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ON MAJORIZATION, FAVARD AND BERWALD INEQUALITIES

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ABSTRACT. In this paper, we obtain extensions of majorization type results and extensions of weighted Favard's and Berwald's inequality. We prove positive semi-definiteness of matrices generated by differences deduced from majorization type results and differences deduced from weighted Favard's and Berwald's inequality. This implies a surprising property of exponentially convexity and log-convexity of these differences which allows us to deduce Lyapunov's and Dresher's inequalities for these differences, which are improvements of majorization type results and weighted Favard's and Berwald's inequalities. Analogous Cauchy's type means, as equivalent forms of exponentially convexity and log-convexity, are also studied and the monotonicity properties are proved.

1. INTRODUCTION

Majorization is a very important topic in mathematics. A complete and superb reference on the subject is the book by Marshall and Olkin [11]. For example, majorization theory is a key tool that allows us to transform complicated non-convex constrained optimization problems that involve matrix-valued variables into simple problems with scalar variables that can be easily solved. We can see such type of applications in [16]. The book by Bhatia [5] contains significant material on majorization theory as well. Other textbooks on matrix and multivariate analysis may also include a section on majorization theory, e.g., [8, Sec.4.3] and [1, Sec.8.10].
In 1947, L. Fuchs gave a weighted generalization of the well-known majorization

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theorem for convex functions and two sequences monotonic in the same sense, see [11, p.419] or [17, p.323]. The following theorem is a simple consequence of Theorem 12.14 in [19] (see also [17, p.328]): Let $x(\tau)$, $y(\tau)$: $[a, b] \to \mathbb{R}$, $x(\tau)$ and $y(\tau)$ are continuous and increasing and let $G : [a, b] \to \mathbb{R}$ be a function of bounded variation.

(a) If

$$\int_{\nu}^{b} x(\tau)\, dG(\tau) \leq \int_{\nu}^{b} y(\tau)\, dG(\tau) \quad \text{for every} \quad \nu \in [a, b], \tag{1.1}$$

and

$$\int_{a}^{b} x(\tau)\, dG(\tau) = \int_{a}^{b} y(\tau)\, dG(\tau)$$

hold, then for every continuous convex function f, we have

$$\int_{a}^{b} f\left[x(\tau)\right]\, dG(\tau) \leq \int_{a}^{b} f\left[y(\tau)\right]\, dG(\tau). \tag{1.2}$$

(b) If (1.1) holds, then (1.2) holds for every continuous increasing convex function f.

Favard [7] proved the following result: Let f be a non-negative continuous concave function on $[a, b]$, not identically zero, and ϕ be a convex function on $[0, 2\widetilde{f}]$, where

$$\widetilde{f} = \frac{1}{b-a} \int_{a}^{b} f(x)\, dx.$$

Then

$$\int_{0}^{1} \phi\left(2\, s\, \widetilde{f}\right) = \frac{1}{2\widetilde{f}} \int_{0}^{2\widetilde{f}} \phi(y)\, dy \geq \frac{1}{b-a} \int_{a}^{b} \phi\left[f(x)\right]\, dx.$$

Favard [7] also proved a following result: Let f be a concave non-negative function on $[a, b] \subset \mathbb{R}$. If $q > 1$, then

$$\frac{1}{b-a} \int_{a}^{b} f^{q}(x) dx \leq \frac{2^{q}}{q+1} \left(\frac{1}{b-a} \int_{a}^{b} f(x) dx \right)^{q}.$$

Some generalizations of the Favard inequality and its reverse are also given in [9, pp.412-413]. Moreover, Berwald (1947) [4] proved the following generalization of Favard's inequality [9, pp.413-414]: Let f be a non-negative, continuous concave function, not identically zero on $[a, b]$, and ψ be a continuous and strictly monotonic function on $[0, y_0]$, where y_0 is sufficiently large. If $\overline{z}$ is the unique positive root of the equation

$$\frac{1}{\overline{z}} \int_{0}^{\overline{z}} \psi(y)\, dy = \frac{1}{b-a} \int_{a}^{b} \psi\left[f(x)\right]\, dx,$$

then for every function $\phi : [0, y_0] \to \mathbb{R}$ which is convex with respect to ψ, we have

$$\int_{0}^{1} \phi\left(s\, \overline{z}\right) = \frac{1}{\overline{z}} \int_{0}^{\overline{z}} \phi(y)\, dy \geq \frac{1}{b-a} \int_{a}^{b} \phi\left[f(x)\right]\, dx.$$

Berwald [4] also proved a following result: If f is a non-negative concave function on $[a, b]$, then for $0 < r < s$ we have

$$\left[\frac{s+1}{b-a}\int_a^b f^s(x)\,dx\right]^{\frac{1}{s}} \leq \left[\frac{r+1}{b-a}\int_a^b f^r(x)\,dx\right]^{\frac{1}{r}}. \tag{1.3}$$

Thunsdroff (1932) proved the following similar result [20]: If f is a non-negative, convex function with $f(a) = 0$, and if $0 < r < s$, then the reverse of the inequality in (1.3) is valid. In [10], some generalizations of Favard's and Berwald's inequalities to the weighted case are given. Other textbooks also include a section on majorization theory, e.g., [3], [9], [15] and [17].

Positive semi-definite matrices have a number of interesting properties. One of these is that all the eigenvalues of a positive semi-definite matrix are real and non-negative. Positive semi-definite matrices are very important in theory of inequalities. So in classical book [3] one of the five chapters (second chapter) is devoted to them. Of course as was noted in [3, p.59-61] a very important positive semi-definite matrix is Grammi matrix. The corresponding determinantal inequality is well known as Gram's inequality. In this paper we show that we can use majorization type results and weighted Favard's and Berwald's inequalities to obtain positive semi-definite matrices that is we can give determinantal form of these inequalities. Very specific form of these determinantal forms enable us to interpret our results in a form of exponentially convex functions([2], [6], [12] and [13], p. 373):

Definition 1.1. A function $h : (a, b) \to \mathbb{R}$ is exponentially convex function if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j\, h\,(x_i + x_j) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $i = 1, ..., n$ such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

Proposition 1.2. *Let $h : (a, b) \to \mathbb{R}$. The following propositions are equivalent.*

(i) *h is exponentially convex.*

(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j\, h\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for every $n \in \mathbb{N}$, for every $\xi_i \in \mathbb{R}$ and every $x_i \in (a, b)$, $1 \leq i \leq n$.

Proof. This follows from well-known Sylvester criterion applied to Definition 1.1. □

Corollary 1.3. *If ϕ is exponentially convex function, then*

$$\det\left[\phi(\frac{x_k + x_l}{2})\right]_{k,l=1}^{n} \geq 0$$

for every $n \in \mathbb{N}$, $x_k \in I$, $k = 1, 2, .., n$.

