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SOME GEOMETRIC CONSTANTS OF ABSOLUTE NORMALIZED NORMS ON $\mathbb{R}^2$

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ABSTRACT. We consider the Banach space $X = (\mathbb{R}^2, \|\cdot\|)$ with a normalized, absolute norm. Our aim in this paper is to calculate the modified Neumann-Jordan constant $C'_{NJ}(X)$ and the Zbăganu constant $C_Z(X)$.

1. INTRODUCTION AND PRELIMINARIES

Let X be a Banach space with the unit ball $B_X = \{x \in X : \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X : \|x\| = 1\}$. Many geometric constants for a Banach space X have been investigated. In this paper we shall consider the following constants;

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \;\middle|\; (x,y) \neq (0,0) \right\},$$

$$C'_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} \;\middle|\; x, y \in S_X \right\},$$

$$C_Z(X) = \sup\left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} \;\middle|\; x, y \in X,\ (x,y) \neq (0,0) \right\}.$$

The constant $C_{NJ}(X)$, called the von Neumann-Jordan constant (hereafter referred to as NJ constant) have been considered in many papers ([3, 8, 10, 12] and so on). The constant $C'_{NJ}(X)$, called the modified von Neumann-Jordan constant (shortly, modified NJ constant) was introduced by Gao in [5] and does not necessarily coincide with $C_{NJ}(X)$ (cf. [1, 4, 7]). The constant $C_Z(X)$ was introduced by Zbăganu ([15]) and was conjectured that $C_Z(X)$ coincides with

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the von Neumann-Jordan constant $C_{NJ}(X)$, but Alonso and Martin [2] gave an example that $C_{NJ}(X) \neq C_Z(X)$ (cf.[6, 9]).

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(a,b)\| = \|(|a|,|b|)\|$ for any $(a,b) \in \mathbb{R}^2$, and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. Let AN_2 denote the family of all absolute normalized norm on $\mathbb{R}^2$, and Ψ_2 denote the family of all continuous convex function ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \leq \psi(t) \leq 1$ for all $0 \leq t \leq 1$. As in [11], it is well known that AN_2 and Ψ_2 are in a one-to-one correspondence under the equation $\psi(t) = \|(1-t,t)\|$ $(0 \leq t \leq 1)$. Denote $\|\cdot\|_\psi$ be an absolute normalized norm associated with a convex function $\psi \in \Psi_2$.

For $\psi, \varphi \in \Psi_2$, we denote $\psi \leq \varphi$ if $\psi(t) \leq \varphi(t)$ for any t in $[0,1]$. Let

$$M_1 = \max_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)} \text{ and } M_2 = \max_{0\leq t\leq 1} \frac{\psi_2(t)}{\psi(t)},$$

where $\psi_2(t) = \|(1-t,t)\|_2 = \sqrt{(1-t)^2 + t^2}$ corresponds to the l_2-norm. In [11], Saito, Kato and Takahashi proved that, if $\psi \geq \psi_2$ (resp. $\psi \leq \psi_2$), then $C_{NJ}(\mathbb{C}^2, \|\cdot\|_\psi) = M_1^2$ (resp. M_2^2).

We put $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ for $\psi \in \Psi_2$. Our aim in this paper is to consider the conditions of ψ that $C_{NJ}(X) = C_Z(X)$ or $C_{NJ}(X) = C'_{NJ}(X)$.

In §2, we consider the modified von Neumann-Jordan constant. We prove that if $\psi \leq \psi_2$, then $C'_{NJ}(X) = C_{NJ}(X) = M_2^2$. If $\psi \geq \psi_2$, then we present the necessarily and sufficient condition that $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_1^2$. Further, we consider the conditions that $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_1^2 M_2^2$. In §3, we study the Zbăganu constant. First, we show that, if $\psi \geq \psi_2$, then $C_Z(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_1^2$. If $\psi \leq \psi_2$, then we give the necessarily and sufficient condition for that $C_Z(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_2^2$. Further we study the conditions that $C_Z(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_1^2 M_2^2$. In §4, we calculate the modified NJ-constant $C'_{NJ}(X)$ and the Zbăganu constant $C_Z(X)$ for some normed liner spaces.

2. The modified NJ constant of R^2

In this section, we consider the Banach space $X = (\mathbb{R}^2, \|\cdot\|_\psi)$. From the definition of the modified NJ constant, it is clear that $C'_{NJ}(X) \leq C_{NJ}(X)$. In this section, we consider the condition that $C'_{NJ}(X) = C_{NJ}(X)$.

