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# COMPOSITION OPERATORS ACTING BETWEEN SOME WEIGHTED MÖBIUS INVARIANT SPACES

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ABSTRACT. In this paper we investigate conditions under which a holomorphic self-map of the unit disk induces a composition operator $C_\phi$ with closed range on the weighted Bloch space $\mathcal{B}_{\log}$. Also, we introduce a new class of functions the so called $F_{\log}(p,q,s)$ spaces. Necessary and sufficient conditions are given for a composition operator $C_\phi$ to be bounded and compact from $\mathcal{B}_{\log}$ to $F_{\log}(p,q,s)$. Moreover, necessary and sufficient conditions for $C_\phi$ from the Dirichlet space $\mathcal{D}$ to the spaces $F_{\log}(p,q,s)$ to be compact are given in terms of the map $\phi$..

## 1. Introduction and preliminaries

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\Delta$ it's boundary, H($\Delta$) be the class of all analytic functions on $\Delta$ and $dA(z)$ the normalized area measure. For each $w \in \Delta$, let $\varphi_w(z)$ denote the Möbius transformations of $\Delta$

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}, \quad for\ z \in \Delta.$$

Let $Aut(\Delta)$ be the group of all conformal automorphisms of $\Delta$. The pseudo-hyperbolic distance between $z$ and $w$ is given by $\sigma(z,w) = |\varphi_z(w)|$. The pseudo-hyperbolic distance is Möbius invariant, that is,

$$\sigma(gz, gw) = \sigma(z,w),$$

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for all $g \in Aut(\Delta)$, the Möbius group of $\Delta$, and all $z, w \in \Delta$. It has the following useful property:

$$1-(\sigma(z,w))^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} = (1-|z|^2)|\varphi_z'(w)|.$$

For $0<\alpha<\infty$, the spaces of all analytic functions $f$ on $\Delta$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\Delta}(1-|z|^2)^\alpha |f'(z)| < \infty,$$

are called $\alpha$-Bloch spaces (see [27]). The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$ (see [4]).

The classical Dirichlet space $\mathcal{D}$ is the space of all functions $f \in \mathcal{D}$ such that

$$\|f\|_{\mathcal{D}}^2 = \int_\Delta |f'(z)|^2 dA(z) < \infty.$$

For $p, s \in (0,\infty)$, $-2<q<\infty$ and $q+s>-1$. An analytic function $f : \Delta \to \mathbb{C}$ defined in the unit disk $\Delta$ belongs to the spaces $F(p,q,s)$ (see [26]) if

$$\|f\|_{F(p,q,s)}^p = \sup_{a\in\Delta}\int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty$$

where $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$ is the Green's function with logarithmic singularity at $a \in \Delta$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad for\ z \in \Delta$. For more information about $F(p,q,s)$ spaces, we refer to [26].

For $0<\alpha<\infty$, the space of analytic functions $f \in \Delta$ such that

$$\|f\|_{\mathcal{B}^\alpha_{\log}} = \sup_{z\in\Delta}(1-|z|^2)^\alpha\left(\log\frac{2}{1-|z|^2}\right)|f'(z)| < \infty,$$

is called weighted $\alpha$-Bloch space $\mathcal{B}^\alpha_{\log}$ (see [16]). If $\alpha = 1$ the space $\mathcal{B}^\alpha_{\log}$ is just the weighted Bloch space $\mathcal{B}_{\log}$. The little weighted Bloch space $\mathcal{B}^\alpha_{\log,0}$ is a subspace of $\mathcal{B}^\alpha_{\log}$ consisting of all $f \in \mathcal{B}^\alpha_{\log}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha\left(\log\frac{2}{1-|z|^2}\right)|f'(z)| = 0.$$

Now, let $0<h<1$, $0\le\theta<2\pi$, and

$$\Omega(h,\theta) = \{re^{it} : 1-h<r<1\} \text{ and } |t-\theta|<h\},$$

$$S(h,\theta) = \{re^{it} : |re^{it}-re^{i\theta}|<h\}.$$

A positive measure $\mu$ on $\Delta$ is a Carleson measure if there is a constant $A$ with

$$\mu(S(h,\theta)) \le A\,h, \text{ where } 0<h<1 \text{ and } 0\le\theta<2\pi.$$

For $0<s<\infty$, we say that a positive measure $\mu$ defined on $\Delta$ is a bounded s-Carleson measure (see [5, 26]) provided $\mu(S(I)) = O(|I|^s)$ for all subarcs $I$ of $\partial\Delta$, where $|I|$ denotes the arc length of $I \subset \partial\Delta$ and $S(I)$ denotes the Carleson box based on $I$, that is,

$$S(I) = \left\{z\in\Delta : \frac{z}{|z|}\in I,\ 1-|z| \le \frac{|I|}{2\pi}\right\}.$$

