) = \frac{\min\{\alpha+s, \alpha+t\}\beta}{s+t} = \beta'.$$

By Theorem 2.1, $T(s,t)$ is $(\mathcal{C}_{\frac{p\max\{2\alpha,s\}}{s\beta'}}, \beta')$-hyponormal. □

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