Proposition 2.1. *Let $\psi \in \Psi_2$. If $\psi \leq \psi_2$, then $C'_{NJ}(X) = C_{NJ}(X) = M_2^2$.*

Proof. For any $x, y \in S_X$, by [11, Lemma 3],

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\leq \|x+y\|_2^2 + \|x-y\|_2^2 \\ &= 2\left(\|x\|_2^2 + \|y\|_2^2\right) \\ &\leq 2M_2^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right) = 4M_2^2.\end{aligned}$$

Now let ψ_2/ψ attain the maximum at $t = t_0$ $(0 \leq t_0 \leq 1)$, and put

$$x = \frac{1}{\psi(t_0)}(1-t_0,\ t_0),\ y = \frac{1}{\psi(t_0)}(1-t_0,\ -t_0).$$

Then $x, y \in S_X$ and

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = \frac{4(1-t_0)^2 + 4t_0^2}{\psi(t_0)^2} = 4\frac{\psi_2(t_0)^2}{\psi(t_0)^2} = 4M_2^2,$$

which implies that $C'_{NJ}(X) = M_2^2$. By [11, Theorem 1], we have this proposition. □

If $\psi \geq \psi_2$, by [11, Theorem 1], then $C_{NJ}(X) = M_1^2$. We now give the necessarily and sufficient condition of $C'_{NJ}(X) = M_1^2$.

Theorem 2.2. *Let $\psi \in \Psi_2$ such that $\psi \geq \psi_2$. Then $C'_{NJ}(X) = M_1^2$ if and only if there exist $s, t \in [0,1]$ $(s < t)$ satisfying one of the following conditions:*

(1) $\psi(s) = \psi_2(s)$, $\psi(t) = \psi_2(t)$ and, if we put $r = \frac{\psi(s)t + \psi(t)s}{\psi(s)+\psi(t)}$, then $\frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1$.

(2) $\psi(s) = \psi_2(s)$, $\psi(t) = \psi_2(t)$ and, if we put $r = \frac{\psi(t)s + \psi(s)t}{\psi(t)+\psi(s)(2t-1)}$, then $\frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1$.

Proof. ($\Longrightarrow$) Suppose that $C'_{NJ}(X) = M_1^2$. First, for any $x, y \in S_X$, by [11, Lemma 3], we have

$$\begin{aligned}\|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\leq M_1^2(\|x+y\|_2^2 + \|x-y\|_2^2) \\ &= 2M_1^2\left(\|x\|_2^2 + \|y\|_2^2\right) \\ &\leq 2M_1^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right) = 4M_1^2.\end{aligned}$$

Since $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ is finite dimensional,

$$C'_{NJ}(X) = \max\left\{\frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{4} \;\middle|\; x, y \in S_X\right\}.$$

Therefore, $C'_{NJ}(X) = M_1^2$ if and only if there exist $x, y \in S_X$ $(x \neq y)$ such that

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 4M_1^2.$$

From the above inequality, the elements $x, y \in S_X$ $(x \neq y)$ satisfy $\|x\|_\psi = \|x\|_2 = 1$, $\|y\|_\psi = \|y\|_2 = 1$ and

$$\frac{\|x+y\|_\psi}{\|x+y\|_2} = \frac{\|x-y\|_\psi}{\|x-y\|_2} = M_1.$$

Since $\|\cdot\|_\psi$ is absolute and $x, y \in S_X$ $(x \neq y)$ satisfy $\|x\|_2 = \|y\|_2 = 1$, it is sufficient to consider the following three cases:

(i) There exist $s, t \in [0,1]$ $(s \neq t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s, s)$ and $y = \frac{1}{\psi_2(t)}(1-t, t)$.

(ii) There exist $s, t \in [0,1]$ $(s < t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s, s)$ and $y = \frac{1}{\psi_2(t)}(-1+t, t)$.

(iii) There exist $s, t \in [0,1]$ $(s > t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s, s)$ and $y = \frac{1}{\psi_2(t)}(-1+t, t)$.

Case (i). We may suppose that $s < t$. Then there exist $\alpha, \beta \in [0, \frac{\pi}{2}]$ $(\alpha < \beta)$ such that

$$x = \frac{1}{\psi_2(s)}(1-s, s) = (\cos\alpha, \sin\alpha),\ y = \frac{1}{\psi_2(t)}(1-t, t) = (\cos\beta, \sin\beta).$$

Since $\|x\|_2 = \|y\|_2 = 1$, we have

$$x + y = (\frac{1-s}{\psi_2(s)} + \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} + \frac{t}{\psi_2(t)}) = ||x+y||_2(\cos\frac{\alpha+\beta}{2}, \sin\frac{\alpha+\beta}{2}).$$

By [13, Propositions 2a and 2b], we remark that

$$\frac{1-s}{\psi_2(s)} \geq \frac{1-t}{\psi_2(t)},\ \frac{s}{\psi_2(s)} \leq \frac{t}{\psi_2(t)}.$$

Since $x - y$ is orthogonal to $x + y$ in the Euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$, we have

$$\begin{aligned} x - y &= (\frac{1-s}{\psi_2(s)} - \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} - \frac{t}{\psi_2(t)}) \\ &= ||x-y||_2(\cos\frac{\alpha+\beta-\pi}{2}, \sin\frac{\alpha+\beta-\pi}{2}) \\ &= ||x-y||_2(\sin\frac{\alpha+\beta}{2}, -\cos\frac{\alpha+\beta}{2}). \end{aligned}$$