If $\mu(S(I)) = o(|I|^s)$ as $|I| \to 0$, then we say that $\mu$ is a compact s-Carleson measure.
A positive Borel measure $\mu$ on $\Delta$ is called an s-logarithmic, p-Carleson measure $(p, s > 0)$ if

$$\sup_{I \subseteq \partial\Delta} \frac{\mu(S(I))(\log \frac{2}{|I|})^p}{|I|^p} < \infty.$$

In [28] it is proved that $\mu$ is an s-logarithmic, p-Carleson measure on $\Delta$ if and only if

$$\sup_{a\in\Delta}\left(\log \frac{2}{1-|a|^2}\right)^s \int_\Delta |\varphi_a'(z)|^p d\mu(z) < \infty.$$

**Definition 1.1.** For $p, s \in (0, \infty)$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\Delta)$ is said to belong to $F_{\log}(p, q, s)$ if

$$\|f\|^p_{F_{\log}(p,q,s)} = \sup_{I\subset\partial\Delta} \frac{(\log \frac{2}{|I|})^p}{|I|^s} \int_{S(I)} |f'(z)|^p (1-|z|^2)^q \big(\log \frac{1}{|z|}\big)^s dA(z) < \infty.$$

By the same proof as done in [26] and for $1 < p < \infty, -2 < q < \infty, 1 < s < \infty$, it is easy to see that $F_{\log}(p, q, s)$ are Banach spaces under the norm

$$\|f\|_{F_{\log}(p,q,s)} = |f(0)| + \sup_{a\in\Delta}\left(\log \frac{2}{1-|a|^2}\right)\left\{\int_\Delta |f'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\right\}^{\frac{1}{p}}.$$

*Remark* 1.2. The interest in the $F_{\log}(p, q, s)$ spaces arises from the fact they cover some well known function spaces, it is immediate that $F_{\log}(2, 0, 1) = BMOA_{\log}$ (see [3, 8]). Also, $F_{\log}(2, 0, p) = \mathcal{Q}^p_{\log}$, where $0 < p < \infty$ (see [11]).

The composition operator $C_\phi : H(\Delta) \to H(\Delta)$ is defined by $C_\phi = f \circ \phi$.
There have been several attempts to study compactness and boundedness of composition operators in many function spaces (see e.g. [1, 2, 6, 7, 9, 10, 13, 14, 15, 17, 18, 30] and others). There are also some studies in several complex variables (see e.g. [21, 24, 29] and others). Most of the previous work in the theory of composition operators dealt with their compactness, relating it to classical function theory. On the other hand there are some studies of closed range composition operators (see [12, 19, 31, 32] and others).
In this paper, we determine when the composition operator $C_\phi$ has a closed range on the weighted Bloch space $\mathcal{B}_{\log}$ and we give a set of necessary conditions and a partial converse for $C_\phi$ on the weighted Bloch space $\mathcal{B}_{\log}$. Also, we characterize boundedness and compactness of the composition operators $C_\phi : \mathcal{B}^\alpha_{\log} \to F_{\log}(p, q, s)$. Finally, we consider the composition operators from the Dirichlet space $\mathcal{D}$ into $F_{\log}(p, q, s)$ spaces.

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that $||T(f)||_Y \le M||f||_X$ for all maps $f \in X$. Moreover, $T : X \to Y$ is said to be compact if it takes bounded sets in $X$ to sets in $Y$ which have compact closure. For Banach spaces $X$ and $Y$ of $H(\Delta)$, $T$ is compact from $X$ to $Y$ if and only if for each bounded sequence $\{x_n\} \in X$, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to some limit in $Y$.

Two quantities $A_f$ and $B_f$, both depending on an analytic function $f$ on $\Delta$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant $C$ not depending on $f$ such that for every analytic function $f$ on $\Delta$ we have:

$$\frac{1}{C}B_f \leq A_f \leq CB_f.$$

If the quantities $A_f$ and $B_f$, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

## 2. Composition operator with closed range on $\mathcal{B}_{\log}$ space

Let $\phi$ be a holomorphic self-map of the unit disk $\Delta$. We write $G = \phi(\Delta)$, and $\tau_\phi(z)$ is defined by

$$\tau_\phi(z) = \frac{(1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)|\phi'(z)|}{(1-|\phi(z)|^2)\big(\log\frac{2}{1-|\phi(z)|^2}\big)}.$$

Yoneda in [25] proved the following results:

**Theorem 2.1.** *Let $\phi$ be a holomorphic function taking $\Delta$ into $\Delta$. Then $C_\phi$ is bounded on $\mathcal{B}_{\log}$ if and only if*

$$\sup_{z\in\Delta}\left(\frac{(1-|z|^2)\log\frac{2}{1-|z|^2}}{(1-|\phi(z)|^2)\log\frac{2}{1-|\phi(z)|^2}}|\phi'(z)|\right) < +\infty.$$

