Ann. Funct. Anal. 2 (2011), no. 2, 130–137
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

LICHNEROWICZ–POISSON COHOMOLOGY AND BANACH LIE ALGEBROIDS

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Communicated by V. Valov

ABSTRACT. In this note the Poisson structures on Banach manifolds are considered. Then a Lichnerowicz–Poisson cohomology is reformulated in the Banach setting and its relation with Banach Lie algebroids cohomology is given in a classical way just as in the finite dimensional case.

1. INTRODUCTION AND PRELIMINARIES

The Poisson structures on Banach manifolds were introduce by A. Odzijewicz and T. Raţiu in [9, 10] just as in the finite dimensional case, and these structures are modeled on the example of a strong symplectic manifold. Also, its elementary properties are presented as well as some comments on the compatibility of the Banach Poisson structure with almost complex, complex, and holomorphic structures, which are reviewed here in the last section. On the other hand in a paper by M. Anastasiei [2], the Banach Lie algebroids are defined as Lie algebroid structures modeled on anchored Banach vector bundles. Taking into account that the Banach Poisson structures defines on the cotangent bundle a structure of Banach Lie algebroid, the our aim in this note is to extend the classical Lichnerowicz–Poisson cohomology for Banach Poisson manifolds and its relation with Banach Lie algebroids cohomology. Finally, the possibility to extend our study at holomorphic Banach Poisson manifolds is discussed.

Given a Banach space $\mathcal{B}$, the notation $\mathcal{B}^*$ will mean the Banach space dual to $\mathcal{B}$. For $\alpha \in \mathcal{B}^*$ and $a \in \mathcal{B}$, we shall denote by $\langle \alpha, a \rangle$ the value of α on a. Thus $\langle \cdot, \cdot \rangle : \mathcal{B}^* \times \mathcal{B} \to \mathbb{R}$ (or $\mathbb{C}$, depending on whether we work with real or complex

Date: Received: 16 October 2011; Accepted: 1 November 2011.
2010 *Mathematics Subject Classification.* Primary 53D17; Secondary 46T05, 57N20, 58A99.
Key words and phrases. Banach Poisson structures, Lichnerowicz–Poisson cohomology, Banach Lie algebroids.

Banach spaces and functions) will denote the natural bilinear continuous duality pairing between $\mathcal{B}$ and its dual $\mathcal{B}^*$, [1].

Definition 1.1. Suppose M is a Hausdorff topological space. A triple $(U, \varphi, \mathcal{B})$ is called a chart of M if U is an open subset of M, $\mathcal{B}$ is a Banach space over $\mathbb{R}$ and $\varphi : U \to \mathcal{B}$ is a homeomorphism onto an open subset of $\mathcal{B}$. If $x \in U$ is a point satisfying $\varphi(x) = 0$, then $(U, \varphi, \mathcal{B})$ is called a chart about x. The charts $(U, \varphi, \mathcal{B})$ and $(V, \psi, \mathcal{B}')$ are said to be $\mathbb{R}$-*smooth compatible* if the homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

between the open subsets of $\mathcal{B}$ and $\mathcal{B}'$, respectively, is smooth. In this case, $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic provided $U \cap V \neq \phi$. An *atlas* of M is a collection of pairwise compatible charts covering M. A maximal atlas (under inclusion) endows M with a structure of a Banach manifold over $\mathbb{R}$.

Definition 1.2. Let M be a Banach manifold and $\mu : TM \to \mathbb{R}_+$ a lower semi-continuous function. Then μ is called a *norm* on TM if the restriction of μ to every tangent spaces T_xM, $x \in M$, is a norm on T_xM with the following property: There is a neighborhood U of $x \in M$ diffeomorphic equivalent to a domain in a Banach space $\mathcal{B}$ such that

$$c||a|| \leq \mu(x, a) \leq C||a|| \text{ for all } (x, a) \in U \times \mathcal{B} \approx TU$$

and suitable constants $0 < c \leq C$. A Banach manifold M together with a fixed norm μ on the tangent bundle TM is called a *normed Banach manifold.*