Thus we have

$$\begin{aligned} ||x+y||_\psi &= ||x+y||_2||(\cos\frac{\alpha+\beta}{2}, \sin\frac{\alpha+\beta}{2})||_\psi \\ &= ||x+y||_2(\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2})\psi(\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}). \end{aligned}$$

Since $||x+y||_\psi = M_1||x+y||_2$, we have

$$M_1 = (\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2})\psi(\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}).$$

Putting $r = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}$, then it is clear that $r = \frac{\psi(s)t + \psi(t)s}{\psi(s)+\psi(t)}$ and $M_1 = \frac{\psi(r)}{\psi_2(r)}$. We also have

$$||x-y||_\psi = ||x-y||_2(\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2})\psi(\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}).$$

Since $||x-y||_\psi = M_1||x-y||_2$, we similarly have

$$M_1 = (\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2})\psi(\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}}) = \frac{\psi(1-r)}{\psi_2(1-r)}.$$

Case (ii). Then there exist $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [\frac{\pi}{2}, \pi]$ such that

$$x = \frac{1}{\psi_2(s)}(1-s, s) = (\cos\alpha, \sin\alpha),\ y = \frac{1}{\psi_2(t)}(-1+t, t) = (\cos\beta, \sin\beta).$$

Since $\|x\|_2 = \|y\|_2 = 1$, we have

$$x + y = (\frac{1-s}{\psi_2(s)} - \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} + \frac{t}{\psi_2(t)}) = ||x+y||_2(\cos\frac{\alpha+\beta}{2}, \sin\frac{\alpha+\beta}{2}).$$

By [13, Propositions 2a and 2b], we remark that

$$\frac{1-s}{\psi_2(s)} \geq \frac{1-t}{\psi_2(t)},\ \frac{s}{\psi_2(s)} \leq \frac{t}{\psi_2(t)}.$$

Since $x - y$ is orthogonal to $x + y$ in the Euclidean space $(\mathbb{R}^2, \|\cdot\|_2)$, we have

$$\begin{aligned} x - y &= (\frac{1-s}{\psi_2(s)} + \frac{1-t}{\psi_2(t)}, \frac{s}{\psi_2(s)} - \frac{t}{\psi_2(t)}) \\ &= ||x-y||_2(\cos\frac{\alpha+\beta-\pi}{2}, \sin\frac{\alpha+\beta-\pi}{2}) \\ &= ||x-y||_2(\sin\frac{\alpha+\beta}{2}, -\cos\frac{\alpha+\beta}{2}). \end{aligned}$$

Since $\cos\frac{\alpha+\beta}{2} \geq 0$ and $\sin\frac{\alpha+\beta}{2} \geq 0$, we have

$$\begin{aligned} ||x+y||_\psi &= ||x+y||_2||(\cos\frac{\alpha+\beta}{2}, \sin\frac{\alpha+\beta}{2})||_\psi \\ &= ||x+y||_2(\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2})\psi(\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}). \end{aligned}$$

Since $||x+y||_\psi = M_1||x+y||_2$, we have

$$M_1 = (\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2})\psi(\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}).$$

Putting $r = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}$, then it is clear that $r = \frac{\psi(t)s + \psi(s)t}{\psi(t) + \psi(s)(2t-1)}$ and $M_1 = \frac{\psi(r)}{\psi_2(r)}$. We also have

$$||x-y||_\psi = ||x-y||_2(\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2})\psi(\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\beta}{2}}).$$

Since $||x-y||_\psi = M_1||x-y||_2$, we similarly have

$$M_1 = (\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2})\psi(\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}}) = \frac{\psi(1-r)}{\psi_2(1-r)}.$$

Case (iii). There exist $s, t \in [0,1]$ $(s > t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s, s)$ and $y = \frac{1}{\psi_2(t)}(-1+t, t)$. Then, we put $s_0 = t$ and $t_0 = s$. We define $x_0,\ y_0$ in S_X by

$$x_0 = \frac{1}{\psi(s_0)}(1-s_0, s_0),\ y_0 = \frac{1}{\psi(t_0)}(-1+t_0, t_0).$$

Then we can reduce Case (ii).

($\Longleftarrow$). If we suppose (1) (resp. (2)), then we put $x = \frac{1}{\psi_2(s)}(1-s, s)$ (resp. $x = \frac{1}{\psi_2(s)}(1-s, s)$) and $y = \frac{1}{\psi_2(t)}(1-t, t)$ (resp. $y = \frac{1}{\psi_2(t)}(-1+t, t)$). Then we have $||x||_\psi = ||x||_2 = 1$, $||y||_\psi = ||y||_2 = 1$, $||x+y||_\psi = M_1||x+y||_2$ and

$||x-y||_\psi = M_1||x-y||_2$. Hence it is clear to prove that $C'_{NJ}(X) = M_1^2$. This completes the proof. □