**Lemma 2.2.** *If $C_\phi$ is bounded on $\mathcal{B}_{\log}$, then for all $f \in \mathcal{B}_{\log}$,*

$$\|f\|_{\mathcal{B}_{\log}} \leq k\left\{\sup(1-|w|^2)\left(\log\frac{2}{1-|w|^2}\right)|f'(w)|, w \in G\right\}$$

*for some constant $k$.*

Now, we give the following result:

**Theorem 2.3.** *If $C_\phi$ is bounded below on $\mathcal{B}_{\log}$, then there exist positive constants $\varepsilon, r$ with $r < 1$ such that, for all $z \in \Delta$, $\sigma(\phi(\Omega_\varepsilon), z) \leq r$ where*

$$\Omega_\varepsilon = \{z \in \Delta, |\tau_\phi(z)| > \varepsilon\}.$$

*Proof.* Since $C_\phi : \mathcal{B}_{\log} \to \mathcal{B}_{\log}$ is bounded below, then there is a constant $k$, $\log 2 < k \leq 1$ such that

$$\|C_\phi f\|_{\mathcal{B}_{\log}} \geq k\|f\|_{\mathcal{B}_{\log}}$$

for $f \in \mathcal{B}_{0,\log}$, for each $w \in \Delta$, let

$$f_w(z) = \frac{w-z}{1-\bar{w}z} - \frac{w-\phi(0)}{1-\bar{w}\phi(0)}.$$

Clearly, $f_w(z)$ is a bounded and continuous analytic function on the closed unit disk and so is in $\mathcal{B}_{0,\log}$. Moreover an easy computation gives $\|f_w\|_{\log} \geq 1$. Thus

$$\|C_\phi f_w\|_{\mathcal{B}_{\log}} \geq k\|f_w\|_{\mathcal{B}_{\log}} \geq k.$$

On the other hand, we also have

$$\begin{aligned}
&(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)|(C_\phi f_w)'(z)| \\
= \;\; &(1-|\varphi_w(\phi(z))|^2)\left(\log\frac{2}{1-|\varphi_w(\phi(z))|^2}\right)|\tau_\phi(z)|,
\end{aligned}$$

and $C_\phi f_w(0)=0$. Then there is a point $z_w\in\Delta$ such that

$$\begin{aligned}
\|C_\phi f_w\|_{\mathcal{B}_{\log}} &\geq (1-|z_w|^2)\left(\log\frac{2}{1-|z_w|^2}\right)|(C_\phi f_w)'(z_w)| \\
&\geq \frac{1}{2}\ \|C_\phi f_w\|_{\mathcal{B}_{\log}} \ \geq\ \frac{k}{2}.
\end{aligned}$$

So, we obtain that

$$(1-|\varphi_w(\phi(z))|^2)\left(\log\frac{2}{1-|\varphi_w(\phi(z))|^2}\right)|\tau_\phi(z)| \geq \ \frac{k}{2}.$$

Thus,

$$(1-|\varphi_w(\phi(z))|^2)\left(\log\frac{2}{1-|\varphi_w(\phi(z))|^2}\right) \geq \ \frac{k}{2},$$

then,

$$|\varphi_w(\phi(z))|^2 \leq \frac{(\sqrt{2}-1)}{e^{\frac{k}{2}}-1}.$$

Let $r=\sqrt{\frac{(\sqrt{2}-1)}{e^{\frac{k}{2}}-1}}<1$ and $\varepsilon=\frac{(\sqrt{2}-1)}{e^{\frac{k}{2}}-1}$. Noting $\sigma(w,\phi(z_w))=|\varphi_w(\phi(z_w))|$, we conclude that

$$\sigma(w,\phi(z_w)) \ < \ r \ \text{ and } \ |\tau_\phi(z_w)|\geq\varepsilon.$$

This completes the proof.

**Theorem 2.4.** *If for some constants $0<r<\frac{1}{3}$, and $\varepsilon>0$, for each $w\in\Delta$, there is a point $z_w\in\Delta$ such that*

$$\sigma(w,\phi(z_w))<r \quad \textit{and} \ \ |\tau_\phi(z_w)|>\varepsilon,$$

*then $C_\phi:\mathcal{B}_{\log}\to\mathcal{B}_{\log}$ is bounded below.*

*Proof.* Let $u=\phi(0)$. Then $\phi=\varphi_u\circ\varphi_u\circ\phi$. Let $\psi=\varphi_u\circ\phi$. Thus $\psi(0)=0,$ and $C_\phi=C_\psi C_{\varphi_u}$. Since $\varphi_u$ is a Möbus transform, $C_{\varphi_u}$ is isometry on $\mathcal{B}_{\log}$. So we need only to prove that $C_\psi$ is bounded on $\mathcal{B}_{\log}$. Moreover $\psi$ still satisfies the conditions of the theorem. In order to prove that $C_\psi$ is bounded on $\mathcal{B}_{\log}$ space it suffices to prove

