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REFINEMENTS OF HÖLDER'S INEQUALITY DERIVED FROM FUNCTIONS $\psi_{p,q,\lambda}$ AND $\phi_{p,q,\lambda}$

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ABSTRACT. We investigate a convex function $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda\psi_q\}$, $(1 \le q < p \le \infty)$, and its corresponding absolute normalized norm $\|.\|_{\psi_{p,q,\lambda}}$. We determine a dual norm and use it for getting refinements of the classical Hölder inequality. Also, we consider a related concave function $\phi_{p,q,\lambda} = \min\{\psi_p, \lambda\psi_q\}$, $(0 < p < q \le 1)$.

1. PRELIMINARIES

Since the end of 20th century several mathematicians have intensively researched properties of the absolute normalized norms on $\mathbf{C}^2$ (see [2], [6], [9], [10], [11]). In this section we give some properties of it which have impact to the classical Hölder inequality for two pairs of numbers.

Let us recall that a norm $\|.\|$ on $\mathbf{C}^2$ is said to be absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $x, y \in \mathbf{C}$ and it is normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbf{C}^2$ is denoted by N_a. Let Ψ denotes the family of all convex functions ψ on $[0, 1]$ with $\psi(0) = \psi(1) = 1$ satisfying

$$\max\{1 - t, t\} \le \psi(t) \le 1, \quad (0 \le t \le 1).$$

Classes N_a and Ψ are in one-to-one correspondence, (see [1]). Namely, if $\|.\| \in N_a$, then the function $\psi(t) = \|(1-t, t)\|$ belongs to Ψ. Conversely, if $\psi \in \Psi$, then the

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mapping

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\frac{|y|}{|x|+|y|}\right), & (x,y)\neq(0,0); \\ 0 & (x,y)=(0,0) \end{cases} \tag{1.1}$$

is a norm and $\|.\|_\psi \in N_a$.

The mostly investigated examples of absolute normalized norms are l_p-norm $\|.\|_p$ and the norm $\|.\|_{\omega,q}$ of the two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ (see [3], [4], [7]). The corresponding convex function ψ_p for the norm $\|.\|_p$ is

$$\psi_p(t) = \begin{cases} [(1-t)^p + t^p]^{\frac{1}{p}}, & 1\le p<\infty; \\ \max\{t, 1-t\}, & p=\infty. \end{cases}$$

A norm $\|.\|_{\omega,q}$ of the two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ is defined as:

$$\|(x,y)\|_{\omega,q} = (x^{*q} + \omega y^{*q})^{1/q}$$

where $0<\omega<1$, $q\ge 1$, (x^*,y^*) is a non-increasing rearrangement of $(|x|,|y|)$. Its corresponding convex function $\psi_{\omega,q}$ is equal to

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{\frac{1}{q}}, & 0\le t\le \frac{1}{2}; \\ (t^q + \omega(1-t)^q)^{\frac{1}{q}}, & \frac{1}{2}\le t\le 1. \end{cases}$$

Another example of particular interest is the following. Let $1\le q<p\le\infty$ and $2^{\frac{1}{p}-\frac{1}{q}} < \lambda < 1$. Then the mapping

$$\|.\|_{p,q,\lambda} = \max\{\|.\|_p, \lambda\|.\|_q\}$$

is a norm from N_a and the corresponding convex function is defined by

$$\psi_{p,q,\lambda}(t) = \max\{\psi_p(t), \lambda\psi_q(t)\}, \quad t\in[0,1].$$

For $\psi\in\Psi$, the dual of the norm $\|.\|_\psi$ is denoted by $\|.\|_\psi^*$. In [5] the following results about dual of the norm were proved: The mapping $\|.\|_\psi^*$ is an absolute normalized norm and the corresponding function $\psi^*\in\Psi$ is given by

$$\psi^*(t) = \sup_{s\in[0,1]} \frac{(1-s)(1-t)+st}{\psi(s)} \tag{1.2}$$

for t with $0\le t\le 1$. Also, the following generalized Hölder inequality holds for convex function $\psi\in\Psi$:

$$|x_1x_2| + |y_1y_2| \le \|(x_1,y_1)\|_\psi \|(x_2,y_2)\|_\psi^*, \quad (x_1,y_1),(x_2,y_2)\in\mathbf{C}^2. \tag{1.3}$$

For example, the dual norm of the space $d^{(2)}(\omega,q)$ was completely determined by Mitani and Saito in [7]. In the same paper they stated that in formula (1.2) the conditions "$s\in[0,1]$ and $t\in[0,1]$" can be replaced with "$s\in[0,1/2]$ and $t\in[0,1/2]$" respectively.