Corollary 1.4. *If $h : (a,b) \to \mathbb{R}^+$ is exponentially convex function then h is a* log-*convex function.*

In this paper, when only one of function is monotonic, then we give majorization type results. We also give generalizations of Favard's and Berwald's inequality and related results. The paper is organized in the following way: In Section 1 we give extension of majorization type results, generalizations of weighted Favard's and Berwald's inequality and related results. In Section 2 we prove positive semi-definiteness of matrices generated by differences deduced from majorization type results and differences deduced from weighted Favard's and Berwald's inequality. This implies a surprising property of exponentially convexity and log-convexity of this differences which allows us to deduce Lyapunov's inequalities for the differences, which are improvements of majorization type results and weighted Favard's and Berwald's inequalities. In Section 3 we introduce new Cauchy's means, as equivalent forms of exponentially convexity and log-convexity and also prove their monotonicity. In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2. Main Results

The following theorem is a slight extension of Lemma 2 in [10] which is proved by L. Maligranda, J. Pečarić, L. E. Persson (1995):

Theorem 2.1. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Suppose that $\varphi : [0,\infty) \to \mathbb{R}$ is a convex function and that*

$$\int_a^x f(t)\, w(t)\, dt \le \int_a^x g(t)\, w(t)\, dt, \quad x \in [a,b] \quad \text{and}$$

$$\int_a^b f(t)\, w(t)\, dt = \int_a^b g(t)\, w(t)\, dt.$$

(1) *If f is a decreasing function on $[a,b]$, then*

$$\int_a^b \varphi\left[f(t)\right]\, w(t)\, dt \le \int_a^b \varphi\left[g(t)\right]\, w(t)\, dt. \tag{2.1}$$

(2) *If g is an increasing function on $[a,b]$, then*

$$\int_a^b \varphi\left[g(t)\right]\, w(t)\, dt \le \int_a^b \varphi\left[f(t)\right]\, w(t)\, dt. \tag{2.2}$$

If φ is strictly convex function and $f \neq g$ (a. e.), then (2.1) and (2.2) are strict.

Proof. As in [10], if we prove the inequalities for $\varphi \in C^1[0,\infty)$, then the general case follows from the pointwise approximation of φ by smooth convex functions. Since φ is a convex function on $[0,\infty)$, it follows that

$$\varphi(u_1) - \varphi(u_2) \ge \varphi'(u_2)\,(u_1 - u_2).$$

If we set

$$F(x) = \int_a^x \left[g(t) - f(t)\right]\, w(t)\, dt,$$

then $F(x) \geq 0$, $x \in [a,b]$, and $F(a) = F(b) = 0$.
Then

$$\begin{aligned}
&\int_a^b \left[\varphi\left[g(t)\right] - \varphi\left[f(t)\right]\right] w(t)\, dt \\
&\quad \geq \int_a^b \varphi'\left[f(t)\right] \left[g(t) - f(t)\right] w(t)\, dt \\
&\quad = \int_a^b \varphi'\left[f(t)\right] dF(t) \\
&\quad = \left[\varphi'\left[f(t)\right] F(t)\right]_a^b - \int_a^b F(t)\, d\left[\varphi'\left[f(t)\right]\right] \\
&\quad = -\int_a^b F(t)\, \varphi''\left[f(t)\right] f'(t)\, dt \geq 0.
\end{aligned}$$

The last inequality follows from the convexity of φ and f being decreasing.
Similarly, we can prove the case when g is increasing.
If φ is strictly convex function and $f \neq g$ $(a.\,e.)$, then

$$\varphi\left[g(t)\right] - \varphi\left[f(t)\right] > \varphi'\left[f(t)\right] \left[g(t) - f(t)\right] \quad \text{(a. e.)}.$$

Which gives strict inequality in (2.1) and (2.2). □

The following Lemma is valid (see for instance [10]):

Lemma 2.2. *Let v be a weight function on $[a,b]$.*

(1) *If h is an increasing function on $[a,b]$, then*

$$\int_a^x h(t)\, v(t)\, dt \int_a^b v(t)\, dt \leq \int_a^b h(t)\, v(t)\, dt \int_a^x v(t)\, dt, \quad x \in [a,b].$$

(2) *If h is a decreasing function on $[a,b]$, then*

$$\int_a^b h(t)\, v(t)\, dt \int_a^x v(t)\, dt \leq \int_a^x h(t)\, v(t)\, dt \int_a^b v(t)\, dt, \quad x \in [a,b].$$

The following theorem is an extension of Theorem 3 in [18] which is proved by J. Pečarić and S. Abramovich (1997):

Theorem 2.3. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is a convex function.*

(1) *Let f/g be a decreasing function on $[a,b]$. If f is an increasing function on $[a,b]$, then*

$$\int_a^b \varphi\left(\frac{f(t)}{\int_a^b f(t)\, w(t)\, dt}\right) w(t)\, dt \leq \int_a^b \varphi\left(\frac{g(t)}{\int_a^b g(t)\, w(t)\, dt}\right) w(t)\, dt. \tag{2.3}$$

If g is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.3).

(2) *Let f/g be an increasing function on $[a, b]$. If g is an increasing function on $[a, b]$, then*

$$\int_a^b \varphi\left(\frac{g(t)}{\int_a^b g(t)\, w(t)\, dt}\right) w(t)\, dt \leq \int_a^b \varphi\left(\frac{f(t)}{\int_a^b f(t)\, w(t)\, dt}\right) w(t)\, dt. \tag{2.4}$$

If f is a decreasing function on $[a, b]$, then the reverse inequality holds in (2.4).

If φ is strictly convex function and $f \neq g$ (a. e.), then the strict inequality holds in (2.3), reverse inequality in (2.3), (2.4) and reverse inequality in (2.4).

Proof. (1) As in [18], using Lemma 2.2 with

$$v(t) = g(t)\, w(t), \qquad h(t) = f(t)/g(t),$$

we obtain

$$\int_a^b f(t)\, w(t)\, dt \int_a^x g(t)\, w(t)\, dt \leq \int_a^x f(t)\, w(t)\, dt \int_a^b g(t)\, w(t)\, dt, \quad x \in [a, b],$$

implies

$$\int_a^x \left(\frac{g(t)}{\int_a^b g(t)\, w(t)\, dt}\right) w(t)\, dt \leq \int_a^x \left(\frac{f(t)}{\int_a^b f(t)\, w(t)\, dt}\right) w(t)\, dt, \quad x \in [a, b].$$

By using Theorem 2.1 and f is increasing on $[a, b]$, we have

$$\int_a^b \varphi\left(\frac{f(t)}{\int_a^b f(t)\, w(t)\, dt}\right) w(t)\, dt \leq \int_a^b \varphi\left(\frac{g(t)}{\int_a^b g(t)\, w(t)\, dt}\right) w(t)\, dt.$$

Similarly, we can prove the case when g is decreasing.
(2) This case is equivalent to the first case switching the role of functions f and g.
Similarly as in Theorem 2.1 for strict inequality, we can get strict inequality in (2.3), reverse inequality in (2.3), (2.4) and reverse inequality in (2.4). □

Remark 2.4. Theorem 2.3 is an generalization of weighted Favard's inequality proved in [10]. This is a consequence of the simple fact that if $x \to \varphi(x)$ is a convex function, then $x \to \varphi(kx)$, $k \in \mathbb{R}$ is also a convex function and substitute $g(t) = t - a$ in (2.3), we have

$$\int_a^b \varphi[f(t)]\, w(t)\, dt \leq \int_a^b \varphi\Big(\frac{\int_a^b f(t)\, w(t)\, dt}{\int_a^b (t - a)\, w(t)\, dt}\,(t - a)\Big)\, w(t)\, dt.$$

If f is a positive increasing concave function, then we get weighted Favard's inequality which is proved by L. Maligranda, J. Pečarić, L. E. Persson (1995) [10].