$$\|C_\psi f\|_{\log} \ \geq \ k,$$

for some constant $k > 0$ and all $f \in \mathcal{B}_{\log}$ with $\|f\|_{\log} = 1$. To do this, let $f \in \mathcal{B}_{\log}$ with norm $\|f\|_{\log} = 1$. For each $z_w \in \Delta$, we have

$$\begin{aligned}
&(1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)\big|(C_\psi f)'(z)\big| \\
&= (1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)\big|f'(\psi(z))\big|\big|\psi'(z)\big| \\
&= \frac{(1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)}{(1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)}(1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)\big|f'(\psi(z))\big|\big|\psi'(z)\big| \\
&= (1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)\big|f'(\psi(z))\big|\frac{(1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)}{(1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)}\big|\psi'(z)\big| \\
&= (1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)\big|f'(\psi(z))\big|\big|\tau_{\psi(z)}\big|.
\end{aligned}$$

Since $\|f\|_{\log} = 1$, noting that $\|f\|_{\log} = |f(0)| + \|f\|_{\mathcal{B}_{\log}}$, then $\|f\|_{\mathcal{B}_{\log}} = (1-|f(0)|)$, there is a point $w \in \Delta$ such that

$$(1-|w|^2)\big(\log\frac{2}{1-|w|^2}\big)|f'(w)| \geq (1-\frac{\frac{1}{3}-r}{2})(1-|f(0)|),$$

where $0 < r < \frac{1}{3}$ and $1 - \frac{\frac{1}{3}-r}{2} < 1$. By Theorem 2.2, we have

$$\begin{aligned}
&\Bigg|\bigg((1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)|f'(z)|\bigg)-\bigg((1-|w|^2)\big(\log\frac{2}{1-|w|^2}\big)|f'(w)|\bigg)\Bigg| \\
&\leq 3\sigma(z,w)\|f\circ\varphi_w\|_{\mathcal{B}_{\log}}.
\end{aligned}$$

Thus whenever $\sigma(\psi(z_w), w) < r < \frac{1}{3}$, we have that

$$\begin{aligned}
&(1-|\psi(z_w)|^2)\big(\log\frac{2}{1-|\psi(z_w)|^2}\big)|f'(\psi(z_w))| \\
&\geq (1-|w|^2)\big(\log\frac{2}{1-|w|^2}\big)|f'(w)| - 3\sigma(\psi(z_w),w)(1-|f(0)|) \\
&\geq (1-|w|^2)\big(\log\frac{2}{1-|w|^2}\big)|f'(w)| - 3r(1-|f(0)|) \\
&\geq (1-\frac{\frac{1}{3}-r}{2} - 3r)(1-|f(0)|) \\
&\geq \frac{5}{6}(1-3r)(1-|f(0)|).
\end{aligned}$$

So,

$$\begin{aligned}
&\|C_\psi f\|_{\log} \geq |f(\psi(0))| + (1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)\big|(C_\psi f)'(z)\big| \\
&\geq |f(0)| + (1-|\psi(z)|^2)\big(\log\frac{2}{1-|\psi(z)|^2}\big)\big|f'(\psi(z))\big|\big|\tau_{\psi(z)}\big| \text{ , for all } z \in \Delta.
\end{aligned}$$

In particular,

$$\begin{aligned}\|C_\psi f\|_{\log} &\geq |f(\psi(0))| + (1-|z|^2)\big(\log\frac{2}{1-|z|^2}\big)\big|(C_\psi f)'(z)\big| \\ &\geq |f(0)| + \frac{5}{6}\,\varepsilon\,(1-3r)(1-|f(0)|) \geq \frac{5}{6}\,\varepsilon\,(1-3r).\end{aligned}$$

Let $k = \frac{5}{6}\,\varepsilon\,(1-3r)$. We have proved that

$$\|C_\psi f\|_{\log} \geq k\,, \quad \text{whenever} \quad \|f\|_{\log} = 1.$$

This completes the proof.

## 3. Composition operators on $F_{\log}(p,q,s)$ spaces

Now we characterize the weighted logarithmic $\alpha$-Bloch spaces $\mathcal{B}^{\alpha}_{\log}$ by the weighted $F_{\log}(p,q,s)$ spaces. The obtained result improve some previous results due to Stroethoff [22] and Zhao [26].

**Theorem 3.1.** *If* $0 < p < \infty$, $-2 < q < \infty$ , $1 < s < \infty$ *and* $\alpha = \frac{q+2}{p}$ *with* $q+s > -1$. *Then the following statements are equivalent:*
(*A*) $f \in \mathcal{B}^{\alpha}_{\log}$.
(*B*) $f \in F_{\log}(p,q,s)$.
(*C*) $\sup\limits_{a\in\Delta}\Big(\log\frac{2}{1-|a|^2}\Big)^p \int_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \infty$.
(*D*) $\sup\limits_{a\in\Delta}\Big(\log\frac{2}{1-|a|^2}\Big)^p \int_\Delta |f'(z)|^p(1-|z|^2)^{\alpha p-2} g^s(z,a) dA(z) < \infty$.