For any $x \in M$ one has canonical isomorphisms $T_xM \approx \mathcal{B}$, $T_x^*M \approx \mathcal{B}^*$ and $T_x^{**}M \approx \mathcal{B}^{**}$ of Banach spaces. Since in general case $\mathcal{B} \subset \mathcal{B}^{**}$ and $\mathcal{B} \neq \mathcal{B}^{**}$ the tangent bundle TM is not isomorphic with twice-dual bundle $T^{**}M$. Hence one has only the canonical inclusion $TM \subset T^{**}M$ isometric on fibers. The isomorphism $TM \approx T^{**}M$ has place only if $\mathcal{B}$ is reflexive. Particularly, when $\mathcal{B}$ is finite dimensional.

According to [9, 10], in the following of this section we briefly present the notion of Banach Poisson structures. Like in the finite dimensional case one defines the Poisson bracket on the space $C^\infty(M)$ as a bilinear smooth antisymmetric map

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying Leibniz and Jacobi identities. Due to the Leibniz property there exists antisymmetric 2-tensor field $\pi \in \Gamma(\Lambda^2 T^{**}M)$ satisfying

$$\{f, g\} = \pi(df, dg) \tag{1.1}$$

for each $f, g \in C^\infty(M)$. In addition from Jacobi property and from

$$\sum_{cycl(f,g,h)} \{\{f, g\}, h\} = [\pi, \pi](df, dg, dh),$$

see [5, 11], one has that the 3-tensor field $[\pi, \pi] \in \Gamma(\Lambda^3 T^{**}M)$, called the Schouten-Nijenhuis bracket of π, satisfies the condition

$$[\pi, \pi] = 0. \tag{1.2}$$

Hence the Poisson bracket can be equivalently described by the antisymmetric 2-tensor field satisfying the differential equation (1.2). One calls π the Poisson tensor. Let us define by

$$\#df := \pi(\cdot, df)$$

the map $\# : T^*M \to T^{**}M$ covering the identity map $id : M \to M$, for any locally defined smooth function f. One has $\#df \in \Gamma(T^{**}M)$, so, opposite to the finite dimensional case, it is not vector field in general. Thus according to [10] we have the following definition:

Definition 1.3. ([10]). A *Banach Poisson manifold* is a pair $(M, \{\cdot,\cdot\})$ consisting of a Banach manifold and a bilinear operation $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the following conditions:

i) $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra;
ii) $\{\cdot,\cdot\}$ satisfies the Leibniz property on each factor;
iii) the vector bundle map $\# : T^*M \to T^{**}M$ covering the identity satisfies $\#(T^*M) \subset TM$.

As we see, the condition iii) allows one to introduce for any function $f \in C^\infty(M)$ the *Hamiltonian vector field* X_f by

$$X_f := \#df.$$

In consequence, after fixing Hamiltonian $h \in C^\infty(M)$ at the above one can consider Banach Hamiltonian system $(M, \{\cdot,\cdot\}, h)$ with equation of motion

$$\frac{df}{dt} = -Z_h(f) = \{h, f\}.$$

Definition 1.4. Let $(M_1, \{\cdot,\cdot\}_1, \pi_1)$ and $(M_2, \{\cdot,\cdot\}_2, \pi_2)$ be two Banach Poisson manifolds. A smooth mapping $\phi : M_1 \to M_2$ is a *Banach Poisson morphism* if:

i) ϕ is Banach morphism;
ii) ϕ is a Poisson type morphism with respect to π_1 and π_2, that is, ϕ satisfies one of the following equivalent properties,
   1. π_1 and π_2 are ϕ-related:
   $$\pi_1(x)(\phi^*\alpha, \phi^*\beta) = \pi_2(\phi(x))(\alpha, \beta),\ \forall\, x \in M_1,\ \forall\, \alpha, \beta \in \Omega^1_{\phi(x)}(M_2);$$
   2. the Hamiltonian vector fields $X_{f\circ\phi}$ and X_f are ϕ-related:
   $$\phi_* Z_{f\circ\phi} = Z_f,\ \forall\, f \in C^\infty(M_2).$$

The Banach Poisson manifolds form a category with morphisms defined above.