We next study the modified NJ constant in the general case. If $\psi \in \Psi$, then by [11, Therem 3], we have

$$\max\{M_1^2, M_2^2\} \le C_{NJ}(X) \le M_1^2 M_2^2.$$

However, by Theorem 2.2, there exist many $\psi \in \Psi$ satisfying $\psi \ge \psi_2$ such that

$$C'_{NJ}(X) < \max\{M_1^2, M_2^2\} = C_{NJ}(X).$$

From [11, Theorem 3], $C_{NJ}(X) = M_1^2 M_2^2$ if either ψ/ψ_2 or ψ_2/ψ attains a maximum at $t = 1/2$. Then, we have the following

Proposition 2.3. *Let $\psi \in \Psi_2$ and let $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. If ψ/ψ_2 attains a maximum at $t = 1/2$, then $C'_{NJ}(X) = C_{NJ}(X) = M_1^2 M_2^2$.*

Proof. Suppose first $M_1 = \psi(1/2)/\psi_2(1/2)$. Take an arbitrary $t \in [0,1]$ and put

$$x = \frac{1}{\psi(t)}(t, 1-t)\ ,\ y = \frac{1}{\psi(t)}(1-t, t).$$

Then $x, y \in S_X$ and

$$\|x+y\|_\psi = \frac{2}{\psi(t)}\psi(\frac{1}{2})\ ,\ \|x-y\|_\psi = \frac{2|2t-1|}{\psi(t)}\psi(\frac{1}{2}).$$

Therefore we have

$$\begin{aligned}\frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{4} &= \left\{(2t-1)^2 + 1\right\}\frac{\psi(1/2)^2}{\psi(t)^2} \\ &= 2\psi_2(t)^2\frac{\psi(1/2)^2}{\psi(t)^2} \\ &= \frac{\psi_2(t)^2}{\psi(t)^2}\frac{\psi(1/2)^2}{\psi_2(1/2)^2} = M_1^2\frac{\psi_2(t)^2}{\psi(t)^2}.\end{aligned}$$

Since t is arbitrary, we have $C'_{NJ}(X) \ge M_1^2 M_2^2$ which prove that $C'_{NJ}(X) = M_1^2 M_2^2$. □

In the case that $M_2 = \psi_2(1/2)/\psi(1/2)$, $C'_{NJ}(X)$ does not necessarily coincide with $M_1^2 M_2^2$. However, we have the following

Theorem 2.4. *Let $\psi \in \Psi_2$ and let $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. Assume that $M_2 = \psi_2(1/2)/\psi(1/2)$ and $M_1 > 1$. Then $C'_{NJ}(X) = M_1^2 M_2^2$ if and only if there exist $s, t \in [0,1]$ $(s < t)$ satisfying one of the following conditions:*

(1) $\psi_2(s) = M_2\psi(s)$, $\psi_2(t) = M_2\psi(t)$ and, if we put $r = \frac{\psi(s)t+\psi(t)s}{\psi(s)+\psi(t)}$, then $\psi(r) = M_1\psi_2(r)$.

(2) $\psi_2(s) = M_2\psi(s)$, $\psi_2(t) = M_2\psi(t)$ and, if we put $r = \frac{\psi(t)s+\psi(s)t}{\psi(t)+\psi(s)(2t-1)}$, then $\psi(r) = M_1\psi_2(r)$.

Proof. ($\Longrightarrow$). For all $x, y \in S_X$, we have

$$\begin{aligned} \|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\le M_1^2 \left(\|x+y\|_2^2 + \|x-y\|_2^2\right) \\ &= 2M_1^2 \left(\|x\|_2^2 + \|y\|_2^2\right) \\ &\le 2M_1^2 M_2^2 \left(\|x\|_\psi^2 + \|y\|_\psi^2\right) = 4M_1^2 M_2^2. \end{aligned}$$

From this inequality, $C'_{NJ}(X) = M_1^2 M_2^2$ if and only if there exist $x, y \in S_X$ $(x \neq y)$ such that

$$\|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 4M_1^2 M_2^2.$$

Suppose that $C'_{NJ}(X) = M_1^2 M_2^2$. Then, the elements $x, y \in S_X$ $(x \neq y)$ satisfy

$$\|x\|_2 = \|y\|_2 = M_2, \ \|x+y\|_\psi = M_1 \|x+y\|_2, \ \|x-y\|_\psi = M_1 \|x-y\|_2.$$

Since $\|\cdot\|_\psi$ is absolute, it is sufficient to consider the following three cases:

(i) There exist $s, t \in [0,1]$ $(s \neq t)$ satisfying $x = \frac{1}{\psi(s)}(1-s, s)$ and $y = \frac{1}{\psi(t)}(1-t, t)$.

(ii) There exist $s, t \in [0,1]$ $(s < t)$ satisfying $x = \frac{1}{\psi(s)}(1-s, s)$ and $y = \frac{1}{\psi(t)}(-1+t, t)$.

(iii) There exist $s, t \in [0,1]$ $(s > t)$ satisfying $x = \frac{1}{\psi(s)}(1-s, s)$ and $y = \frac{1}{\psi(t)}(-1+t, t)$.