*Proof.* The proof is similar to the main results in [22, 27], so it will be omitted.

Theorem 3.1, will be needed to study composition operators between $F_{\log}(p,q,s)$ and weighted $\mathcal{B}^{\alpha}_{\log}$ spaces.

**Lemma 3.2.** *Let* $0 < \alpha < \infty$, *there are two functions* $f_1, f_2 \in \mathcal{B}^{\alpha}_{\log}$ *such that*

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{(1-|z|^2)^\alpha(\log\frac{2}{1-|z|^2})}$$

*where* $C$ *is a positive constant.*

*Proof.* The proof of this lemma is similar to that of Lemma 3.1 in [11] or Lemma 2.2 in [16] with some simple modifications, so it will be omitted.
We need the following notation.

$$\Phi_\phi(\alpha,p,s;a) = \Big(\log\frac{2}{1-|a|^2}\Big)^p \int_\Delta |\phi'(z)|^p \frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\big(\log\frac{2}{1-|\phi(z)|^2}\big)^p} dA(z),$$

for $0 < p, \alpha < \infty$ and $1 < s < \infty$. Now, we will give the following theorem:

**Theorem 3.3.** *Let $0 < p, \alpha < \infty$ let $1 < s < \infty$. If $\phi$ is an analytic self-map of the unit disk, then the induced composition operator $C_\phi$ maps $\mathcal{B}^{\alpha}_{\log}$ into $F_{\log}(p, \alpha p - 2, s)$ boundedly if and only if*

$$\sup_{a\in\Delta} \Phi_\phi(\alpha, p, s; a) < \infty. \tag{3.1}$$

*Proof.* Let $f \in \mathcal{B}^{\alpha}_{\log}$ with $\|f\|_{\mathcal{B}^{\alpha}_{\log}} \leq 1$, then in view of Theorem 3.1 , we obtain

$$\begin{aligned}
&\|C_\phi f\|^p_{F_{\log}(p,\,\alpha p-2,s)}\\
=\ & \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta |(f\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
=\ & \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta |f'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
\leq\ & \|f\|^p_{\mathcal{B}^{\alpha}_{\log}}\ \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta |\phi'(z)|^p\frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log\frac{2}{1-|\phi(z)|^2}\right)^p}\,dA(z)\\
=\ & \|f\|^p_{\mathcal{B}^{\alpha}_{\log}}\ \sup_{a\in\Delta}\Phi_\phi(\alpha,p,s;a)\ <\infty.
\end{aligned}$$

For the other direction we use the fact that for each function $f \in \mathcal{B}^{\alpha}_{\log}$, the analytic function $C_\phi(f) \in F_{\log}(p, \alpha p - 2, s)$. Then using the functions of Lemma 3.2 we get the following:

$$\begin{aligned}
&2^p\left\{\|C_\phi f_1\|^p_{F_{\log}(p,\alpha p-2,s)}+\|C_\phi f_2\|^p_{F_{\log}(p,\alpha p-2,s)}\right\}\\
=\ & 2^p\sup_{a\in\Delta}\left\{\left(\log\frac{2}{1-|a|^2}\right)^p\right.\\
&\left.\times\int_\Delta\Big[|(f_1\circ\phi)'(z)|^p+|(f_2\circ\phi)'(z)|^p\Big](1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\right\}\\
\geq\ & \sup_{a\in\Delta}\left\{\left(\log\frac{2}{1-|a|^2}\right)^p\right.\\
&\left.\times\int_\Delta\Big[|(f_1\circ\phi)'(z)|+|(f_2\circ\phi)'(z)|\Big]^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\right\}\\
\geq\ & \sup_{a\in\Delta}\left\{\left(\log\frac{2}{1-|a|^2}\right)^p\right.\\
&\left.\times\int_\Delta\Big[|(f_1'(\phi(z))|+|(f_2(\phi(z))|\Big]^p|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\right\}\\
\geq\ & C\ \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta|\phi'(z)|^p\frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log\frac{2}{1-|a|^2}\right)^p}dA(z)\\
\geq\ & C\ \sup_{a\in\Delta}\Phi_\phi(\alpha,p,s;a).
\end{aligned}$$

Hence $C_\phi$ is bounded, then (3.1) holds. The proof is completed.

Now, we describe compactness in the following result.