In [8] authors considered a family $\tilde\Psi$ of concave functions $\tilde\psi$ on $[0,1]$ with $\tilde\psi(0) = \tilde\psi(1) = 1$. The mapping $\|.\|_{\tilde\psi}$ defined by (1.1) satisfies the inverse Minkowski inequality, i.e. for $u,v,z,w\in\mathbf{C}$ the following is valid

$$\|(|u|+|z|, |v|+|w|)\|_{\tilde\psi} \ge \|(|u|,|v|)\|_{\tilde\psi} + \|(|z|,|w|)\|_{\tilde\psi}.$$

Furthermore, if $\tilde{\psi} \in \tilde{\Psi}$, then $\frac{\tilde{\psi}(t)}{t}$ is non-increasing on $\langle 0,1]$ and $\frac{\tilde{\psi}(t)}{1-t}$ is non-decreasing on $[0,1\rangle$. Moreover, if $0 \le p \le r, 0 \le q \le s$, we have

$$\|(p,q)\|_\psi \le \|(r,s)\|_\psi.$$

For a concave function $\psi \in \tilde{\Psi}$ let us define a function ψ_* by

$$\psi_*(t) = \inf_{0\le s\le 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for $0 \le t \le 1$. The corresponding map $\|.\|_{*\psi}$ is defined by (1.1). Similar concluding as in [7] gives us that if ψ is symmetric with respect to $t = 1/2$, then ψ_* is also symmetric with respect to $t = 1/2$ and

$$\psi_*(t) = \inf_{0\le s\le 1/2} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for $1/2 \le t \le 1$, ([8]). Also, the inverse generalized Hölder inequality holds, i.e.

$$\|(x_1,y_1)\|_\psi \|(x_2,y_2)\|_{*\psi} \le |x_1x_2| + |y_1y_2|, \tag{1.4}$$

where $x_1, x_2, y_1, y_2 \in \mathbf{C}$, $\psi \in \tilde{\Psi}$.

Examples of concave functions from the family $\tilde{\Psi}$ are the following:

(i) ψ_p for $p \in \langle 0,1\rangle$.

(ii) $\psi_{\omega,q}$ for $q \in \langle 0,1\rangle$ and $\omega \ge 1$, (see [8]).

(iii) Let $0 < p < q \le 1$ and $\lambda \in \langle 1, 2^{\frac{1}{p}-\frac{1}{q}}\rangle$. The function $\phi_{p,q,\lambda} = \min\{\psi_p, \lambda\psi_q\}$ belongs to the family $\tilde{\Psi}$.

In this paper we consider functions $\psi_{p,q,\lambda}$ and $\phi_{p,q,\lambda}$. Firstly, in the following section we consider the function $\psi_{p,q,\lambda}$. We will determine the dual function $\psi^*_{p,q,\lambda}$ and the corresponding map $\|.\|^*_{\psi_{p,q,\lambda}}$. Also, we will state and analyse the generalized Hölder inequality and the Cauchy inequality which appear in that case and find out some refinements of the classical Hölder and the Cauchy inequalities. The last section is devoted to the function $\phi_{p,q,\lambda}$ which belongs to $\tilde{\Psi}$. We will calculate $\|.\|_{\phi_{p,q,\lambda}}$, ϕ_{p,q,λ_*}, the corresponding map $\|.\|_{*\phi_{p,q,\lambda}}$ and investigate inequalities which arise from the inverse generalized Hölder inequality (1.4).

2. Function $\psi_{p,q,\lambda}$

2.1. **Case $p < \infty$.** Let $1 \le q < p < \infty$ and $\lambda \in \langle 2^{1/p-1/q}, 1\rangle$. Let us consider a function $\psi_{p,q,\lambda}(s) = \max_{s\in[0,1]}\{\psi_p(s), \lambda\psi_q(s)\}$. It is a function from Ψ.

Let $s_0 \in [0,1/2]$ be a point such that $\psi_p(s_0) = \lambda\psi_q(s_0)$, i.e. $[(1-s_0)^p + s_0^p]^{1/p} = \lambda[(1-s_0)^q + s_0^q]^{1/q}$. Then we have

$$\psi_{p,q,\lambda}(t) = \begin{cases} \psi_p(t) & \text{for } t \in [0,s_0] \cup [1-s_0,1]; \\ \lambda\psi_q(t) & \text{for } t \in [s_0, 1-s_0] \end{cases}$$

and the corresponding absolute normalized norm is

$$\|(x,y)\|_{\psi_{p,q,\lambda}} = \max\{\|(x,y)\|_p, \lambda\|(x,y)\|_q\} = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{for } \frac{y^*}{x^*} \le k \\ \lambda(|x|^q + |y|^q)^{1/q} & \text{for } \frac{y^*}{x^*} \ge k \end{cases}$$

where $k = \frac{s_0}{1-s_0}$.