2. Lichnerowicz–Poisson cohomology and Banach Lie algebroids

In this section the classical Lichnerowicz–Poisson cohomology, see [7, 11, 12], is reformulated in the Banach setting and its relation with Banach Lie algebroids cohomology is given just as in the finite dimensional case.

Let (M, π) be a Banach Poisson manifold. We put $\mathcal{V}^p(M)$ the space of all C^∞ p-vector fields of M, $(\mathcal{V}^1(M) = \mathcal{X}(M))$ and $\Omega^p(M)$ the space of all C^∞ p-differential forms on M, $(\Omega^1(M) = \mathcal{X}^*(M))$. There is a morphism

$$\#_\pi : \Omega^1(M) \to \mathcal{V}^1(M),\ \#_\pi(\alpha)(\beta) = \pi(\alpha, \beta),\ \text{for } \alpha, \beta \in \Omega^1(M).$$

It can be extended to $\Omega^p(M)$ as follows: let I^* be the adjoint of $I = \#_\pi$ (it is easy to see that $I^* = -I$), then

$$I : \Omega^p(M) \to \mathcal{V}^p(M),\ I(\alpha)(\alpha_1, \dots, \alpha_p) = \alpha(I^*\alpha_1, \dots, I^*\alpha_p), \tag{2.1}$$

for $\alpha \in \Omega^p(M)$ and $\alpha_1 \dots, \alpha_p \in \Omega^1(M)$. Like in the finite dimensional case the Banach Poisson bracket from (1.1) induces a bracket of 1-forms which is the unique natural extension of the formula $\{df, dg\} = d\{f, g\}$, and is given by

$$\{\alpha, \beta\} = L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}(\alpha) - d\pi(\alpha, \beta), \tag{2.2}$$

where L_X denotes the Lie derivative.

Now, we consider the differential operator

$$\sigma : \mathcal{V}^p(M) \to \mathcal{V}^{p+1}(M),\ \sigma(X) = -[X, \pi].$$

Using the standard properties of the Schouten-Nijenhuis bracket, see [11], it is easy to prove:

i) $\sigma^2 = 0$;
ii) $\sigma(X_1 \wedge X_2) = \sigma(X_1) \wedge X_2 + (-1)^{\deg X_1} X_1 \wedge \sigma(X_2)$;
iii) $\sigma([X_1, X_2]) = -[\sigma(X_1), X_2] - (-1)^{\deg X_1}[X_1, \sigma(X_2)]$,

for $X_1, X_2 \in \mathcal{V}^\bullet(M)$ and where $\deg X$ denotes the degree of the multivector X. So, we obtain a differential complex

$$\dots \longrightarrow \mathcal{V}^p(M) \xrightarrow{\sigma} \mathcal{V}^{p+1}(M) \xrightarrow{\sigma} \dots$$

whose cohomology groups, denoted by $H^p_{LP}(M, \pi)$, will be called *Lichnerowicz–Poisson cohomology groups* of Banach Poisson manifold (M, π).

We also notice that for $X \in \mathcal{V}^p(M)$, one has

$$(\sigma X)(\alpha_0, \dots, \alpha_p) = \sum_{i=0}^{p} (-1)^i \#_\pi(\alpha_i)(X(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_p)) \tag{2.3}$$

$$+ \sum_{i<j=1}^{p} (-1)^{i+j} X(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_p),$$

where $\alpha_i \in \Omega^1(M)$, and $\widehat{\ }$ denotes the absence of an argument.