As in the proof of Theorem 2.2, we can prove this theorem. This completes the proof. □

3. The Zbăganu constant of $\mathbb{R}^2$

The Zbăganu constant $C_Z(X)$ in [15] is defined by

$$C_Z(X) = \sup\left\{ \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} \;\middle|\; x, y \in X, \ (x,y) \neq (0,0) \right\}.$$

Then it is clear that $C_Z(X) \le C_{NJ}(X)$ for any Banach space X. In this section, we consider the condition that $C_Z(X) = C_{NJ}(X)$ for $X = (\mathbb{R}^2, \|\cdot\|_\psi)$. Then, we have the following

Proposition 3.1. *Let $\psi \in \Psi_2$. If $\psi \ge \psi_2$, then $C_Z(X) = C_{NJ}(X) = M_1^2$.*

Proof. For any $x, y \in X$,

$$\begin{aligned} 2\|x+y\|_\psi \|x-y\|_\psi &\le \|x+y\|_\psi^2 + \|x-y\|_\psi^2 \\ &\le M_1^2 \left(\|x+y\|_2^2 + \|x-y\|_2^2\right) \\ &= 2M_1^2 \left(\|x\|_2^2 + \|y\|_2^2\right) \\ &\le 2M_1^2 \left(\|x\|_\psi^2 + \|y\|_\psi^2\right). \end{aligned}$$

Since ψ/ψ_2 attains the maximum at $t = t_0$ $(0 \le t_0 \le 1)$, we put $x = (1-t_0, 0)$ and $y = (0, t_0)$, respectively. Then we have

$$\begin{aligned} \|x+y\|_\psi^2 + \|x-y\|_\psi^2 &= 2\psi(t_0)^2 \\ &= 2M_1^2 \psi_2(t_0)^2 \\ &= 2M_1^2 \left(\|x\|_\psi^2 + \|y\|_\psi^2\right). \end{aligned}$$

Since $\|x+y\|_\psi = \psi(t_0) = \|x-y\|_\psi$, we have

$$2\|x+y\|_\psi\|x-y\|_\psi = \|x+y\|_\psi^2 + \|x-y\|_\psi^2 = 2M_1^2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right).$$

Therefore we have

$$\frac{\|x+y\|_\psi\|x-y\|_\psi}{\|x\|_\psi^2+\|y\|_\psi^2} = M_1^2,$$

which implies that $C_Z(X) = M_1^2$. □

We next consider the case that $\psi \le \psi_2$. We remark that the Zbăganu constant $C_Z(X)$ is in the following form;

$$C_Z(X) = \sup\left\{\frac{4\|x\|\|y\|}{\|x+y\|^2+\|x-y\|^2} \;\middle|\; x,y\in X,\ (x,y)\ne(0,0)\right\}.$$

Then we have the following

Theorem 3.2. *Let $\psi \in \Psi_2$. Assume that $\psi \le \psi_2$. Then $C_Z(X) = M_2^2$ if and only if there exist $s,t \in [0,1]$ $(s<t)$ satisfying one of the following conditions:*

(1) *$\psi(s) = \psi_2(s)$, $\psi(t) = \psi_2(t)$ and, if we put $r = \frac{\psi(s)t+\psi(t)s}{\psi(s)+\psi(t)}$, then $\frac{\psi_2(r)}{\psi(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_2$.*

(2) *$\psi(s) = \psi_2(s)$, $\psi(t) = \psi_2(t)$ and, if we put $r = \frac{\psi(t)s+\psi(s)t}{\psi(t)+\psi(s)(2t-1)}$, then $\frac{\psi_2(r)}{\psi(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_2$.*

Proof. For any $x,y\in X$,

$$\begin{aligned}4\|x\|_\psi\|y\|_\psi &\le 2\left(\|x\|_\psi^2+\|y\|_\psi^2\right)\\ &\le 2\left(\|x\|_2^2+\|y\|_2^2\right)\\ &= \|x+y\|_2^2+\|x-y\|_2^2\\ &\le M_2^2\left(\|x+y\|_\psi^2+\|x-y\|_\psi^2\right).\end{aligned}$$

Since $X = (\mathbb{R}^2, \|\cdot\|_\psi)$ is finite dimensional,

$$C_Z(X) = \max\left\{\frac{4\|x\|_\psi\|y\|_\psi}{\|x+y\|_\psi^2+\|x-y\|_\psi^2} \;\middle|\; x,y\in X,\ (x,y)\ne(0,0)\right\}.$$

Then $C_Z(X) = M_2^2$ if and only if there exist $x,y\in S_X$ $(x\ne y)$ such that

$$\frac{4\|x\|_\psi\|y\|_\psi}{\|x+y\|_\psi^2+\|x-y\|_\psi^2} = M_2^2.$$

From the above inequality, $\|x\|_2 = \|x\|_\psi = \|y\|_\psi = \|y\|_2$ and

$$\frac{\|x+y\|_2}{\|x+y\|_\psi} = \frac{\|x-y\|_2}{\|x-y\|_\psi} = M_2^2.$$

Hence we may assume that

$$\|x\|_2 = \|x\|_\psi = \|y\|_\psi = \|y\|_2 = 1.$$

As in the proof of Theorem 2.2, it is sufficient to consider the following three cases:

(i) There exist $s,t \in [0,1]$ $(s \neq t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s,s)$ and $y = \frac{1}{\psi_2(t)}(1-t,t)$.