The function $\psi_{p,q,\lambda}$ is symmetric, so by Propositions 2 and 3 from [7], the function $\psi^*_{p,q,\lambda}$ is also symmetric and

$$\psi^*_{p,q,\lambda}(t) = \sup_{s\in[0,1/2]} \frac{(1-s)(1-t)+st}{\psi_{p,q,\lambda}(s)}, \quad t \in [0, \frac{1}{2}]. \tag{2.1}$$

Let us calculate the function $\psi^*_{p,q,\lambda}$. From (2.1) we have that $\psi^*_{p,q,\lambda}(t) = \max\{A, B\}$ where

$$A = \max_{s\in[0,s_0]} \frac{(1-s)(1-t)+st}{\psi_p(s)}, \quad B = \max_{s\in[s_0,1/2]} \frac{(1-s)(1-t)+st}{\lambda\psi_q(s)}.$$

Firstly, we consider

$$h_t(s) = \left[\frac{((1-s)(1-t)+st)^p}{(1-s)^p+s^p}\right]^{\frac{1}{p}} = (f_t(s))^{\frac{1}{p}}.$$

The first derivative of $f_t(s)$ is equal

$$f_t'(s) = \frac{p(1-s-t+2st)^{p-1}(t(1-s)^{p-1}-(1-t)s^{p-1})}{((1-s)^p+s^p)^2}$$

and the unique stationary point on $\langle 0, 1/2\rangle$ is

$$s_1 = \frac{t^{\frac{1}{p-1}}}{t^{\frac{1}{p-1}}+(1-t)^{\frac{1}{p-1}}} = \frac{1}{1+\left(\frac{1-t}{t}\right)^{\frac{1}{p-1}}}.$$

The function h_t increases for $s \le s_1$ and decreases for $s \ge s_1$. Consider now another function

$$g_t(s) = \frac{1}{\lambda}\left[\frac{((1-s)(1-t)+st)^q}{(1-s)^q+s^q}\right]^{\frac{1}{q}}.$$

The point of the local maximum of g_t on interval $\langle 0, 1/2\rangle$ is

$$s_2 = \frac{t^{\frac{1}{q-1}}}{t^{\frac{1}{q-1}}+(1-t)^{\frac{1}{q-1}}} = \frac{1}{1+\left(\frac{1-t}{t}\right)^{\frac{1}{q-1}}}.$$

Since $0 \le t \le \frac{1}{2}$, $q < p$ we have $s_2 \le s_1$ and there are exist three different orders: $s_0 \le s_2 \le s_1$, $s_2 \le s_1 \le s_0$ and $s_2 \le s_0 \le s_1$.

Case a) If $s_0 \le s_2 \le s_1$, then:

$$A = \max_{s\in[0,s_0]} h_t(s) = h_t(s_0) = \frac{(1-s_0)(1-t)+s_0t}{((1-s_0)^p+s_0^p)^{\frac{1}{p}}} = C_{s_0}(t),$$

$$B = \max_{s\in[s_0,1/2]} g_t(s) = g_t(s_2) = \frac{1}{\lambda}\left[\frac{((1-s_2)(1-t)+s_2t)^q}{(1-s_2)^q+s_2^q}\right]^{\frac{1}{q}}$$
$$= \frac{1}{\lambda}\left((1-t)^{\frac{q}{q-1}}+t^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}.$$

The Hölder inequality for pairs $(1-s_0, s_0)$ and $(1-t, t)$ with exponents q and $\frac{q}{1-q}$ states:

$$(1-s_0)(1-t)+s_0t \le ((1-s_0)^q+s_0^q)^{\frac{1}{q}}\left((1-t)^{\frac{q}{1-q}}+t^{\frac{q}{1-q}}\right)^{\frac{1-q}{q}}.$$

Using the previous result we get that

$$C_{s_0}(t)=\frac{(1-s_0)(1-t)+s_0t}{\lambda((1-s_0)^q+s_0^q)^{\frac{1}{q}}}\le\frac{1}{\lambda}((1-t)^{\frac{q}{q-1}}+t^{\frac{q}{q-1}})^{\frac{q-1}{q}}$$

and in this case

$$\psi^*_{p,q,\lambda}(t)=\frac{1}{\lambda}((1-t)^{\frac{q}{q-1}}+t^{\frac{q}{q-1}})^{\frac{q-1}{q}}.$$

Case b) If $s_2\le s_1\le s_0$, then

$$A=\max_{s\in[0,s_0]}h_t(s)=h_t(s_1)=\frac{(1-s_1)(1-t)+s_1t}{((1-s_1)^p+s_1^p)^{\frac{1}{p}}}=[(1-t)^{\frac{p}{p-1}}+t^{\frac{p}{p-1}}]^{\frac{p-1}{p}}$$

$$B=\max_{s\in[s_0,1/2]}g_t(s)=g_t(s_0)=\frac{(1-s_0)(1-t)+s_0t}{\lambda((1-s_0)^q+s_0^q)^{\frac{1}{q}}}=C_{s_0}(t).$$

As above, using once more the Hölder inequality we see that

$$\psi^*_{p,q,\lambda}(t)=\max\{[(1-t)^{\frac{p}{p-1}}+t^{\frac{p}{p-1}}]^{\frac{p-1}{p}},C_{s_0}(t)\}=[(1-t)^{\frac{p}{p-1}}+t^{\frac{p}{p-1}}]^{\frac{p-1}{p}}.$$