Now, the definitions given above have some easy consequences such as:

i) $H^0_{LP}(M, \pi) = \{f \in C^\infty(M) \,:\, \forall\, g \in C^\infty(M),\ X_g f = 0\}$, since $\sigma f = X_f$.
ii) $H^1_{LP}(M, \pi) = \mathcal{V}^1_\pi(M)/\mathcal{V}^1_{\mathcal{H}}(M)$, where

$$\mathcal{V}^1_\pi = \{X \in \mathcal{V}^1(M) \,:\, L_X\pi = 0\} \text{ and } \mathcal{V}^1_{\mathcal{H}}(M) = \{X_f \,:\, f \in C^\infty(M)\},$$

since $\sigma X = -L_X\pi$.
iii) $\sigma\pi = 0$, and π defines a *fundamental class* $[\pi] \in H^2_{LP}(M, \pi)$.
iv) We have a natural homomorphism $\psi : H^p(M, \mathbb{R}) \to H^p_{LP}(M, \pi)$, which is defined by the extension $\#_\pi$ to p-forms from (2.1), since (2.3) shows that

$$\sigma(I(\alpha)) = (-1)^p I(d\alpha).$$

Next, we consider the notion of Banach Lie algebroid, see [2], and its cohomology ring. Let M be a smooth Banach manifold modeled on Banach space $\mathcal{B}_M$ and let $p : E \to M$ be a Banach vector bundle whose type fiber is a Banach space $\mathcal{B}_E$. We denote by $\tau : TM \to M$ the tangent bundle of M.

Definition 2.1. We say that $p: E \to M$ is a *anchored* vector bundle if there exists a vector bundle morphism $\rho : E \to TM$. The morphism ρ will be called the anchor map.

The anchored vector bundles over the same base M form a category. The objects are the pairs (E, ρ_E) with ρ_E the anchor of E and a morphism $\phi : (E, \rho_E) \to (F, \rho_F)$ is a vector bundle morphism $\phi : E \to F$ which verifies the condition $\rho_F \circ \phi = \rho_E$.

Let $p : E \to M$ be an anchored Banach vector bundle with the anchor $\rho : E \to TM$ and the induced morphism $\rho_E : \Gamma(E) \to \mathcal{X}(M)$. Assume there exists defined a bracket $[\cdot,\cdot]_E$ on the space $\Gamma(E)$ that provides a structure of real Lie algebra on $\Gamma(E)$.

Definition 2.2. ([2]). The triplet $(E, \rho_E, [\cdot,\cdot]_E)$ is called a *Banach Lie algebroid* if

i) $\rho_E : (\Gamma(E, [\cdot,\cdot]_E) \to (\mathcal{X}(M), [\cdot,\cdot])$ is a Lie algebra homomorphism, i.e.

$$[s_1, s_2]_E = [\rho_E(s_1), \rho_E(s_2)];$$

ii) $[s_1, fs_2]_E = f[s_1, s_2]_E + \rho_E(s_1)(f)s_2,$

for every $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

Example 2.3.
i) The tangent bundle $\tau : TM \to M$ is a Banach Lie algebroid with the anchor the identity map and the usual Lie bracket of vector fields on M.
ii) For any submersion $p : E \to M$, the vertical bundle $VE = \ker p_*$ over E is an anchored Banach vector bundle. As the Lie bracket of two vertical vector fields is again a vertical vector field it follows that $(VE, i, [\cdot,\cdot]_{VE})$, where $i : VE \to TE$ is the inclusion map, is a Banach Lie algebroid. This applies, in particular, to any Banach vector bundle $p : E \to M$.
iii) If $(\mathcal{G}, [\cdot,\cdot])$ is a Banach Lie algebra then $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}} = [\cdot,\cdot], \rho = 0)$ is a Banach Lie algebroid over a point.
iv) Let (M, π) be a Banach Poisson manifold. Then it is possible to define a Lie algebra structure $\{\cdot,\cdot\}$ on the space of 1-forms on M, see the formula (2.2), in such a way that the triple $(T^*M, \{\cdot,\cdot\}, \#_\pi)$ is a Banach Lie algebroid over M. For $f, g \in C^\infty(M)$ and 1-forms $\alpha, \beta \in \Omega^1(M)$ we have
   1. $d\{f, g\} = \{df, dfg\}$;
   2. $\#_\pi(\{\alpha, \beta\}) = [\#_\pi(\alpha), \#_\pi(\beta)]$;
   3. $\{f\alpha, \beta\} = f\{\alpha, \beta\} - \#_\pi(\beta)f\alpha$,

   so, the triple $(T^*M, \{\cdot,\cdot\}, \#_\pi)$ is a Banach Lie algebroid over M.