(ii) There exist $s,t \in [0,1]$ $(s < t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s,s)$ and $y = \frac{1}{\psi_2(t)}(-1+t,t)$.

(iii) There exist $s,t \in [0,1]$ $(s > t)$ satisfying $x = \frac{1}{\psi_2(s)}(1-s,s)$ and $y = \frac{1}{\psi_2(t)}(-1+t,t)$.

As in the proof of Theorem 2.2, we can similarly prove this theorem. □

We next study the Zbăganu constant $C_Z(X)$ in general case. If $\psi \in \Psi$, by [11, Theorem 3], then we have

$$\max\{M_1^2, M_2^2\} \leq C_Z(X) \leq C_{NJ}(X) \leq M_1^2 M_2^2.$$

However, by Theorem 3.2, there exist many $\psi \in \Psi$ satisfying $\psi \geq \psi_2$ such that

$$C_Z(X) < C_{NJ}(X) \leq \max\{M_1^2, M_2^2\}.$$

From [11, Theorem 3], $C_{NJ}(X) = M_1^2 M_2^2$ if either ψ/ψ_2 or ψ_2/ψ attains a maximum at $t = 1/2$. Then, we have the following

Proposition 3.3. *Let $\psi \in \Psi_2$ and let $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. If $M_2 = \frac{\psi_2(1/2)}{\psi(1/2)}$, then $C_Z(X) = C_{NJ}(X) = M_1^2 M_2^2$.*

Proof. From the definition, we have $C_Z(X) \leq C_{NJ}(X) = M_1^2 M_2^2$. Take an arbitrary $t \in [0,1]$ and put $x = (t, 1-t)$ and $y = (1-t, t)$. Then $\|x\|_\psi = \|y\|_\psi = \psi(t)$ and $\|x+y\|_\psi = \|(1,1)\|_\psi = 2\psi(1/2)$, $\|x-y\|_\psi = \|(2t-1, 1-2t)\|_\psi = 2|2t-1|\psi(1/2)$. Hence we have

$$\begin{aligned}\frac{4\|x\|_\psi \|y\|_\psi}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} &= \frac{2\left(\|x\|_\psi^2 + \|y\|_\psi^2\right)}{\|x+y\|_\psi^2 + \|x-y\|_\psi^2} \\ &= \frac{\psi(t)^2}{\left(1+(2t-1)^2\right)\psi(1/2)^2} \\ &= \frac{\psi(t)^2}{2\psi_2(t)^2 \psi(1/2)^2} \\ &= \frac{\psi(t)^2}{\psi_2(t)^2}\frac{\psi_2(1/2)^2}{\psi(1/2)^2} = M_2^2 \frac{\psi(t)^2}{\psi_2(t)^2}\end{aligned}$$

Since t is arbitrary, we have $C_Z(X) \geq M_1^2 M_2^2$. Therefore we have $C_Z(X) = M_1^2 M_2^2$. This completes the proof. □

In case that $M_1 = \psi(1/2)/\psi_2(1/2)$, we have the following theorem as in the proof of Theorem 2.2 and so omit the proof.

Theorem 3.4. *Let $\psi \in \Psi_2$ and let $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. If $M_1 = \frac{\psi(1/2)}{\psi_2(1/2)}$ and $M_2 > 1$, then $C_Z(X) = M_1^2 M_2^2$ if and only if there exist $s,t \in [0,1]$ $(s < t)$ satisfying one of the following conditions:*

(1) $\psi_2(s) = M_2\psi(s)$, $\psi_2(t) = M_2\psi(t)$ *and, if we put* $r = \frac{\psi(s)t+\psi(t)s}{\psi(s)+\psi(t)}$, *then* $\psi(r) = M_1\psi_2(r)$.

(2) $\psi_2(s) = M_2\psi(s)$, $\psi_2(t) = M_2\psi(t)$ *and, if we put* $r = \frac{\psi(t)s+\psi(s)t}{\psi(t)+\psi(s)(2t-1)}$, *then* $\psi(r) = M_1\psi_2(r)$.

4. Examples

In this section, we calculate $C'_{NJ}(X)$ and $C_Z(X)$ of some Banach spaces $X = (\mathbb{R}^2, \|\cdot\|_\psi)$, where $\psi \in \Psi$. First, we consider the case that $\psi = \psi_p$.

Example 4.1. Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. We put $t = \min(p, q)$. Then $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_p) = C_Z(\mathbb{R}^2, ||\cdot||_p) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_p) = 2^{\frac{2}{t}-1}$.