Case c) If $s_2\le s_0\le s_1$, then

$$A=\max_{s\in[0,s_0]}h_t(s)=h_t(s_0)=C_{s_0}(t),$$

$$B=\max_{s\in[s_0,1/2]}g_t(s)=g_t(s_0)=C_{s_0}(t)$$

and

$$\psi^*_{p,q,\lambda}(t)=C_{s_0}(t).$$

Let us express conditions of above cases in terms of the variable t.

a) If $s_2\ge s_0$, then

$$\frac{1}{1+\left(\frac{1-t}{t}\right)^{\frac{1}{q-1}}}\ge s_0$$

which is equivalent to the inequality

$$t\ge t_2=\frac{1}{1+\left(\frac{1-s_0}{s_0}\right)^{q-1}}.$$

b) If $s_1\le s_0$, then

$$\frac{1}{1+\left(\frac{1-t}{t}\right)^{\frac{1}{p-1}}}\le s_0$$

or equivalently

$$t\le t_1=\frac{1}{1+\left(\frac{1-s_0}{s_0}\right)^{p-1}}.$$

Note that since $q<p, 0<s_0<1/2$ we have $0<t_1<t_2\le\frac{1}{2}$.

c) And finally, if $s_2\le s_0\le s_1$, then $t_1\le t\le t_2$.

The following theorem is a result of the above discussion.

Theorem 2.1. *Let $1 \le q < p < \infty$ and $\lambda \in \langle 2^{1/p-1/q}, 1\rangle$. Then the function $\psi^*_{p,q,\lambda}$ is equal*

$$\psi^*_{p,q,\lambda}(t) = \begin{cases} ((1-t)^{p'} + t^{p'})^{1/p'}, & 0 \le t \le t_1; \\ C_{s_0}(t) & t_1 \le t \le t_2; \\ \dfrac{1}{\lambda}((1-t)^{q'} + t^{q'})^{1/q'}, & t_2 \le t \le 1-t_2; \\ C_{s_0}(1-t) & 1-t_2 \le t \le 1-t_1; \\ ((1-t)^{p'} + t^{p'})^{1/p'}, & 1-t_1 \le t \le 1, \end{cases}$$

where p' and q' are conjugate exponents of p and q, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$,

$$t_1 = \frac{1}{1 + \left(\frac{1-s_0}{s_0}\right)^{p-1}}, \quad t_2 = \frac{1}{1 + \left(\frac{1-s_0}{s_0}\right)^{q-1}}$$

and

$$C_{s_0}(t) = \frac{(1-s_0)(1-t) + s_0 t}{((1-s_0)^p + s_0^p)^{\frac{1}{p}}} = \frac{(1-s_0)(1-t) + s_0 t}{\lambda((1-s_0)^q + s_0^q)^{\frac{1}{q}}}.$$

Let us determine the norm which corresponds to the function $\psi^*_{p,q,\lambda}$. Let $t = \frac{|y|}{|x|+|y|} \in [0, t_1]$. Since $t_1 \le \frac{1}{2}$, then $|y| \le |x|$, i.e. $x^* = |x|, y^* = |y|$ and the inequality $0 \le t \le t_1$ gives $\frac{y^*}{x^*+y^*} \le \frac{1}{1+\left(\frac{1-s_0}{s_0}\right)^{p-1}}$ or $\frac{y^*}{x^*} \le \left(\frac{s_0}{1-s_0}\right)^{p-1}$. Using formula (1.1) we get that in this case $\|(x,y)\|^*_{\psi_{p,q,\lambda}} = (|x|^{p'} + |y|^{p'})^{1/p'}$. Calculations for other cases are similar. So, we have the following result.

The norm $\|(x,y)\|^*_{\psi_{p,q,\lambda}}$ is equal to

$$\|(x,y)\|^*_{\psi_{p,q,\lambda}} = \begin{cases} (|x|^{p'} + |y|^{p'})^{1/p'}, & \frac{y^*}{x^*} \le k^{p-1}; \\ \dfrac{1}{(k^p_\lambda + 1)^{\frac{1}{p}}}(ky^* + x^*), & k^{p-1} \le \frac{y^*}{x^*} \le k^{q-1}; \\ \dfrac{1}{\lambda}(|x|^{q'} + |y|^{q'}|)^{1/q'}, & \frac{y^*}{x^*} \ge k^{q-1} \end{cases}$$

where $k = \frac{s_0}{1-s_0}$ and (x^*, y^*) is a non-increasing rearrangement of $(|x|, |y|)$.