There exists a canonical cohomology theory associated to a Banach Lie algebroid. Let $(E, \rho_E, [\cdot,\cdot]_E)$ be a Banach Lie algebroid over a Banach manifold M. The space $C^\infty(M)$ is a $\Gamma(E)$-module relative to the representation

$$\Gamma(E) \times C^\infty(M) \to C^\infty(M),\ (s, f) \mapsto \rho_E(s)f.$$

Following the well-known Chevalley-Eilenberg cohomology theory [3], we can introduce a cohomology complex associated to the Banach Lie algebroid as follows.

A p-linear mapping

$$\omega^p : \Gamma(E) \times \ldots \times \Gamma(E) \to C^\infty(M)$$

is called a $C^\infty(M)$-valued p-cochain. Let $C^p(\Gamma(E); C^\infty(M))$ denote the vector space of these cochains.

The operator $d_E : C^p(\Gamma(E); C^\infty(M)) \to C^{p+1}(\Gamma(E); C^\infty(M))$ given by

$$(d_E\omega^p)(s_0, \ldots, s_p) = \sum_{i=0}^{p} (-1)^i \rho_E(s_i)(\omega^p(s_0, \ldots, \widehat{s_i}, \ldots, s_p))$$

$$+ \sum_{i<j=1}^{p} (-1)^{i+j} \omega^p([s_i, s_j]_E, s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_p),$$

for $\omega^p \in C^p(\Gamma(E); C^\infty(M))$ and $s_0, \ldots, s_p \in \Gamma(E)$, defines a coboundary since $d_E \circ d_E = 0$. Hence, $(C^p(\Gamma(E); C^\infty(M)), d_E)$, $p \geq 1$ is a differential complex and the corresponding cohomology spaces $H^p(\Gamma(E), C^\infty(M))$ are called the cohomology groups of $\Gamma(E)$ with coefficients in $C^\infty(M)$.

Lemma 2.4. *If $\omega^p \in C^p(\Gamma(E); C^\infty(M))$ is skew-symmetric and $C^\infty(M)$-linear, then $d_E\omega^p$ also is skew-symmetric.*

From now on, the subspace of skew-symmetric and $C^\infty(M)$-linear cochains of the space $C^p(\Gamma(E); C^\infty(M))$ will be denoted by $\Omega^p(\Gamma(E); C^\infty(M))$.

The Banach Lie algebroid cohomology of E is the cohomology of the subcomplex $(\Omega^p(\Gamma(E); C^\infty(M)), d_E)$, $p \geq 1$.

Using (2.3) it follows an analogue result of the finite dimensional case:

Proposition 2.5. *Let (M, π) be a Banach Poisson manifold. Then the Banach Lie algebroid cohomology of $(T^*M, \{\cdot,\cdot\}, \#_\pi)$ is the Lichnerowicz–Poisson cohomology of (M, π).*

3. The holomorphic case

Let us suppose that the Banach Poisson manifold $(M, \{\cdot,\cdot\}, \pi)$ has an almost complex structure, that is, there is a smooth vector bundle map $J : TM \to TM$ covering the identity, which satisfies $J^2 = -id$. The complex Poisson structure π is said to be *compatible with the complex structure* J if the following diagram commutes:

$$\begin{array}{ccc} T^*M & \xrightarrow{\#} & TM \\ {\scriptstyle J^*}\downarrow & & \downarrow{\scriptstyle J} \\ T^*M & \xrightarrow{\#} & TM \end{array},$$

that is,

$$J \circ \# = \# \circ J^*. \tag{3.1}$$

The decomposition

$$\pi = \pi_{(2,0)} + \pi_{(1,1)} + \pi_{(0,2)}$$

induced by the complex structure J and the reality of π, implies that the compatibility condition (3.1) is equivalent to

$$\pi_{(1,1)} = 0 \text{ and } \overline{\pi_{(2,0)}} = \pi_{(0,2)}, \tag{3.2}$$

where the overline denotes the complex conjugate.