**Theorem 3.4.** *Let $0 < p, \alpha < \infty$ and let $1 < s < \infty$. If $\phi$ is an analytic self-map of $\Delta$, then the induced composition operator $C_\phi : \mathcal{B}^\alpha_{\log} \to F_{\log}(p, \alpha p - 2, s)$ is compact if and only if $\phi \in F_{\log}(p, \alpha p - 2, s)$ and*

$$\lim_{r\to 1} \sup_{a\in\Delta} \Phi_\phi(\alpha, p, s; a) = 0. \tag{3.2}$$

*Proof.* Let $C_\phi : \mathcal{B}^\alpha_{\log} \to F_{\log}(p, \alpha p - 2, s)$ be compact. This means that

$$\phi \in F_{\log}(p, \alpha p - 2, s).$$

Let $f_n(z) = \frac{z^n}{n}$. Since $\|f_n\|_{\mathcal{B}^\alpha_{\log}} \le M$ ($M = \frac{2^\alpha}{e\alpha}$) and $f_n(z) \to 0$ as $n \to \infty$, locally uniformly on $\Delta$, then by the compactness of $C_\phi$, $\|C_\phi(f_n)\|_{F_{\log}(p,\alpha p-2,s)} \to 0$ as $n \to \infty$. This means that for each $r \in (0,1)$ and for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that if $n \ge N$, then

$$N^{\alpha p} r^{p(N-1)} \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where $\Omega_r = \{z \in \Delta, |\phi(z)| > r\}$, if we choose $r$ so that $(N^{\alpha p}\, r^{p(N-1)}) = 1$, then

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{3.3}$$

Let now $f$ with $\|f\|_{\mathcal{B}^\alpha_{\log}} \le 1$. We consider the functions $f_t(z) = f(tz), t \in (0,1)$. Then $f_t \to f$ uniformly on compact subset of the unit disk as $t \to 1$ and the family $(f_t)$ is bounded on $\mathcal{B}^\alpha_{\log}$, thus

$$\|(f_t \circ \phi) - (f \circ \phi)\| \to 0.$$

Due to compactness of $C_\phi$ we get that, for $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Delta} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where $F_t(\phi(z)) = (f\circ\phi)'(z) - (f_t \circ \phi)'(z)$. Thus, if we fix $t$, then

$$\begin{aligned}
& \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |(f\circ\phi)'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & 2^p \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
& +2^p \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |(f_t\circ\phi)'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & \varepsilon 2^p + 2^p \|f_t'\|^p_{H^\infty} \\
& \times \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & \varepsilon 2^p + \varepsilon 2^p |f_t'\|^p_{H^\infty},
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\Omega_r} |(f\circ\phi)'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & \varepsilon 2^p (1 + |f_t'\|^p_{H^\infty}),
\end{aligned}$$

where we have used (3.3). On the other hand, for each $\|f\|_{\mathcal{B}^\alpha_{\log}} \leq 1$ and $\varepsilon > 0$, there exists a $\delta$ depending on $f, \varepsilon$, such that for $r \in [\delta, 1)$,

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}|(f\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon. \quad (3.4)$$

Since $C_\phi$ is compact, then it maps the unit ball of $\mathcal{B}^\alpha_{\log}$ to a relatively compact subset of $F_{\log}(p, \alpha p-2, s)$. Thus for each $\varepsilon > 0$ there exists a finite collection of functions $f_1, f_2, \ldots, f_n$ in the unit ball of $\mathcal{B}^\alpha_{\log}$ such that for each $\|f\|_{\mathcal{B}^\alpha_{\log}} \leq 1$, there is $k \in \{1, 2, 3, \ldots, n\}$ such that

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Delta}|F_k(\phi(z))|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where $F_k(\phi(z)) = (f\circ\phi)'(z) - (f_k\circ\phi)'(z)$.
Using also (3.4), we get for $\delta = \max_{1\leq k\leq n}\delta(f_k, \varepsilon) and r \in [\delta, 1), that$

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}|(f_k\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon.$$

Hence for any $f, \|f\|_{\mathcal{B}^\alpha_{\log}} \leq 1$, combining the two relations as above we get that

$$\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}|(f\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon 2^p.$$

Therefore, we get that (3.2) holds.
For the sufficiency we use that $\phi \in F_{\log}(p,\, \alpha p-2, s)$ and (3.2) holds. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $\mathcal{B}^\alpha_{\log}$, such that $f_n \to 0$ as $n \to \infty$, uniformly on the compact subsets of the unit disk. Let also $r \in (0, 1)$ and $\Phi_r = \{z \in \Delta, |\phi(z)| \leq r\}$. Then

$$\begin{aligned}
& \|f_n\circ\phi\|^p_{F_{\log}(p,\alpha p-2,s)} \\
\leq\ & 2^p|f_n(\phi(0))| \\
& +2^p\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Phi_r}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
& +2^p\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
=\ & 2^p\,(I_1+I_2+I_3).
\end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $I_1 = |f_n(\phi(0))|$ goes to zero as $n \to \infty$ and for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for each $n > N$,

$$\begin{aligned}
I_2 &= \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Phi_r}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
&\leq \varepsilon\,\|\phi\|^p_{F_{\log}(p,\,\alpha p-2,s)}.
\end{aligned}$$

We also observe that

$$\begin{aligned} I_3 &= \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s\,dA(z)\\ &\le \|f_n\|^p_{\mathcal{B}^\alpha_{\log}}\sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_{\Omega_r}\frac{|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log\frac{2}{1-|\phi(z)|^2}\right)^p}dA(z). \end{aligned}$$