As in [18], there are non-concave functions f, which satisfy assumption of Theorem 2.3. For instance,

$$f(x) = (1 + x^p)^{\frac{1}{p}}, \qquad p > 0, \qquad x \geq 0$$

is a non-concave function but $f(x)/x$ is a decreasing.
The following corollary is an application of Theorem 2.3.

Corollary 2.5. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Also let $\varphi(x) = x^p$, where $p > 1$ or $p < 0$.*

(1) *Let f/g be a decreasing function on $[a,b]$. If f is an increasing function on $[a,b]$, then*
$$\frac{\int_a^b f^p(t)\,w(t)\,dt}{\int_a^b g^p(t)\,w(t)\,dt} \le \left(\frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b g(t)\,w(t)\,dt}\right)^p. \tag{2.5}$$
If g is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.5).

(2) *Let f/g be an increasing function on $[a,b]$. If g is an increasing function on $[a,b]$, then*
$$\left(\frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b g(t)\,w(t)\,dt}\right)^p \le \frac{\int_a^b f^p(t)\,w(t)\,dt}{\int_a^b g^p(t)\,w(t)\,dt}. \tag{2.6}$$
If f is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.6).

If $\varphi(x) = x^p$, $0 < p < 1$, then the reverse inequality holds in (2.5), reverse inequality in (2.5), (2.6) and reverse inequality in (2.6).

Proof. Since f/g is a decreasing function on $[a,b]$ and f is an increasing function on $[a,b]$, then using Theorem 2.3 (2.3) and substitute $\varphi(x) = x^p$, where $p > 1$ or $p < 0$, we have
$$\int_a^b \left(\frac{f(t)}{\int_a^b f(t)\,w(t)\,dt}\right)^p w(t)\,dt \le \int_a^b \left(\frac{g(t)}{\int_a^b g(t)\,w(t)\,dt}\right)^p w(t)\,dt,$$
or equivalently
$$\frac{\int_a^b f^p(t)\,w(t)\,dt}{\int_a^b g^p(t)\,w(t)\,dt} \le \left(\frac{\int_a^b f(t)\,w(t)\,dt}{\int_a^b g(t)\,w(t)\,dt}\right)^p.$$
Similarly, we can prove the other cases. □

Remark 2.6. If we substitute $g(t) = t - a$, $w(t) \equiv 1$ and f is a positive increasing concave function in (2.5), then we have classical Favard's inequality (see [17, p. 212])
$$\frac{1}{b-a}\int_a^b f^p(t)\,dt \le \frac{2^p}{p+1}\Big(\frac{1}{b-a}\int_a^b f(t)\,dt\Big)^p.$$
The following theorem is a slight extension of Theorem 2 in [18] which is proved by J. Pečarić and S. Abramovich (1997):

Theorem 2.7. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Suppose $\varphi, \psi : [0,\infty) \to \mathbb{R}$ are such that ψ is a strictly* *increasing function and φ is a convex function with respect to ψ i.e., $\varphi \circ \psi^{-1}$ is convex. Suppose also that*

$$\int_a^x \psi\,[f(t)]\; w(t)\,dt \;\le\; \int_a^x \psi\,[g(t)]\; w(t)\,dt, \quad x \in [a,b], \quad \textit{and} \tag{2.7}$$

$$\int_a^b \psi\,[f(t)]\; w(t)\,dt \;=\; \int_a^b \psi\,[g(t)]\; w(t)\,dt. \tag{2.8}$$

(1) *If f is a decreasing function on $[a,b]$, then*

$$\int_a^b \varphi\,[f(t)]\; w(t)\,dt \;\le\; \int_a^b \varphi\,[g(t)]\; w(t)\,dt. \tag{2.9}$$

(2) *If g is an increasing function on $[a,b]$, then*

$$\int_a^b \varphi\,[g(t)]\; w(t)\,dt \;\le\; \int_a^b \varphi\,[f(t)]\; w(t)\,dt. \tag{2.10}$$

If $\varphi \circ \psi^{-1}$ is strictly convex function and $f \neq g$ (a.e.), then the strict inequality holds in (2.9) and (2.10).

Proof. Without loss of generality, it is sufficient to prove the case when $\psi(t) = t$, but this case is proved in Theorem 2.1. □

Theorem 2.8. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Suppose φ, $\psi : [0,\infty) \to \mathbb{R}$ are such that ψ is a continuous and strictly increasing function and φ is a convex function with respect to ψ i.e., $\varphi \circ \psi^{-1}$ is convex.*
Let z_1 be such that

$$\int_a^b \psi\,[z_1\, g(t)]\; w(t)\,dt \;=\; \int_a^b \psi\,[f(t)]\; w(t)\,dt. \tag{2.11}$$

(1) *Let f/g be a decreasing function on $[a,b]$. If f is an increasing function on $[a,b]$, then*

$$\int_a^b \varphi\,[f(t)]\; w(t)\,dt \;\le\; \int_a^b \varphi\,[z_1\, g(t)]\; w(t)\,dt. \tag{2.12}$$

If g is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.12).

(2) *Let f/g be an increasing function on $[a,b]$. If g is an increasing function on $[a,b]$, then*

$$\int_a^b \varphi\,[z_1\, g(t)]\; w(t)\,dt \;\le\; \int_a^b \varphi\,[f(t)]\; w(t)\,dt. \tag{2.13}$$

If f is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.13).

If $\varphi \circ \psi^{-1}$ is strictly convex function and $f \neq z_1\, g$ (a.e.), then the strict inequality holds in (2.12), reverse inequality in (2.12), (2.13) and reverse inequality in (2.13).

Proof. Since ψ is continuous, then $F(z) = \int_a^b \psi[z\, g(t)]\, w(t)\, dt$ is also continuous. Therefore by using $f > 0$ and ψ is strictly increasing, we have $F(0) = \int_a^b \psi(0)\, w(t)\, dt < \int_a^b \psi[f(t)]\, w(t)\, dt$. Since in both cases (1) and (2), f/g is bounded above, we take any $z_0 > f/g$ or $f < z_0\, g$. So, $F(z_0) = \int_a^b \psi[z_0\, g(t)]\, w(t)\, dt > \int_a^b \psi[f(t)]\, w(t)\, dt$. This shows the existence of z_1.