Example 2.2. Let us investigate the simplest case when $q = 1$ and $p = 2$. Then $\lambda \in \langle \frac{1}{\sqrt{2}}, 1\rangle$, $t_1 = s_0 = \frac{1-\sqrt{2\lambda^2-1}}{2}$, $t_2 = \frac{1}{2}$ and $k = \frac{\lambda^2 - \sqrt{2\lambda^2-1}}{1-\lambda^2}$. In that case

$$\psi_{2,1,\lambda}(t) = \begin{cases} ((1-t)^2 + t^2)^{1/2}, & 0 \le t \le s_0 \text{ or } 1-s_0 \le t \le 1; \\ 1, & s_0 \le t \le 1-s_0, \end{cases}$$

$$\psi^*_{2,1,\lambda}(t) = \begin{cases} ((1-t)^2 + t^2)^{1/2}, & 0 \le t \le s_0; \\ \dfrac{2s_0 - 1}{\lambda}t + \dfrac{1-s_0}{\lambda}, & s_0 \le t \le \frac{1}{2}; \\ \dfrac{1-2s_0}{\lambda}t + \dfrac{s_0}{\lambda}, & \frac{1}{2} \le t \le 1-s_0; \\ ((1-t)^2 + t^2)^{1/2}, & 1-s_0 \le t \le 1. \end{cases}$$

The corresponding norms are

$$\|(x,y)\|_{\psi_{2,1,\lambda}} = \begin{cases} (x^2+y^2)^{1/2}, & \frac{y^*}{x^*} \le k; \\ \lambda(|x|+|y|), & \frac{y^*}{x^*} \ge k, \end{cases}$$

$$\|(x,y)\|^*_{\psi_{2,1,\lambda}} = \begin{cases} (x^2+y^2)^{1/2}, & \frac{y^*}{x^*} \le k; \\ \dfrac{1-s_0}{\lambda}(ky^*+x^*), & \frac{y^*}{x^*} \ge k. \end{cases}$$

2.2. **Case $p=\infty$.** If $p=\infty$ and $\lambda \in \langle 2^{-\frac{1}{q}}, 1\rangle$, then

$$\|(x,y)\|_{\psi_{\infty,q,\lambda}} = \max\{\|x,y\|_\infty, \lambda\|(x,y)\|_q\}$$

and

$$\psi_{\infty,q,\lambda}(t) = \begin{cases} 1-t, & t\in[0,s_0]; \\ \lambda\psi_q(t), & t\in[s_0,1-s_0]; \\ t, & t\in[1-s_0,1] \end{cases}$$

where s_0 is a point from $[0,1]$ such that $1-s_0 = \lambda((1-t)^q+t^q)^{1/q}$. An easy calculation, similar to the calculation in the first part of this section gives us that

$$\psi^*_{\infty,q,\lambda}(t) = \begin{cases} C_{s_0}(t) & 0\le t\le t_2; \\ \dfrac{1}{\lambda}((1-t)^{q'}+t^{q'})^{1/q'}, & t_2\le t\le 1-t_2; \\ C_{s_0}(1-t) & 1-t_2\le t\le 1 \end{cases}$$

and

$$\|(x,y)\|^*_{\psi_{\infty,q,\lambda}} = \begin{cases} ky^*+x^*, & \frac{y^*}{x^*} \le k^{q-1}; \\ \dfrac{1}{\lambda}(|x|^{q'}+|y|^{q'}|)^{1/q'}, & \frac{y^*}{x^*} \ge k^{q-1} \end{cases}$$

where q', t_2 and k are defined as in the case when $p\ne\infty$.

Let us observe that $\psi^*_{\infty,q,\lambda} = \lim_{p\to\infty}\psi^*_{p,q,\lambda}$ and $\|.\|^*_{\psi_{\infty,q,\lambda}} = \lim_{p\to\infty}\|.\|^*_{\psi_{p,q,\lambda}}$. For $q=1$ the function $\psi^*_{\infty,1,\lambda}$ was calculated in [5].

2.3. **Refinements of the Hölder inequality and the Cauchy inequality.** As a consequence of the generalized Hölder inequality (1.3) and results about norms $\|.\|_{\psi_{p,q,\lambda}}$ and $\|.\|^*_{\psi_{p,q,\lambda}}$ we have six inequalities which are valid in the different regions. As we will see two of them have a form of the classical Hölder inequality and some of them are better than the classical Hölder inequality. Because of simplicity, let x_1,x_2,y_1,y_2 be non-negative real numbers. These six inequalities are the following:

(1) If $\frac{y_1^*}{x_1^*}\le k$ and $\frac{y_2^*}{x_2^*}\le k^{p-1}$, then

$$x_1x_2+y_1y_2 \le (x_1^p+y_1^p)^{1/p}(x_2^{p'}+y_2^{p'})^{1/p'}.$$

(2) If $\frac{y_1^*}{x_1^*}\le k$ and $k^{p-1}\le\frac{y_2^*}{x_2^*}\le k^{q-1}$, then

$$x_1x_2+y_1y_2 \le \frac{1}{(k^p+1)^{1/p}}(x_1^p+y_1^p)^{1/p}(ky_2^*+x_2^*).$$

(3) If $\frac{y_1^*}{x_1^*} \le k$ and $\frac{y_2^*}{x_2^*} \ge k^{q-1}$, then

$$x_1x_2 + y_1y_2 \le \frac{1}{\lambda}(x_1^p + y_1^p)^{1/p}(x_2^{q'} + y_2^{q'})^{1/q'}.$$