Suppose that $1 \le p \le 2$. Since $\psi_p \ge \psi_2$, we have $C_Z(\mathbb{R}^2, ||\cdot||_p) = 2^{\frac{2}{p}-1}$ by Proposition 3.1. On the other hand, as in Theorem 2.2, we take $s = 0$ and $t = 1$. Since $r = \frac{\psi(0)\cdot 1+\psi(1)\cdot 0}{\psi(0)+\psi(1)} = \frac{1}{2}$ and $M_1 = \psi_p(1/2)/\psi_2(1/2) = 2^{\frac{1}{p}-\frac{1}{2}}$, by Theorem 2.2, we have $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_p) = M_1^2 = 2^{\frac{2}{p}-1}$.

If $2 \le p \le \infty$, then we similarly have, by Proposition 2.1 and Theorem 3.2, $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_p) = C_Z(\mathbb{R}^2, ||\cdot||_p) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_p) = 2^{\frac{2}{p}-1}$.

In [14, Example], C. Yang and H. Li calculated the modified NJ constant of the following normed linear space. From our theorems, we have

Example 4.2. Let $\lambda > 0$ and $X_\lambda = \mathbb{R}^2$ endowed with norm

$$||(x, y)||_\lambda = (||(x, y)||_p^2 + \lambda||(x, y)||_q^2)^{1/2}.$$

(i) If $2 \le p \le q \le \infty$, then $C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = C_Z(X_\lambda) = \frac{2(\lambda+1)}{2^{2/p}+\lambda 2^{2/q}}$.

(ii) If $1 \le p \le q \le 2$, then $C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = C_Z(X_\lambda) = \frac{2^{2/p}+\lambda 2^{2/q}}{2(\lambda+1)}$.

To see this, first, we remark that (p, q) is not necessarily a Hölder pair. We define the normalized norm $||\cdot||_\lambda^0$ by

$$||(x, y)||_\lambda^0 = \frac{||(x, y)||_\lambda}{\sqrt{1+\lambda}}.$$

Then $||\cdot||_\lambda^0$ is absolute and so put the corresponding function $\psi_\lambda(t) = ||(1-t, t)||_\lambda^0$.
(i) Suppose that $2 \le p \le q \le \infty$. Since $\psi_\lambda \le \psi_2$, by Proposition 2.1, we have $C_{NJ}(X_\lambda) = C'_{NJ}(X_\lambda) = M_2^2 = \frac{2(\lambda+1)}{2^{2/p}+\lambda 2^{2/q}}$. On the other hand, in Theorem 3.2, we take $s = 0$ and $t = 1$. Then we have $r = 1/2$ and $\frac{\psi_2(1/2)}{\psi_\lambda(1/2)} = M_2$. Thus we have $C_Z(X_\lambda) = M_2^2 = \frac{2^{2/p}+\lambda 2^{2/q}}{2(\lambda+1)}$.
(ii) Suppose that $1 \le p \le q \le 2$. Since $\psi_\lambda \ge \psi_2$, by Theorem 2.2 and Proposition 3.1, we similarly have (ii).

Example 4.3. Put

$$\psi(t) = \begin{cases} \psi_2(t) & (0 \le t \le 1/2), \\ \left(2-\sqrt{2}\right)t + \sqrt{2} - 1 & (1/2 \le t \le 1). \end{cases}$$

Then $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) < C_Z(\mathbb{R}^2, \|\cdot\|_\psi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = 2\sqrt{2}(\sqrt{2}-1)$.

In fact, $\psi \in \Psi_2$ and the norm of $\|\cdot\|_\psi$ is

$$\|(a,b)\|_\psi = \begin{cases} \sqrt{|a|^2+|b|^2} & (|a| \geq |b|) \\ \left(\sqrt{2}-1\right)|a|+|b| & (|a| \leq |b|). \end{cases}$$

Since $\psi \geq \psi_2$, by Proposition 3.1, we have $C_Z\left(\mathbb{R}^2, \|\cdot\|_\psi\right) = M_1^2 = 2\sqrt{2}\left(\sqrt{2}-1\right)$.

We assume that $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) = M_1^2$. By Theorem 2.2, we can choose $r \in [0,1]$ such that $\frac{\psi(r)}{\psi_2(r)} = \frac{\psi(1-r)}{\psi_2(1-r)} = M_1$. This is impossible by the definition of ψ. Therefore we have $C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\psi) < M_1^2$.

Example 4.4. Let $1/2 \leq \beta \leq 1$. We define a convex function $\psi_\beta \in \Psi_2$ by

$$\psi_\beta(t) = \max\{1-t, t, \beta\}.$$

By [11, Example 4], we have

$$C_{NJ}(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = \begin{cases} \dfrac{\beta^2+(1-\beta)^2}{\beta^2} & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ 2(\beta^2+(1-\beta)^2) & (\beta \in (\frac{1}{\sqrt{2}}, 1]). \end{cases}$$

Indeed,

$$M_1 = \begin{cases} 1 & (\beta \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]) \\ \frac{\psi_\beta(1/2)}{\psi_2(1/2)} = \frac{\beta}{1/\sqrt{2}} = \sqrt{2}\beta & (\beta \in (\frac{1}{\sqrt{2}}, 1]) \end{cases}$$

and

$$M_2 = \frac{\psi_2(\beta)}{\psi_\beta(\beta)} = \frac{1}{\beta}\{(1-\beta)^2+\beta^2\}^{1/2}.$$

If $1/2 \leq \beta \leq 1/\sqrt{2}$, then $\psi_\beta \leq \psi_2$ and so, by Proposition 2.1, we have

$$C'_{NJ}(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = M_2^2 = \frac{\beta^2+(1-\beta)^2}{\beta^2}.$$

By Theorem 3.2, we have $C_Z(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) < M_2^2$.