(1) As in [18], because f/g is decreasing, ψ is a strictly increasing function and

$$\int_a^b \psi\left[z_1\, g(t)\right]\, w(t)\, dt = \int_a^b \psi\left[f(t)\right]\, w(t)\, dt,$$

there is an $x_0 \in [a, b]$ such that

$$f(x)/g(x) \geq z_1,\ x \in [a, x_0] \quad \text{and} \quad f(x)/g(x) \leq z_1,\ x \in [x_0, b], \tag{2.14}$$

hence

$$\int_a^x \psi\left[z_1\, g(t)\right]\, w(t)\, dt \leq \int_a^x \psi\left[f(t)\right]\, w(t)\, dt,\ \ x \in [a, b]. \tag{2.15}$$

We give the proof of inequality (2.15) as in [10] for the convenience of a reader. If $a \leq x \leq x_0$, then the inequality (2.15) follows immediately from (2.14). If $x_0 \leq x \leq b$, then, by using equality (2.11) and the second inequality in (2.14), we have

$$\begin{aligned}
&\int_a^x \psi\left[z_1\, g(t)\right]\, w(t)\, dt \\
&\quad = \int_a^b \psi\left[z_1\, g(t)\right]\, w(t)\, dt - \int_x^b \psi\left[z_1\, g(t)\right]\, w(t)\, dt \\
&\quad = \int_a^b \psi\left[f(t)\right]\, w(t)\, dt - \int_x^b \psi\left[z_1\, g(t)\right]\, w(t)\, dt \\
&\quad \leq \int_a^b \psi\left[f(t)\right]\, w(t)\, dt - \int_x^b \psi\left[f(t)\right]\, w(t)\, dt \\
&\quad = \int_a^x \psi\left[f(t)\right]\, w(t)\, dt.
\end{aligned}$$

By using the inequality (2.15), the equality (2.11), the assumption that $\varphi \circ \psi^{-1}$ is convex, f is increasing and Theorem 2.7, we obtain

$$\int_a^b \varphi\left[f(t)\right]\, w(t)\, dt \leq \int_a^b \varphi\left[z_1\, g(t)\right]\, w(t)\, dt.$$

By using the inequality (2.15), the equality (2.11), the assumption that $\varphi \circ \psi^{-1}$ is convex, g is decreasing and Theorem 2.7, we obtain

$$\int_a^b \varphi\left[z_1\, g(t)\right]\, w(t)\, dt \leq \int_a^b \varphi\left[f(t)\right]\, w(t)\, dt.$$

(2) We can prove analogously the following inequality by using similar procedure as the first case

$$\int_a^x \psi\left[f(t)\right]\, w(t)\, dt \leq \int_a^x \psi\left[z_1\, g(t)\right]\, w(t)\, dt,\ \ x \in [a, b]. \tag{2.16}$$

By using the inequality (2.16), the equality (2.11), the assumption that $\varphi \circ \psi^{-1}$ is a convex function, g is increasing and Theorem 2.7, we obtain

$$\int_a^b \varphi\left[z_1\, g(t)\right]\, w(t)\, dt \leq \int_a^b \varphi\left[f(t)\right]\, w(t)\, dt.$$

By using the inequality (2.16), the equality (2.11), the assumption that $\varphi \circ \psi^{-1}$ is a convex function, f is decreasing and Theorem 2.7, we obtain

$$\int_a^b \varphi\left[f(t)\right]\, w(t)\, dt \leq \int_a^b \varphi\left[z_1\, g(t)\right]\, w(t)\, dt.$$

Similarly as in Theorem 2.7 for strict inequality, we can get strict inequality in (2.12), reverse inequality in (2.12), (2.13) and reverse inequality in (2.13). □

Remark 2.9. Theorem 2.8 is an extension of weighted Berwald's inequality. If we substitute $g(t) = (t-a)/(b-a)$ in (2.12), then

$$\int_a^b \varphi\left[f(t)\right]\, w(t)\, dt \;\leq\; \int_a^b \varphi\left(\frac{t-a}{b-a}\, z_1\right)\, w(t)\, dt.$$

If $z_1 > 0$, where z_1 is defined as in Theorem 2.8 and f is a positive increasing concave function, then we get weighted Berwald's inequality which is proved by L. Maligranda, J. Pečarić, L. E. Persson (1995) [10].

The following corollary is an application of Theorem 2.8.

Corollary 2.10. *Let w be a weight function on $[a,b]$ and let f and g be positive functions on $[a,b]$. Also let $\psi(x) = x^q$, $\varphi(x) = x^p$ such that $q \leq p$, $q \neq 0$, $p \neq 0$.*

(1) *Let f/g be a decreasing function on $[a,b]$. If f is an increasing function on $[a,b]$, then*

$$\left(\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b g^p(t)\, w(t)\, dt}\right)^{\frac{1}{p}} \;\leq\; \left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt}\right)^{\frac{1}{q}}. \tag{2.17}$$

If g is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.17).

(2) *Let f/g be an increasing function on $[a,b]$. If g is an increasing function on $[a,b]$, then*

$$\left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt}\right)^{\frac{1}{q}} \;\leq\; \left(\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b g^p(t)\, w(t)\, dt}\right)^{\frac{1}{p}}. \tag{2.18}$$

If f is a decreasing function on $[a,b]$, then the reverse inequality holds in (2.18).

Proof. Since f/g is a decreasing function on $[a,b]$ and f is an increasing function on $[a,b]$. For $q \leq p$, $q \neq 0$, $p > 0$, $\varphi(x) = x^p$ is a convex function with respect to $\psi(x) = x^q$, then using Theorem 2.8 (2.12), we have

$$\int_a^b f^p(t)\, w(t)\, dt \;\leq\; \int_a^b \left(z_1\, g(t)\right)^p\, w(t)\, dt.$$

Using (2.11), z_1 can be written as

$$z_1 = \left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt} \right)^{\frac{1}{q}}. \tag{2.19}$$

Substitute the value of z_1, we get

$$\int_a^b f^p(t)\, w(t)\, dt \le \left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt} \right)^{\frac{p}{q}} \int_a^b g^p(t)\, w(t)\, dt\,,$$

or equivalently

$$\left(\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b g^p(t)\, w(t)\, dt} \right)^{\frac{1}{p}} \le \left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt} \right)^{\frac{1}{q}}.$$

For $q \le p$, $q \ne 0$, $p < 0$, $\varphi(x) = x^p$ is a concave function with respect to $\psi(x) = x^q$, then using Theorem 2.8 (reverse inequality in (2.12)), we have

$$\int_a^b (z_1\, g(t))^p\; w(t)\, dt \le \int_a^b f^p(t)\, w(t)\, dt.$$

Substitute the value of z_1, we get

$$\left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt} \right)^{\frac{p}{q}} \int_a^b g^p(t)\, w(t)\, dt \le \int_a^b f^p(t)\, w(t)\, dt,$$

or equivalently

$$\left(\frac{\int_a^b f^p(t)\, w(t)\, dt}{\int_a^b g^p(t)\, w(t)\, dt} \right)^{\frac{1}{p}} \le \left(\frac{\int_a^b f^q(t)\, w(t)\, dt}{\int_a^b g^q(t)\, w(t)\, dt} \right)^{\frac{1}{q}}.$$

Similarly, we can prove the other cases. □

Remark 2.11. If we take $g(t) = t - a$, $w(t) \equiv 1$ and f is a positive concave function on $[a, b]$, then the decreasing rearrangement f^* is also concave function on $[a, b]$ (see [10]), and using Corollary (2.10) with f^*, we obtain

$$\left[\frac{p+1}{b-a} \int_a^b f^*(x)^p\, dx \right]^{\frac{1}{p}} \le \left[\frac{q+1}{b-a} \int_a^b f^*(x)^q\, dx \right]^{\frac{1}{q}}.$$

Equimeasurability of f with f^* then gives the classical Berwald inequality (1.3).