(4) If $\frac{y_1^*}{x_1^*} \ge k$ and $\frac{y_2^*}{x_2^*} \le k^{p-1}$, then

$$x_1x_2 + y_1y_2 \le \lambda(x_1^q + y_1^q)^{1/q}(x_2^{p'} + y_2^{p'})^{1/p'}.$$

(5) If $\frac{y_1^*}{x_1^*} \ge k$ and $k^{p-1} \le \frac{y_2^*}{x_2^*} \le k^{q-1}$, then

$$x_1x_2 + y_1y_2 \le \frac{\lambda}{(k^p+1)^{1/p}}(x_1^q + y_1^q)^{1/q}(ky_2^* + x_2^*).$$

(6) If $\frac{y_1^*}{x_1^*} \ge k$ and $\frac{y_2^*}{x_2^*} \ge k^{q-1}$, then

$$x_1x_2 + y_1y_2 \le (x_1^q + y_1^q)^{1/q}(x_2^{q'} + y_2^{q'})^{1/q'}.$$

In the following theorem we give a refinement of the classical Hölder inequality.

Theorem 2.3. *Let $x_1, x_2, y_1, y_2 > 0$, $1 \le q < p$ and $k = \frac{s_0}{1-s_0}$. Let p' and q' be a conjugate exponents of p and q respectively, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

If $\frac{y_1^*}{x_1^*} \le k$ and $k^{p-1} \le \frac{y_2^*}{x_2^*} \le k^{q-1}$, then

$$x_1x_2 + y_1y_2 \le \frac{1}{(k^p+1)^{1/p}}(x_1^p + y_1^p)^{1/p}(ky_2^* + x_2^*) \le (x_1^p + y_1^p)^{1/p}(x_2^{p'} + y_2^{p'})^{1/p'}. \quad (2.2)$$

If $\frac{y_1^*}{x_1^*} \ge k$ and $k^{p-1} \le \frac{y_2^*}{x_2^*} \le k^{q-1}$, then

$$x_1x_2 + y_1y_2 \le \frac{\lambda}{(k^p+1)^{1/p}}(x_1^q + y_1^q)^{1/q}(ky_2^* + x_2^*) \le (x_1^q + y_1^q)^{1/q}(x_2^{q'} + y_2^{q'})^{1/q'}. \quad (2.3)$$

Proof. Assume that $\frac{y_1^*}{x_1^*} \le k$ and $k^{p-1} \le \frac{y_2^*}{x_2^*} \le k^{q-1}$. The first inequality in (2.2) is a consequence of the generalized Hölder inequality (1.3). The second inequality is equivalent to $\frac{1}{(k^p+1)^{1/p}}(ky_2^* + x_2^*) \le (x_2^{p'} + y_2^{p'})^{1/p'}$. Furthemore, the previous inequality is equivalent to

$$\frac{(1+kt)^{p'}}{1+t^{p'}} \le (1+k^p)^{\frac{1}{p-1}} \quad (2.4)$$

for $t \in [k^{p-1}, k^{q-1}]$. Function $y(t) = \frac{(1+kt)^{p'}}{1+t^{p'}}$ is non-increasing on $[k^{p-1}, k^{q-1}]$ and $y(k^{p-1}) = (1+k^p)^{\frac{1}{p-1}}$. So, inequality (2.4) holds and refinement (2.2) is proved.

Inequality (2.3) is proved similarly. □

When $p = 2$ and $q = 1$, then using the generalized Hölder inequality (1.3) we have the following inequalites:

(1) If $\frac{y_1^*}{x_1^*} \le k$ and $\frac{y_2^*}{x_2^*} \le k$, then

$$x_1x_2 + y_1y_2 \le (x_1^2 + y_1^2)^{1/2}(x_2^2 + y_2^2)^{1/2}.$$

(2) If $\frac{y_1^*}{x_1^*} \ge k$, $\frac{y_2^*}{x_2^*} \le k$, then

$$x_1x_2 + y_1y_2 \le \lambda(|x_1| + |y_1|)(x_2^2 + y_2^2)^{1/2}.$$

(3) If $\frac{y_1^*}{x_1^*} \le k$, $\frac{y_2^*}{x_2^*} \ge k$, then

$$x_1x_2 + y_1y_2 \le \frac{1-s_0}{\lambda}(x_1^2+y_1^2)^{1/2}(x_2^* + ky_2^*).$$

(4) If $\frac{y_1^*}{x_1^*} \ge k$, $\frac{y_2^*}{x_2^*} \ge k$, then

$$x_1x_2 + y_1y_2 \le (1-s_0)(|x_1|+|y_1|)(x_2^* + ky_2^*).$$

The first inequality is the classical Cauchy inequality. As a consequence of (2.2), the third one is a refinement of the Cauchy inequality, i.e. we have

$$x_1x_2 + y_1y_2 \le \frac{1-s_0}{\lambda}(x_1^2+y_1^2)^{1/2}(x_2^* + ky_2^*) \le (x_1^2+y_1^2)^{1/2}(x_2^2+y_2^2)^{1/2} \tag{2.5}$$

for $\frac{y_1^*}{x_1^*} \le k$, $\frac{y_2^*}{x_2^*} \ge k$.