Under the assumption that (3.2) holds, then for every $n>N$ and for every $\varepsilon>0$ there exists $r_1$ such that for every $r>r_1$, $I_3<\varepsilon$. Thus if $\phi\in F_{\log}(p,\alpha p-2,s)$, we obtain

$$\|f_n\circ\phi\|^p_{F_{\log}(p,\alpha p-2,s)} \le 2^p\left\{0+\varepsilon\,\|\phi\|^p_{F_{\log}(p,\,\alpha p-2,s)}+\varepsilon\right\}\le\varepsilon C.$$

Combining the above, we get that $\|C_\phi(f_n)\|^p_{F_{\log}(p,\,\alpha p-2,s)}\to 0$ as $n\to\infty$, which proves compactness. The proof of our theorem is therefore established.

Now we consider the composition operators from the Dirichlet space $\mathcal{D}$ into $F_{\log}(p,q,s)$ spaces. Our result is stated as follows.

**Theorem 3.5.** *Let $2\le p<\infty, 1<s<\infty, -2<q<\infty$ and $q+s>-1$. If $\phi$ is an analytic self-map of $\Delta$, then the composition operator $C_\phi:\mathcal{D}\to F_{\log}(p,q,s)$ is compact if and only if*

$$\lim_{|a|\to 1}\|C_\phi\varphi_a\|_{F_{\log}(p,q,s)}=0. \tag{3.5}$$

*Proof.* Assume that $C_\phi:\mathcal{D}\to F_{\log}(p,q,s)$ is compact. Since $\{\varphi_a : a\in\Delta\}$ is a bounded set in $\mathcal{D}$ and $\varphi_a-a\to 0$ uniformly on compact sets as $|a|\to 1$, the compactness of $C_\phi$ yields that

$$\|C_\phi\varphi_a\|_{F_{\log}(p,q,s)}\longrightarrow 0 \quad as \quad |a|\to 1.$$

Conversely, let $\{f_n\}\in\mathcal{D}$ be a bounded sequence. Since $f_n\in\mathcal{D}\subset\mathcal{B}$, for $z\in\Delta$

$$|f_n(z)|\le\sup_n\|f_n\|_{\mathcal{D}}\left(1+\frac{1}{2}\log\frac{1+|z|}{1-|z|}\right).$$

Hence, $\{f_n\}$ is a normal family. Thus, there is a subsequence $\{f_{n_k}\}$, which converges to $f$ analytic on $\Delta$ and both $f_{n_k}\to f$ and $f'_{n_k}\to f'$ uniformly on compact subsets of $\Delta$. It is easy to show that $f\in\mathcal{D}$. We replace $f$ by $C_\phi f$, we remark that $C_\phi$ is compact by showing

$$\|C_\phi f_{n_k}-C_\phi f\|_{F_{\log}(p,q,s)}\longrightarrow 0 \quad as \;\; |k|\to\infty.$$

We write

$$\begin{aligned}
&\|C_\phi\varphi_a\|^p_{F_{\log}(p,q,s)}\\
&= \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta |(\varphi_a\circ\phi)'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&= \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p\int_\Delta \frac{(1-|a|^2)^p}{|1-\overline{a}\phi(z)|^{2p}}|\phi'(z)|^p(1-|z|^2)^q g^s(z,a)dA(z)\\
&= \sup_{a\in\Delta}\int_\Delta \frac{(1-|a|^2)^p}{|1-\overline{a}w|^{2p}}\big(N^{a,p,q,s}_{\log}(\phi,w)\big)dA(w).
\end{aligned}$$

Here,

$$N^{a,p,q,s}_{\log}(\phi,w)=\left(\log\frac{2}{1-|a|^2}\right)^p\sum_{z\in\phi^{-1}(w)}|\phi'(z)|^{p-2}(1-|z|^2)^q g^s(z,a)$$

is the counting function. Thus (3.5) is equivalent to

$$\lim_{|a|\to 1}\sup_{a\in\Delta}\int_\Delta \frac{(1-|a|^2)^p}{|1-\overline{a}w|^{2p}}\big(N^{a,p,q,s}_{\log}(\phi,w)\big)dA(w)=0.$$

Hence by [5] or [23], for any $\varepsilon>0$ there exists $\delta$, where $0<\delta<1$, such that for $0<h<\delta$ and all $a\in\Delta$,

$$\sup_{a\in\Delta}\int_{S(h,\theta)} N^{a,p,q,s}_{\log}(\phi,w)dA(w)\ <\ \varepsilon\, h^p,$$

where $S(h,\theta)$ is a Carleson box. For $F_{n_k}(z)=f'_{n_k}(z)-f'(z)$, the mean value property for analytic functions $f'_{n_k}$ and $f'$ yields that,

$$F_{n_k}(w)=\frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}}F_{n_k}(z)dA(z).$$