3. Exponentially Convexity, Lyapunov's and Dresher's type of inequalities

Throughout the paper we will frequently use the following family of convex functions with respect to $\psi(x) = x^q$ $(q > 0)$ on $(0, \infty)$:

$$\varphi_s(x) := \begin{cases} \frac{q^2}{s(s-q)}\, x^s, & s \ne 0, q; \\ -q\, \log x, & s = 0; \\ q\, x^q\, \log x, & s = q. \end{cases} \tag{3.1}$$

The following lemma is equivalent to the definition of convex function (see [17, p.2]).

Lemma 3.1. *If ϕ is convex on an interval $I \subseteq \mathbb{R}$, then*

$$\phi(s_1)\,(s_3 - s_2) + \phi(s_2)\,(s_1 - s_3) + \phi(s_3)\,(s_2 - s_1) \geq 0$$

holds for every $s_1 < s_2 < s_3$, $s_1, s_2, s_3 \in I$.

The following theorem gives positive semi-definite matrix, exponentially convex function and log-convex function for difference deduced from generalized Berwald's inequality given in Theorem 2.8 and also Lyapunov's inequality for this difference.

Theorem 3.2. *Let w be a weight function on $[a,b]$ and let f and g be two positive functions on $[a,b]$. Suppose f/g is a decreasing function on $[a,b]$, f is an increasing function on $[a,b]$ and*

$$\Omega_s := \begin{cases} \frac{q^2}{s(s-q)}\left[\left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b g^q(t)\,w(t)\,dt}\right)^{\frac{s}{q}} \int_a^b g^s(t)\,w(t)\,dt - \int_a^b f^s(t)\,w(t)\,dt\right], & s \neq 0, q; \\ -\log\left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b g^q(t)\,w(t)\,dt}\right) \int_a^b w(t)\,dt - q\int_a^b \log g(t)\,w(t)\,dt & \\ \qquad + q\int_a^b \log f(t)\,w(t)\,dt, & s = 0; \\ \left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b g^q(t)\,w(t)\,dt}\right) \log\left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b g^q(t)\,w(t)\,dt}\right) \int_a^b g^q(t)\,w(t)\,dt + & \\ \qquad q\left(\frac{\int_a^b f^q(t)\,w(t)\,dt}{\int_a^b g^q(t)\,w(t)\,dt}\right) \int_a^b g^q(t)\,\log g(t)\,w(t)\,dt & \\ \qquad - q\int_a^b f^q(t)\,\log f(t)\,w(t)\,dt, & s = q. \end{cases}$$

Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and every $s_1, ..., s_n \in \mathbb{R}$, the matrix $\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite, that is,*

$$\det\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0 \tag{3.2}$$

for $k = 1, ..., n$.

(b) *The function $s \to \Omega_s$ is exponentially convex.*

(c) *The function $s \to \Omega_s$ is a* log*-convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\Omega_s^{t-r} \leq \Omega_r^{t-s}\,\Omega_t^{s-r}. \tag{3.3}$$

Proof. (a) Consider the function

$$\phi(x) = \sum_{i,j}^{k} u_i u_j\,\varphi_{s_{ij}}(x)$$

for $k = 1, ..., n$, $x > 0$, $u_i \in \mathbb{R}$, $s_{ij} \in \mathbb{R}$, where $s_{ij} = \frac{s_i+s_j}{2}$ and $\varphi_{s_{ij}}$ is defined in (3.1).

Here, we shall show that $\phi(x)$ is convex with respect to $\psi(x) = x^q$ $(q > 0)$.
Set

$$F(x) = \phi(x^{\frac{1}{q}}) = \sum_{i,j}^{k} u_i u_j \, \varphi_{s_{ij}}(x^{\frac{1}{q}}).$$

We have

$$\begin{aligned} F''(x) &= \sum_{i,j}^{k} u_i u_j \, x^{\frac{s_{ij}}{q} - 2} \\ &= \left(\sum_{i}^{k} u_i \, x^{\frac{s_i}{2q} - 1} \right)^2 \geq 0, \;\; x > 0. \end{aligned}$$

Therefore, $\phi(x)$ is convex with respect to $\psi(x) = x^q$ $(q > 0)$ for $x > 0$. Using Theorem 2.8,

$$\int_a^b \phi\left[z_1 \, g(t)\right] \, w(t) \, dt \geq \int_a^b \phi\left[f(t)\right] \, w(t) \, dt,$$

where z_1 is given in (2.19), we have

$$\begin{aligned} &\int_a^b \left(\sum_{i,j}^{k} u_i u_j \, \varphi_{s_{ij}} \left[z_1 \, g(t)\right] \right) w(t) \, dt \\ &\qquad - \int_a^b \left(\sum_{i,j}^{k} u_i u_j \, \varphi_{s_{ij}} \left[f(t)\right] \right) w(t) \, dt \geq 0, \end{aligned}$$

or equivalently

$$\sum_{i,j}^{k} u_i u_j \, \Omega_{s_{ij}} \geq 0.$$

From last inequality, it follows that the matrix $\left[\Omega_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k}$ is a positive semi-definite matrix, that is, (3.2) is valid.
(b) Note that Ω_s is continuous for $s \in \mathbb{R}$ since

$$\lim_{s \to 0} \Omega_s = \Omega_0 \;\text{ and }\; \lim_{s \to q} \Omega_s = \Omega_q.$$

Then by using Proposition 1.2, we get exponentially convexity of the function $s \to \Omega_s$.
(c) For $k = 2$, (3.2) becomes

$$\Omega_{p_1} \, \Omega_{p_2} \geq \Omega_{p_{12}}^2 = \Omega_{\frac{p_1+p_2}{2}}^2,$$

that is Ω_s is log-convex in the Jensen sense for $s \in \mathbb{R}$.
Since Ω_s is continuous, therefore it is log-convex. We can also prove log-convexity by using Corollary 1.3. Since Ω_s is log-convex, i.e., $s \mapsto \log \Omega_s$ is convex, by Lemma 3.1 for $-\infty < r < s < t < \infty$, then we get

$$\log \Omega_s^{t-r} \leq \log \Omega_r^{t-s} + \log \Omega_t^{s-r},$$

which is equivalent to (3.3). $\square$

Remark 3.3. If we take $g(t) = t - a$, then Ω_s converts to Γ_s which is given in [14] and also if we take $g(t) = b - t$, then Ω_s converts to Φ_s which is also given in [14].