Assume that $1/\sqrt{2} < \beta \leq 1$. Since $M_1 = \frac{\psi_\beta(1/2)}{\psi_2(1/2)}$, we have, by Proposition 2.3,

$$C'_{NJ}(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = M_1^2 M_2^2 = 2(\beta^2+(1-\beta)^2).$$

On the other hand, we take $s = \beta$ and $t = 1-\beta$ in Theorem 3.4. Then we have $r = \frac{\psi(\beta)(1-\beta)+\psi(1-\beta)\beta}{\psi(\beta)+\psi(1-\beta)} = 1/2$. By Theorem 3.4, we have

$$C_Z(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = M_1^2 M_2^2 = 2(\beta^2+(1-\beta)^2).$$

Example 4.5. We consider ψ_β in Example 4.4 in case of $\beta = 1/\sqrt{2}$. Then we have

$$C'_{NJ}(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = M_2^2 = 2\sqrt{2}(\sqrt{2}-1).$$

On the other hand, we have $C_Z(\mathbb{R}^2, \|\cdot\|_{\psi_\beta}) = M_2^2 = 2\sqrt{2}(\sqrt{2}-1)$.

For this ψ_β, define a convex function $\varphi \in \Psi_2$ by

$$\varphi(t) = \begin{cases} \psi_\beta(t) & (0 \leq t \leq 1/2), \\ \psi_2(t) & (1/2 \leq t \leq 1). \end{cases}$$

As in Example 4.2, we similarly have

$$C_Z(\mathbb{R}^2, \|\cdot\|_\varphi) < C'_{NJ}(\mathbb{R}^2, \|\cdot\|_\varphi) = C_{NJ}(\mathbb{R}^2, \|\cdot\|_\varphi) = M_2^2 = 2\sqrt{2}\left(\sqrt{2}-1\right).$$

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References

1. J. Alonso, P. Martin and P.L. Papini, *Wheeling around von Neumann-Jordan constant in Banach Spaces*, Studia Math. **188** (2008), no. 2, 135–150.
2. J. Alonso and P. Martin, *A counterexample for a conjecture of G. Zbâganu about the Neumann-Jordan constant*, Rev. Roumaine Math. Pures Appl. **51** (2006), 135–141.
3. J.A. Clarkson, *The von Neumann-Jordan constant for the Lebesgue spaces*, Ann. of Math. (2) **38** (1937), no. 1, 114–115.
4. J. Gao, *A Pythagorean approach in Banach spaces*, J. Inequal. Appl. (2006), Art. ID 94982, 1–11.
5. J. Gao and K. Lau, *On the geometry of spheres in normed linear spaces*, J. Austral. Math. Soc., **48**(1990), 101–112.
6. J. Gao and S. Saejung, *Normal structure and the generalized James and Zbăganu constant*, Nonlinear Anal. **71** (2009), no. 7-8, 3047–3052.
7. J. Gao and S. Saejung, *Some geometric measures of spheres in Banach spaces*, Appl. Math. Comput. **214** (2009), no. 1, 102–107.
8. P. Jordan and J. von Neumann, *On inner products in linear metric spaces*, Ann. of Math. (2) **36** (1935), no. 3, 719–723.
9. E. Llorens-Fuster, E.M. Mazcuñán-Navarro and S. Reich, *The Ptolemy and Zbăganu constants of normed spaces*, Nonlinear Anal. **72** (2010), no. 11, 3984–3993.
10. M. Kato and Y. Takahashi, *On sharp estimates concerning von Neumann-Jordan and James constants for a Banach space*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **82** (2010), 1–17.
11. K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), no. 2, 515–532.
12. Y. Takahashi and M. Kato, *A simple inequality for the von Neumann-Jordan and James constants of a Banach space*, J. Math. Anal. Appl. **359** (2009), no. 2, 602–609.
13. Y. Takahashi, M. Kato and K. -S. Saito, *Strict convexity of absolute norms on* $\mathbb{C}^2$ *and direct sums of Banach spaces*, J. Inequal. Appl., **7**(2002), 179–186.
14. C. Yang and H. Li, *An inequality between Jordan-con Neumann constant and James constant*, Appl. Math. Lett. **23** (2010), no. 3, 277–281.
15. G. Zbăganu, *An inequality of M. Rădulescu and S. Rădulescu which characterizes inner product spaces*, Rev. Roumaine Math. Puers Appl. **47** (2001), 253–257.

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