But for $\frac{y_1^*}{x_1^*} \ge k$, $\frac{y_2^*}{x_2^*} \le k$ (conditions of case (2)) we can put better inequality. Let us prove that

$$\frac{1-s_0}{\lambda}(x_1^* + ky_1^*) \le \lambda(|x_1|+|y_1|). \tag{2.6}$$

Putting $t = \frac{y_1^*}{x_1^*}$ the previous inequality becomes $(1-s_0)(1+kt) \le \lambda^2(1+t)$ or $t(\lambda^2-(1-s_0)k) \ge 1-s_0-\lambda^2$. Since $s_0 \le 1/2$, then $\lambda^2-(1-s_0)k = \lambda^2 - s_0 = 2s_0^2 - 3s_0 + 1 \ge 0$ and we can write $t \ge \frac{1-s_0-\lambda^2}{\lambda^2-s_0} = \frac{s_0}{1-s_0} = k$. The last inequality $t \ge k$ is true in the case (2), so we prove (2.6).

If we change the places of (x_1, y_1) and (x_2, y_2) in (2.5), then for $\frac{y_1^*}{x_1^*} \ge k$, $\frac{y_2^*}{x_2^*} \le k$ we get

$$x_1x_2 + y_1y_2 \le \frac{1-s_0}{\lambda}(x_2^2+y_2^2)^{1/2}(x_1^* + ky_1^*) \le (x_1^2+y_1^2)^{1/2}(x_2^2+y_2^2)^{1/2}$$

This and (2.6) shows that we have improved the inequality from the case (2) and this improved inequality

$$x_1x_2 + y_1y_2 \le \frac{1-s_0}{\lambda}(x_2^2+y_2^2)^{1/2}(x_1^* + ky_1^*)$$

is also an refinement of the classical Cauchy inequality.

So in the case (1) we have the classical Cauchy inequality and in cases (2) and (3) we have refinements of the classical Cauchy inequality.

3. Function $\phi_{p,q,\lambda}$

Let $0 < p < q < 1$ and $\lambda \in \langle 1, 2^{1/p-1/q}\rangle$. The function $\phi_{p,q,\lambda}$ is defined as $\phi_{p,q,\lambda}(s) = \min_{s\in[0,1]}\{\psi_p(s), \lambda\psi_q(s)\}$. It is a function from $\tilde{\Psi}$ and it is easy to see that

$$\phi_{p,q,\lambda} = \begin{cases} \psi_p(t) & \text{for } t \in [0, s_0]; \\ \lambda\psi_q(t) & \text{for } t \in [s_0, \frac{1}{2}] \end{cases}$$

which is expanded to the whole interval $[0,1]$ by symmetrization with respect to the point $1/2$, where $s_0 \in [0, 1/2]$ is a point such that $\psi_p(s_0) = \lambda\psi_q(s_0)$. Also,

$$\|(x,y)\|_{\phi_{p,q,\lambda}} = \min\{\|(x,y)\|_p, \lambda\|(x,y)\|_q\} = \begin{cases} (|x|^p+|y|^p)^{1/p} & \text{for } \frac{y^*}{x^*} \le k; \\ \lambda(|x|^q+|y|^q)^{1/q} & \text{for } \frac{y^*}{x^*} \ge k \end{cases}$$

where $k = \frac{s_0}{1-s_0}$. In the following text we will use notation $\phi = \phi_{p,q,\lambda}$.

Using the same method as in the Section 2, we state the following theorem:

Theorem 3.1. *Let $0 < p < q < 1$ and $\lambda \in \langle 1, 2^{1/p-1/q}\rangle$. Then the function ϕ_* is equal*

$$\phi_*(t) = \begin{cases} ((1-t)^{p'} + t^{p'})^{1/p'}, & 0 \le t \le 1 - t_1; \\ C_{s_0}(1-t) & 1 - t_1 \le t \le 1 - t_2; \\ \dfrac{1}{\lambda}((1-t)^{q'} + t^{q'})^{1/q'}, & 1 - t_2 \le t \le t_2; \\ C_{s_0}(t) & t_2 \le t \le t_1; \\ ((1-t)^{p'} + t^{p'})^{1/p'}, & t_1 \le t \le 1, \end{cases}$$

where p' and q' are conjugate exponents of p and q, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$,

$$t_1 = \frac{1}{1 + \left(\frac{1-s_0}{s_0}\right)^{p-1}}, \quad t_2 = \frac{1}{1 + \left(\frac{1-s_0}{s_0}\right)^{q-1}}$$

and

$$C_{s_0}(t) = \frac{(1-s_0)(1-t) + s_0 t}{((1-s_0)^p + s_0^p)^{\frac{1}{p}}} = \frac{(1-s_0)(1-t) + s_0 t}{\lambda((1-s_0)^q + s_0^q)^{\frac{1}{q}}}.$$