The following theorem gives the Dresher's inequality for difference deduced from generalized Berwald's inequality given in Theorem 2.8.

Theorem 3.4. *Let Ω_s be defined as in* Theorem 3.2 *and $t, s, u, v \in \mathbb{R}$ such that $s \leq u, t \leq v, s \neq t, u \neq v$. Then*

$$\left(\frac{\Omega_t}{\Omega_s}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Omega_v}{\Omega_u}\right)^{\frac{1}{v-u}}. \tag{3.4}$$

Proof. Similar to the proof of Theorem 2.2 in [14]. □

Remark 3.5. Similarly as in Theorem 3.2 and Theorem 3.4, we can get positive semi-definite matrices, exponentially convex functions, log-convex functions, Lyapunov's inequalities and Dresher's inequalities for the cases when f/g is a decreasing function and g is a decreasing function, f/g is an increasing function and f is a decreasing function, and f/g is an increasing function and g is an increasing function by using Theorem 2.8.

The following theorem gives positive semi-definite matrix, exponentially convex function and log-convex function for difference deduced from majorization type results given in Theorem 2.7 and also Lyapunov's inequality for this difference.

Theorem 3.6. *Let w be a weight function on $[a, b]$ and let f and g be two positive functions on $[a, b]$. Suppose f is a decreasing function on $[a, b]$ and*

$$\overline{\Gamma}_s := \begin{cases} \frac{q^2}{s(s-q)} \left[\int_a^b g^s(t)\, w(t)\, dt \,-\, \int_a^b f^s(t)\, w(t)\, dt\right], & s \neq 0, q; \\ q \left[\int_a^b \log f(t)\, w(t)\, dt \,-\, \int_a^b \log g(t)\, w(t)\, dt\right], & s = 0; \\ q \left[\int_a^b g^q(t)\, \log g(t)\, w(t)\, dt \,-\, \int_a^b f^q(t)\, \log f(t)\, w(t)\, dt\right], & s = q, \end{cases}$$

such that conditions (2.7) and (2.8) are satisfied. Then the following statements are valid:

(a) *For every $n \in \mathbb{N}$ and every $s_1, \ldots, s_n \in \mathbb{R}$, the matrix $\left[\overline{\Gamma}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^n$ is a positive semi-definite, that is,*

$$\det\left[\overline{\Gamma}_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^k \geq 0$$

for $k = 1, \ldots, n$.

(b) *The function $s \to \overline{\Gamma}_s$ is exponentially convex.*

(c) *The function $s \to \overline{\Gamma}_s$ is a* log-*convex on $\mathbb{R}$ and the following inequality holds for $-\infty < r < s < t < \infty$:*

$$\left(\overline{\Gamma}_s\right)^{t-r} \leq \left(\overline{\Gamma}_r\right)^{t-s} \left(\overline{\Gamma}_t\right)^{s-r}.$$

Proof. As in the proof of Theorem 3.2, we use Theorem 2.7 instead of Theorem 2.8. □

The following theorem gives the Dresher's inequality for difference deduced from majorization type results given in Theorem 2.7.

Theorem 3.7. *Let* $\overline{\Gamma}_s$ *be defined as in* Theorem 3.6 *and* $t, s, u, v \in \mathbb{R}$ *such that* $s \leq u, t \leq v, s \neq t, u \neq v$. *Then*

$$\left(\frac{\overline{\Gamma}_t}{\overline{\Gamma}_s}\right)^{\frac{1}{t-s}} \leq \left(\frac{\overline{\Gamma}_v}{\overline{\Gamma}_u}\right)^{\frac{1}{v-u}}. \tag{3.5}$$

Proof. Similar to the proof of Theorem 2.2 in [14]. □

Remark 3.8. Similarly as in Theorem 3.6 and Theorem 3.7, we can get positive semi-definite matrix, exponentially convex function, log-convex function, Lyapunov's inequality and Dresher's inequality the case when g is an increasing function given in Theorem 2.7.

Remark 3.9. We can get positive semi-definite matrices, exponentially convex functions, log-convex functions and Lyapunov's inequalities for differences deduced from generalized Favard's inequality (see Theorem 2.3) and majorization type results (see Theorem 2.1) by substituting $q = 1$ in Theorem 3.2 and Theorem 3.6 respectively. We can also get Dresher's inequalities for differences deduced from generalized Favard's inequality and majorization type results by substituting $q = 1$ in Theorem 3.4 and Theorem 3.7 respectively.

4. **Mean Value Theorems**

Let us note that (3.4) and (3.5) have the form of some known inequalities between means (eg. Stolarsky means, Gini means, etc). Here we will prove that expressions on both sides of (3.4) and (3.5) are also means. The proofs in the remaining cases are analogous.

Theorem 4.1. *Let* w *be a weight function on* $[a, b]$, f *and* g *be two positive functions on* $[a, b]$, $\psi \in C^2([0, \infty))$ *and* $\varphi \in C^2([0, z_1])$. *Let* f/g *be a decreasing function on* $[a, b]$ *and* f *be an increasing function on* $[a, b]$. *Also let* $\psi'(y) > 0$ *for* $y \in [0, z_1]$ *and* z_1 *is defined as in* Theorem 2.8. *Then there exists* $\xi \in [0, z_1]$ *such that*

$$\begin{aligned}
&\int_a^b \varphi\left[z_1\, g(t)\right]\, w(t)\, dt \;-\; \int_a^b \varphi\left[f(t)\right]\, w(t)\, dt \\
=\;& \frac{\psi'(\xi)\; \varphi''(\xi) \;-\; \varphi'(\xi)\; \psi''(\xi)}{2\, \left(\psi'(\xi)\right)^3} \Big[\int_a^b \psi^2\left[z_1\, g(t)\right]\, w(t)\, dt \\
-\;& \int_a^b \psi^2\left[f(t)\right]\, w(t)\, dt\Big].
\end{aligned}$$

Proof. Similar to the proof of Theorem 4.2 in [14]. □

Theorem 4.2. *Let* w *be a weight function on* $[a, b]$, f *and* g *be two positive functions on* $[a, b]$, $\psi \in C^2([0, \infty))$ *and* $\varphi_1, \varphi_2 \in C^2([0, z_1])$. *Let* f/g *be a decreasing function on* $[a, b]$ *and* f *be an increasing function on* $[a, b]$. *Also let* $\psi'(y) > 0$ *for*

$y \in [0, z_1]$ and $f \neq z_1\, g\,(a.\,e.)$, where z_1 is defined as in Theorem 2.8*. Then there exists $\xi \in [0, z_1]$ such that*

$$\frac{\psi'(\xi)\,\varphi_1''(\xi)\;-\;\varphi_1'(\xi)\,\psi''(\xi)}{\psi'(\xi)\,\varphi_2''(\xi)\;-\;\varphi_2'(\xi)\,\psi''(\xi)} \;=\; \frac{\int_a^b \varphi_1\left[z_1\,g(t)\right]\,w(t)\,dt\;-\;\int_a^b \varphi_1\left[f(t)\right]\,w(t)\,dt}{\int_a^b \varphi_2\left[z_1\,g(t)\right]\,w(t)\,dt\;-\;\int_a^b \varphi_2\left[f(t)\right]\,w(t)\,dt} \tag{4.1}$$

provided that $\psi'(y)\,\varphi_2''(y)\;-\;\varphi_2'(y)\,\psi''(y)\;\neq\;0$ for every $y \in [0, z_1]$.