Then by Jensen's inequality (see [20] theorem 3.3), we have

$$|F_{n_k}(w)|^p=\frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}}|F_{n_k}(z)|^p dA(z).$$

Note that if $|w-z|<\frac{1-|w|}{2}$, then we have that $w\in S(2(1-|z|),\theta)$ and also $\frac{1}{(1-|w|)^2}\le\frac{C}{(1-|z|)^2}$ (see [23]). Then, by Fubini's theorem (see [20] theorem 8.8), for

$F_{n_k}(z) = f'_{n_k}(z) - f'(z)$, we deduce that

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{\Delta} |F_{n_k}(w)|^p \big(N^{a,p,q,s}_{\log}(\phi, w)\big) dA(w)\\
\leq\ & \sup_{a\in\Delta}\int_{\Delta}\left\{\frac{4}{\pi(1-|w|)^2}\int_{|w-z|<\frac{1-|w|}{2}} |F_{n_k}(z)|^p dA(z)\right\} N^{a,p,q,s}_{\log}(\phi, w)dA(w)\\
\leq\ & C\sup_{a\in\Delta}\int_{\Delta}\frac{|F_{n_k}(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)} N^{a,p,q,s}_{\log}(\phi, w)dA(w)dA(z)\\
=\ & C\sup_{a\in\Delta}\left\{\left(\log\frac{2}{1-|a|^2}\right)^p\right.\\
&\left.\times\int_{|z|>1-\frac{\delta}{2}}\frac{|F_{n_k}(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)} N^{a,p,q,s}_{\log}(\phi, w)dA(w)dA(z)\right\}\\
&+C\sup_{a\in\Delta}\left\{\left(\log\frac{2}{1-|a|^2}\right)^p\right.\\
&\left.\times\int_{|z|\leq 1-\frac{\delta}{2}}\frac{|F_{n_k}(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)} N^{a,p,q,s}_{\log}(\phi, w)\ dA(w)dA(z)\right\}.
\end{aligned}$$

For one hand, since $f_{n_k}, f \in \mathcal{D}\subset\mathcal{B}$, $2\leq p<\infty$ and $F_{n_k}(z) = f'_{n_k}(z) - f'(z)$, we have

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{|z|>1-\frac{\delta}{2}}\frac{|F_{n_k}(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)} N^{a,p,q,s}_{\log}(\phi, w)dA(w)dA(z)\\
\leq\ & \varepsilon\ 2^p\sup_{a\in\Delta}\int_{|z|>1-\frac{\delta}{2}} |F_{n_k}(z)|^p(1-|z|)^{p-2}dA(z)\\
\leq\ & \varepsilon C\|f_{n_k}-f\|^{p-2}_{\mathcal{B}}\sup_{a\in\Delta}\int_{|z|>1-\frac{\delta}{2}} |F_{n_k}(z)|^2 dA(z)\\
\leq\ & \varepsilon C\|f_{n_k}-f\|^{p-2}_{\mathcal{B}}\|f_{n_k}-f\|^2_{\mathcal{D}}\\
\leq\ & \varepsilon C_1\|f_{n_k}-f\|^2_{\mathcal{D}},
\end{aligned}$$

where $C$ and $C_1$ are positive constants. On the other hand,

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{|z|\leq 1-\frac{\delta}{2}}\frac{|F_{n_k}(z)|^p}{(1-|z|)^2}\int_{S(2(1-|z|),\theta)} N^{a,p,q,s}_{\log}(\phi, w)dA(w)dA(z)\\
\leq\ & C\sup_{a\in\Delta}\int_{\Delta} N^{a,p,q,s}_{\log}(\phi, w)dA(w)\int_{|z|\leq 1-\frac{\delta}{2}} |F_{n_k}(z)|^p dA(z)\\
\leq\ & \varepsilon C,
\end{aligned}$$

for $n$ large enough and since $F_{n_k}(z) = (f'_{n_k}(z) - f'(z)) \longrightarrow 0$ uniformly on $\{z\in\Delta : |z|\leq 1-\frac{\delta}{2}\}$. Therefore, for sufficiently large $k$, the above discussion gives

$$
\begin{aligned}
&\|C_\phi f_{n_k} - C_\phi f\|^p_{F_{\log}(p,q,s)} \\
= \;& \sup_{a\in\Delta}\left(\log\frac{2}{1-|a|^2}\right)^p \int_\Delta |(f_{n_k}\circ\phi)'(z) - (f\circ\phi)'(z)|^p (1-|z|^2)^q g^s(z,a)\,dA(z) \\
= \;& \sup_{a\in\Delta}\int_\Delta |f'_{n_k}(z) - f'(z)|^p (1-|z|^2)^q N^{a,p,q,s}_{\log}(\phi,w)\; dA(w) < \varepsilon C.
\end{aligned}
$$

It follows that $C_\phi$ is a compact operator. Therefore, the proof is completed.

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