In view of (3.2), $[\pi, \pi] = 0$ is equivalent to

$$[\pi_{(2,0)}, \pi_{(2,0)}] = 0 \text{ and } [\pi_{(2,0)}, \overline{\pi_{(2,0)}}] = 0. \tag{3.3}$$

If (3.1) holds, the triple $(M, \{\cdot,\cdot\}, \pi_{(2,0)}, J)$ is called an *almost complex Banach Poisson manifold*. If J is given by a complex analytic structure $M_{\mathbb{C}}$ on M it will be called a *complex Banach Poisson manifold*. For finite dimensional complex manifolds these structures were introduced and studied by A. Lichnerowicz [8].

Let $\Omega^{p,q}(M_{\mathbb{C}})$ and $\mathcal{V}^{p,q}(M_{\mathbb{C}})$ be the space of (p,q)-forms and (p,q)-vector fields, respectively, on the complex Banach Poisson manifold (M, π). We also denote by $\Omega^p_{\mathcal{O}}(M_{\mathbb{C}})$ and $\mathcal{V}^p_{\mathcal{O}}(M_{\mathbb{C}})$ the space of holomorphic p-forms and p-vector fields, that are holomorphic $(p,0)$-forms and holomorphic $(p,0)$-vector fields, respectively. If

$$\#(\Omega^1_{\mathcal{O}}(M_{\mathbb{C}})) \subset \mathcal{V}^1_{\mathcal{O}}(M_{\mathbb{C}}), \tag{3.4}$$

that is, complex Hamiltonian vector field Z_f is holomorphic if f is holomorphic function, then, in addition to (3.2) and (3.3), one has $\pi_{(2,0)} \in \mathcal{V}^2_{\mathcal{O}}(M_{\mathbb{C}})$. As expected, the compatibility condition (3.4) is stronger than (3.1). Note that (3.4) implies the second condition in (3.3). Thus the compatibility condition (3.4) induces on the underlying complex Banach manifold $M_{\mathbb{C}}$ a holomorphic Poisson tensor $\pi_{\mathcal{O}} := \pi_{(2,0)}$. A pair $(M_{\mathbb{C}}, \pi_{\mathcal{O}})$ consisting of an analytic complex Banach manifold $M_{\mathbb{C}}$ and a holomorphic skew symmetric contravariant two-tensor field $\pi_{\mathcal{O}}$ such that $[\pi_{\mathcal{O}}, \pi_{\mathcal{O}}] = 0$ and (3.4) holds will be called a *holomorphic Banach Poisson manifold*.

Consider now a holomorphic Banach Poisson manifold $(M_{\mathbb{C}}, \pi_{\mathcal{O}})$. Denote by $M_{\mathbb{R}}$ the underlying real Banach manifold and define the real two-vector field $\pi_{\mathbb{R}} := Re\pi_{\mathcal{O}}$. It is easy to see that $(M_{\mathbb{R}}, \pi_{\mathbb{R}})$ is a real Banach Poisson manifold compatible with the complex Banach manifold structure of M. Summarizing, we have that there are two procedures that are inverses of each other: a holomorphic Banach Poisson manifold corresponds in a bijective manner to a real Banach Poisson manifold whose Poisson tensor is compatible with the underlying complex manifold structure. One can call these constructions the *complexification* and *realification* of Banach Poisson structures on complex manifolds. In the finite dimensional case such strucures were studied by [6].

Now, in similar manner as in the previous section, we can construct a holomorphic Lichnerowicz–Poisson cohomology for holomorphic Banach Poisson manifolds and its relation with holomorphic Banach Lie algebroids cohomology. A such study for the finite dimensional case may be found in [4].

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