Proof. Similar to the proof of Theorem 4.3 in [14]. □

Corollary 4.3. *Let w be a weight function on $[a, b]$ and let f and g be two positive functions on $[a, b]$. Also let f/g be a decreasing function on $[a, b]$, f be an increasing function on $[a, b]$ and $f \neq z_1\, g\,(a.\,e.)$, where z_1 is defined as in* Theorem 2.8 *for $\psi(x) = x^q$ $(q > 0)$ or explicitly z_1 is given in* (2.19)*, then for distinct $s, t, q \in \mathbb{R}\backslash\{0\}$, there exists $\xi \in (0, z_1]$ such that*

$$\xi^{t-s} \;=\; \frac{s(s-q)}{t(t-q)}\,\frac{\int_a^b \left(z_1\,g(r)\right)^t\,w(r)\,dr\;-\;\int_a^b f^t(r)\,w(r)\,dr}{\int_a^b \left(z_1\,g(r)\right)^s\,w(r)\,dr\;-\;\int_a^b f^s(r)\,w(r)\,dr}. \tag{4.2}$$

Proof. Set $\varphi_1(x) = x^t$, $\varphi_2(x) = x^s$ and $\psi(x) = x^q$, $t \neq s \neq 0, q$ in (4.1), then we get (4.2). □

Remark 4.4. Since the function $\xi \to \xi^{t-s}$ is invertible, then from (4.2) we have

$$0 \;<\; \left(\frac{s(s-q)}{t(t-q)}\,\frac{\int_a^b \left(z_1\,g(r)\right)^t\,w(r)\,dr\;-\;\int_a^b f^t(r)\,w(r)\,dr}{\int_a^b \left(z_1\,g(r)\right)^s\,w(r)\,dr\;-\;\int_a^b f^s(r)\,w(r)\,dr}\right)^{\frac{1}{t-s}} \;\leq\; z_1. \tag{4.3}$$

In fact, a similar result can also be given for (4.1). Namely, suppose that $\Lambda(y) = \left(\psi'(y)\,\varphi_1''(y)\;-\;\varphi_1'(y)\,\psi''(y)\right)/\left(\psi'(y)\,\varphi_2''(y)\;-\;\varphi_2'(y)\,\psi''(y)\right)$ has inverse function. It follows from (4.1) that

$$\xi \;=\; \Lambda^{-1}\left(\frac{\int_a^b \varphi_1\left[z_1\,g(r)\right]\,w(r)\,dr\;-\;\int_a^b \varphi_1\left[f(r)\right]\,w(r)\,dr}{\int_a^b \varphi_2\left[z_1\,g(r)\right]\,w(r)\,dr\;-\;\int_a^b \varphi_2\left[f(r)\right]\,w(r)\,dr}\right).$$

By the inequality (4.3), we can consider

$$M_{t,s} \;=\; \left(\frac{\Omega_t}{\Omega_s}\right)^{\frac{1}{t-s}} \quad \text{for } \; s, t \in \mathbb{R}$$

as means in broader sense. Moreover we can extend these means in other cases. So by limit we have

$$\begin{aligned}
&\log M_{s,s} \;= \\
&\quad \frac{z_1^s\,\log z_1\,\int_a^b g^s(r)\,w(r)\,dr\;+\;z_1^s\,\int_a^b g^s(r)\,\log g(r)\,w(r)\,dr}{z_1^s\,\int_a^b g^s(r)\,w(r)\,dr\;-\;\int_a^b f^s(r)\,w(r)\,dr} \\
&\quad -\,\frac{\int_a^b f^s(r)\,\log f(r)\,w(r)\,dr}{z_1^s\,\int_a^b g^s(r)\,w(r)\,dr\;-\;\int_a^b f^s(r)\,w(r)\,dr}\;-\;\frac{2s\;-\;q}{s\,(s\;-\;q)}, \quad s \neq 0, q.
\end{aligned}$$

$$\log M_{q,q} = \frac{z_1^q \log^2 z_1^q \frac{1}{q^2} \int_a^b g^q(r)\, w(r)\, dr + 2\, z_1^q \log z_1 \int_a^b g^q(r) \log g(r)\, w(r)\, dr}{2 \left(z_1^q \log z_1 \int_a^b g^q(r)\, w(r)\, dr + z_1^q \int_a^b g^q(r) \log g(r)\, w(r)\, dr - \int_a^b f^q(r) \log f(r)\, w(r)\, dr \right)}$$
$$+ \frac{z_1^q \int_a^b g^q(r) \log^2 g(r)\, w(r)\, dr - \int_a^b f^q(r) \log^2 f(r)\, w(r)\, dr}{2 \left(z_1^q \log z_1 \int_a^b g^q(r)\, w(r)\, dr + z_1^q \int_a^b g^q(r) \log g(r)\, w(r)\, dr - \int_a^b f^q(r) \log f(r)\, w(r)\, dr \right)}$$
$$- \frac{1}{q}.$$

$$\log M_{0,0} = \frac{\log^2 z_1^q \frac{1}{q^2} \int_a^b w(r)\, dr + 2 \log z_1 \int_a^b \log g(r)\, w(r)\, dr}{2 \left(\log z_1 \int_a^b w(r)\, dr + \int_a^b \log g(r)\, w(r)\, dr + \int_a^b \log f(r)\, w(r)\, dr \right)}$$
$$+ \frac{\int_a^b \log^2 g(r)\, w(r)\, dr + \int_a^b \log^2 f(r)\, w(r)\, dr}{2 \left(\log \gamma \frac{1}{q} \int_a^b w(r)\, dr + \int_a^b \log g(r)\, w(r)\, dr + \int_a^b \log f(r)\, w(r)\, dr \right)} + \frac{1}{q}.$$

Theorem 4.5. *Let $t \leq u$, $r \leq s$, then the following inequality is valid*

$$M_{t,r} \leq M_{u,s}. \tag{4.4}$$

Proof. By similar procedure as in the proof of Theorem 4.6 in [14], since Ω_s is log-convex, we get (3.4) and (4.4) follows immediately from (3.4). □

Denote,

$$m_{f,g} = \min\{m_f, m_g\} \quad \text{and} \quad M_{f,g} = \max\{M_f, M_g\},$$

where, m_f and m_g denote minimums of f and g respectively, and M_f and M_g denote maximums of f and g respectively.