The corresponding map $\|.\|_{*\phi}$ is equal

$$\|(x,y)\|_{*\phi} = \begin{cases} (|x|^{p'} + |y|^{p'})^{1/p'}, & \frac{y^*}{x^*} \le k^{1-p}; \\ \dfrac{1}{(k^p+1)^{\frac{1}{p}}}(ky^* + x^*), & k^{1-p} \le \frac{y^*}{x^*} \le k^{1-q}; \\ \dfrac{1}{\lambda}(|x|^{q'} + |y|^{q'}|)^{1/q'}, & \frac{y^*}{x^*} \ge k^{1-q} \end{cases}$$

where $k = \frac{s_0}{1-s_0}$ and (x^*, y^*) is a non-increasing rearrangement of $(|x|, |y|)$.

As a consequence of the inverse generalized Hölder inequality (1.4) and results about $\|.\|_\phi$ and $\|.\|_{*\phi}$ we have six inequalities which are valid in the different regions. As we will see two of them have a form of the classical inverse Hölder inequality. Because of simplicity, let x_1, x_2, y_1, y_2 be non-negative real numbers. These inequalities are the following:

(1) If $\frac{y_1^*}{x_1^*} \le k$ and $\frac{y_2^*}{x_2^*} \le k^{1-p}$, then

$$x_1x_2 + y_1y_2 \ge (x_1^p + y_1^p)^{1/p}(x_2^{p'} + y_2^{p'})^{1/p'}.$$

(2) If $\frac{y_1^*}{x_1^*} \le k$ and $k^{1-p} \le \frac{y_2^*}{x_2^*} \le k^{1-q}$, then

$$x_1x_2 + y_1y_2 \ge \frac{1}{(k^p+1)^{1/p}}(x_1^p + y_1^p)^{1/p}(ky_2^* + x_2^*).$$

(3) If $\frac{y_1^*}{x_1^*} \le k$ and $\frac{y_2^*}{x_2^*} \ge k^{1-q}$, then

$$x_1x_2 + y_1y_2 \ge \frac{1}{\lambda}(x_1^p + y_1^p)^{1/p}(x_2^{q'} + y_2^{q'})^{1/q'}.$$

(4) If $\frac{y_1^*}{x_1^*} \ge k$ and $\frac{y_2^*}{x_2^*} \le k^{1-p}$, then

$$x_1x_2 + y_1y_2 \ge \lambda(x_1^q + y_1^q)^{1/q}(x_2^{p'} + y_2^{p'})^{1/p'}.$$

(5) If $\frac{y_1^*}{x_1^*} \geq k$ and $k^{1-p} \leq \frac{y_2^*}{x_2^*} \leq k^{1-q}$, then

$$x_1x_2 + y_1y_2 \geq \frac{\lambda}{(k^p+1)^{1/p}}(x_1^q + y_1^q)^{1/q}(ky_2^* + x_2^*).$$

(6) If $\frac{y_1^*}{x_1^*} \geq k$ and $\frac{y_2^*}{x_2^*} \geq k^{1-q}$, then

$$x_1x_2 + y_1y_2 \geq (x_1^q + y_1^q)^{1/q}(x_2^{q'} + y_2^{q'})^{1/q'}.$$

In the following theorem we give a refinement of the classical inverse Hölder inequality.

Theorem 3.2. *Let $x_1, x_2, y_1, y_2 > 0$, $0 < p < q < 1$ and $k = \frac{s_0}{1-s_0}$. Let p' and q' be a conjugate exponents of p and q respectively.*

If $\frac{y_1^*}{x_1^*} \leq k$ and $k^{1-p} \leq \frac{y_2^*}{x_2^*} \leq k^{1-q}$, then

$$x_1x_2 + y_1y_2 \geq \frac{1}{(k^p+1)^{1/p}}(x_1^p + y_1^p)^{1/p}(ky_2^* + x_2^*) \geq (x_1^p + y_1^p)^{1/p}(x_2^{p'} + y_2^{p'})^{1/p'}.$$

If $\frac{y_1^*}{x_1^*} \geq k$ and $k^{1-p} \leq \frac{y_2^*}{x_2^*} \leq k^{1-q}$, then

$$x_1x_2 + y_1y_2 \geq \frac{\lambda}{(k^p+1)^{1/p}}(x_1^q + y_1^q)^{1/q}(ky_2^* + x_2^*) \geq (x_1^q + y_1^q)^{1/q}(x_2^{q'} + y_2^{q'})^{1/q'}.$$

Proof. The proof is similar to the proof of Theorem 2.3. □

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