Theorem 4.6. *Let w be a weight function on $[a, b]$, f and g be two positive functions on $[a, b]$ such that conditions (2.7) and (2.8) are satisfied, $\psi \in C^2([0, \infty))$ and $\varphi \in C^2([m_{f,g}, M_{f,g}])$. Also let f be a decreasing function on $[a, b]$ and $\psi'(y) > 0$ for $y \in [m_{f,g}, M_{f,g}]$. Then there exists $\xi \in [m_{f,g}, M_{f,g}]$ such that*

$$\int_a^b \varphi[g(t)]\, w(t)\, dt - \int_a^b \varphi[f(t)]\, w(t)\, dt$$
$$= \frac{\psi'(\xi)\, \varphi''(\xi) - \varphi'(\xi)\, \psi''(\xi)}{2\, (\psi'(\xi))^3} \Big[\int_a^b \psi^2[g(t)]\, w(t)\, dt$$
$$- \int_a^b \psi^2[f(t)]\, w(t)\, dt \Big].$$

Proof. Similar to the proof of Theorem 4.2 in [14]. □

Theorem 4.7. *Let w be a weight function on $[a, b]$, f and g be two positive functions on $[a, b]$ such that conditions (2.7) and (2.8) are satisfied, $\psi \in C^2([0, \infty))$ and $\varphi_1, \varphi_2 \in C^2([m_{f,g}, M_{f,g}])$. Also let f be a decreasing function on $[a, b]$,*

$\psi'(y) > 0$ for $y \in [m_{f,g}, M_{f,g}]$ and $f \neq g\,(a.e.)$. Then there exists $\xi \in [m_{f,g}, M_{f,g}]$ such that

$$\frac{\psi'(\xi)\,\varphi_1''(\xi) - \varphi_1'(\xi)\,\psi''(\xi)}{\psi'(\xi)\,\varphi_2''(\xi) - \varphi_2'(\xi)\,\psi''(\xi)} = \frac{\int_a^b \varphi_1\,[g(t)]\;w(t)\,dt - \int_a^b \varphi_1\,[f(t)]\;w(t)\,dt}{\int_a^b \varphi_2[g(t)]\,w(t)\,dt - \int_a^b \varphi_2\,[f(t)]\;w(t)\,dt} \tag{4.5}$$

provided that $\psi'(y)\,\varphi_2''(y) - \varphi_2'(y)\,\psi''(y) \neq 0$ for every $y \in [m_{f,g}, M_{f,g}]$.

Proof. Similar to the proof of Theorem 4.3 in [14]. □

Corollary 4.8. *Let w be a weight function on $[a,b]$ and let f and g be two positive functions on $[a,b]$ such that conditions (2.7) and (2.8) are satisfied. Also let f be a decreasing function on $[a,b]$ and $f \neq g\,(a.e.)$, then for distinct $s,t,q \in \mathbb{R}\backslash\{0\}$, there exists $\xi \in [m_{f,g}, M_{f,g}]$ such that*

$$\xi^{t-s} = \frac{s(s-q)}{t(t-q)}\,\frac{\int_a^b g^t(r)\,w(r)\,dr - \int_a^b f^t(r)\,w(r)\,dr}{\int_a^b g^s(r)\,w(r)\,dr - \int_a^b f^s(r)\,w(r)\,dr}. \tag{4.6}$$

Proof. Set $\varphi_1(x) = x^t$, $\varphi_2(x) = x^s$ and $\psi(x) = x^q$, $t \neq s \neq 0, q$ in (4.5), then we get (4.6). □

Remark 4.9. Since the function $\xi \to \xi^{t-s}$ is invertible, then from (4.6) we have

$$m_{f,g} \le \left(\frac{s(s-q)}{t(t-q)}\,\frac{\int_a^b g^t(r)\,w(r)\,dr - \int_a^b f^t(r)\,w(r)\,dr}{\int_a^b g^s(r)\,w(r)\,dr - \int_a^b f^s(r)\,w(r)\,dr}\right)^{\frac{1}{t-s}} \le M_{f,g}. \tag{4.7}$$

In fact, similar result can also be given for (4.5). Namely, suppose that $\Lambda(y) = (\psi'(y)\,\varphi_1''(y) - \varphi_1'(y)\,\psi''(y))\,/\,(\psi'(y)\,\varphi_2''(y) - \varphi_2'(y)\,\psi''(y))$ has inverse function. It follows from (4.5) that

$$\xi = \Lambda^{-1}\left(\frac{\int_a^b \varphi_1\,[g(r)]\;w(r)\,dr - \int_a^b \varphi_1\,[f(r)]\;w(r)\,dr}{\int_a^b \varphi_2\,[g(r)]\;w(r)\,dr - \int_a^b \varphi_2\,[f(r)]\;w(r)\,dr}\right).$$

By the inequality (4.7), we can consider

$$\overline{M}_{t,s} == \left(\frac{\overline{\Gamma}_t}{\overline{\Gamma}_s}\right)^{\frac{1}{t-s}} \quad \text{for } s,t \in \mathbb{R}, \;\; s \neq t,$$

as means in broader sense. Moreover we can extend these means in other cases. So by limit we have

$$\log \overline{M}_{t,s} = \frac{\int_a^b g^s(r)\,\log g(r)\,w(r)\,dr - \int_a^b f^s(r)\,\log f(r)\,w(r)\,dr}{\int_a^b g^s(r)\,w(r)\,dr - \int_a^b f^s(r)\,w(r)\,dr} - \frac{2s-q}{s(s-q)}, \quad s \neq q.$$

$$\log \overline{M}_{q,q} = \frac{\int_a^b g^q(r)\,\log^2 g(r)\,w(r)\,dr - \int_a^b f^q(r)\,\log^2 f(r)\,w(r)\,dr}{2\left[\int_a^b g^q(r)\,\log g(r)\,w(r)\,dr - \int_a^b f^q(r)\,\log f(r)\,w(r)\,dr\right]} - \frac{1}{q}.$$

$$\log \overline{M}_{0,0} = \frac{\int_a^b \log^2 g(r)\, w(r)\, dr \;-\; \int_a^b \log^2 f(r)\, w(r)\, dr}{2\left[\int_a^b \log g(r)\, w(r)\, dr \;-\; \int_a^b \log f(r)\, w(r)\, dr\right]} + \frac{1}{q}.$$

Theorem 4.10. *Let $t \le u$, $r \le s$, then the following inequality is valid*

$$\overline{M}_{t,r} \le \overline{M}_{u,s}. \tag{4.8}$$

Proof. By similar procedure as in the proof of Theorem 4.6 in [14], since $\overline{\Gamma}_s$ is log-convex, we get (3.5) and (4.8) follows immediately from (3.5). □

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