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# A GENERAL ITERATIVE ALGORITHM FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Let $E$ be a real $q$-uniformly smooth Banach space whose duality map is weakly sequentially continuous. Let $T : E \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $A : E \to E$ be an $\eta$-strongly accretive map which is also $\kappa$-Lipschitzian. Let $f : E \to E$ be a contraction map with coefficient $0 < \alpha < 1$. Let a sequence $\{y_n\}$ be defined iteratively by $y_0 \in E,\ y_{n+1} = \alpha_n \gamma f(y_n) + (I - \alpha_n \mu A)Ty_n, n \geq 0$, where $\{\alpha_n\},\ \gamma$ and $\mu$ satisfy some appropriate conditions. Then, we prove that $\{y_n\}$ converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality $\langle(\gamma f - \mu A)x^*, j(y - x^*)\rangle \leq 0,\ \forall\ y \in F(T)$. Convergence of the correspondent implicit scheme is also proved without the assumption that $E$ has weakly sequentially continuous duality map. Our results are applicable in $l_p$ spaces, $1 < p < \infty$.

## 1. Introduction

Let $E$ be a real Banach space and $E^*$ be the dual space of $E$. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a gauge function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let $\varphi$ be a gauge function, a generalized duality mapping with respect to $\varphi$, $J_\varphi : E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\},$$

where $\langle ., .\rangle$ denotes the duality pairing between element of $E$ and that of $E^*$. If $\varphi(t) = t$, then $J_\varphi$ is simply called the normalized duality mapping and is denoted by $J$. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi(x)$.
If however $\varphi(t) = t^{q-1}$, for some $q > 1$, then $J_\varphi$ is still called the generalized

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duality mapping and is denoted by $J_q$ (see, for example [9, 10]).

The space $E$ is said to have weakly (sequentially) continuous duality map if there exists a gauge function $\varphi$ such that $J_\varphi$ is singled valued and (sequentially) continous from $E$ with weak topology to $E^*$ with weak* topology. It is well known that all $l_p$ spaces, $(1 < p < \infty)$ have weakly sequentially continuous duality mappings. It is well known (see, for example, [16]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$, and that if $E^*$ is strictly convex then $J_q$ is single valued.

A mapping $A : D(A) \subset E \to E$ is said to be *accretive* if $\forall x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq 0, \tag{1.1}$$

where $D(A)$ denotes the domain of $A$. $A$ is called $\eta-$*strongly accretive* if $\forall x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ and $\eta \in (0,1)$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \eta\|x-y\|^q. \tag{1.2}$$

$A$ is $\kappa-$*Lipschitzian* if for some $\kappa > 0$, $\|A(x) - A(y)\| \leq \kappa\|x-y\| \ \forall\ x, y \in D(A)$. A mapping $T : E \to E$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x-y||,\ \forall x, y \in E.$$

A point $x \in E$ is called *a fixed point* of $T$ if $Tx = x$. The set of fixed points of $T$ is denoted by $F(T) := \{x \in E : Tx = x\}$. In Hilbert spaces, accretive operators are called *monotone* where inequalities (1.1) and (1.2) hold with $j_q$ replaced by the identity map on $H$.

Moudafi [5] introduced the viscosity approximation method for nonexpansive mappings. Let $f$ be a contraction on $H$, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n,\ n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Xu [12] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.3) strongly converges to the unique solution $x^*$ in $F(T)$ of the variational inequality

$$\langle (I-f)x^*, x - x^*\rangle \geq 0,\ \text{for } x \in F(T).$$

In [13], it is proved, under some conditions on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrary,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n,\ n \geq 0, \tag{1.4}$$

converges strongly to $x^* \in F(T)$ which is the unique solution of the minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where $A$ is a strongly positive bounded linear operator. That is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}||x||^2,\ \forall x \in H.$$

Combining the iterative method (1.3) and (1.4), Marino and Xu [7] consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \; n \geq 0. \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $x^* \in F(T)$ which solves the variational inequality

$$\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \; x \in F(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where $h$ is a potential function for $\gamma f$ (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$. The variational inequality problem: Find a point $x^* \in K$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \; \forall y \in K$$

is equivalent to the following fixed point equation

$$x^* = P_K(x^* - \delta Ax^*), \tag{1.6}$$

where $\delta > 0$ is an arbitrary fixed constant, $A$ is a nonlinear operator on $K$ and $P_K$ is the *nearest point projection map* from $H$ onto $K$, i.e., $P_K x = y$ where $\|x - y\| = \inf_{u \in K} \|x - u\|$ for $x \in H$. Consequently, under appropriate conditions on $A$ and $\delta$, fixed point methods can be used to find or approximate a solution of the variational inequality. Considerable efforts have been devoted to this problem (see, for example, [14, 17] and the references contained therein). For instance, if $A$ is strongly monotone and Lipschitz then, a mapping $B : H \to H$ defined by $Bx = P_K(x - \delta Ax)$, $x \in H$ with $\delta > 0$ sufficiently small is a strict contraction. Hence, the *Picard iteration,* $x_0 \in H$, $x_{n+1} = Bx_n$, $n \geq 0$ of the classical Banach contraction mapping principle converges to the unique solution of the variational inequality. It has been observed that the projection operator $P_K$ in the fixed point formulation (1.6) may make the computation of the iterates difficult due to possible complexity of the convex set $K$. In order to reduce the possible difficulty with the use of $P_K$, Yamada [17] introduced the following hybrid descent method for solving the variational inequality:

$$x_{n+1} = Tx_n - \lambda_n \mu A(Tx_n), \; n \geq 0, \tag{1.7}$$

where $T$ is a nonexpansive mapping, $A$ is an $\eta-$strongly monotone and $\kappa-$Lipschitz operator with $\eta > 0, \; \kappa > 0, \; 0 < \mu < \frac{2\eta}{\kappa^2}$. He proved that if $\{\lambda_n\}$ satisfies appropriate conditions then, $\{x_n\}$ converges strongly to the unique solution $x^*$ of the variational inequality

$$\langle Ax^*, x - x^* \rangle \geq 0, \; x \in F(T).$$

Very recently, Tian [6] combined the Yamada's method (1.7) with the iterative method (1.5) and introduced the following general iterative method in Hilbert spaces:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n,\ n \geq 0. \tag{1.8}$$

Then, he proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution $x^* \in F(T)$ of the variational inequality

$$\langle (\gamma f - \mu A)x^*, x - x^* \rangle \leq 0,\ x \in F(T).$$

We remark immediately here that the results of Tian [6] improved the results of Yamada [17], Moudafi [5], Xu [12] and Marino and Xu [13] in Hilbert spaces.

In this paper, motivated and inspired by the above research results, our purpose is to extend the result of Tian [6] to $q$-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Thus, our results are applicable in $l_p$ spaces, $1 < p < \infty$. Furthermore, our results extend the results of Moudafi [5], Xu [12] and Marino and Xu [13] to Banach spaces much more general than Hilbert.

## 2. Preliminaries

Let $E$ be a real Banach space. Let $K$ be a nonempty closed convex and bounded subset of a Banach space $E$ and let the diameter of $K$ be defined by $d(K) := sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := sup\{\|x - y\| : y \in K\}$ and let $r(K) := inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of $K$ relative to itself. The *normal structure coefficient* $N(E)$ of $E$ (see, for example, [1]) is defined by $N(E) := inf\Big\{\frac{d(K)}{r(K)}$ : $K$ is a closed convex and bounded subset of E with $d(K) > 0\Big\}$. A space $E$ such that $N(E) > 1$ is said to have uniform normal structure. It is known that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, for example, [2, 4]).

Let $\mu$ be a linear continuous functional on $\ell^\infty$ and let $a = (a_1, a_2, \ldots) \in \ell^\infty$. We will sometimes write $\mu_n(a_n)$ in place of the value $\mu(a)$. A linear continuous functional $\mu$ such that $||\mu|| = 1 = \mu(1)$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for every $a = (a_1, a_2, \ldots) \in \ell^\infty$ is called a *Banach limit*. It is known that if $\mu$ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in \ell^\infty$ (see, for example, [2, 3]).

Let $E$ be a normed space with dimE $\geq 2$. The *modulus of smoothness* of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x + y\| \ + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}.$$

The space $E$ is called *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$. For some positive constant $q$, $E$ is called $q-$*uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(t) \le ct^q,\ t > 0$. It is known that

$$L_p(or\ l_p) \text{ spaces are } \begin{cases} 2- \text{ uniformly smooth, if, } 2 \le p < \infty \\ p- \text{ uniformly smooth, if, } 1 < p \le 2. \end{cases}$$

It is well known that if $E$ is smooth then the duality mapping is singled-valued, and if $E$ is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

We shall make use of the following well known results.

**Lemma 2.1.** *Let $E$ be a real normed space. Then*

$$||x + y||^2 \le ||x||^2 + 2\langle y, j(x + y)\rangle,$$

*for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.*

**Lemma 2.2.** (Xu, [15]) *Let $E$ be a real q-uniformly smooth Banach space for some $q > 1$, then there exists some positive constant $d_q$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q\ \forall\ x, y \in E \text{ and } j_q(x) \in J_q(x).$$

**Lemma 2.3.** (Xu, [11]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \ge 0,$$

*where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup\ \sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.*

**Lemma 2.4.** (Lim and Xu, [4]) *Suppose $E$ is a Banach space with uniform normal structure, $K$ is a nonempty bounded subset of $E$, and $T : K \to K$ is uniformly $k-$Lipschitzian mapping with $k < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset $C$ of $K$ with the following property $(P)$ :*

$$(P)\ x \in C \text{ implies } \omega_w(x) \subset C,$$

*where $\omega_w(x)$ is the $\omega-$limit set of $T$ at $x$, i.e., the set*

$$\{y \in E : y = \text{weak} - \lim_j T^{n_j}x \text{ for some } n_j \to \infty\}.$$

*Then, $T$ has a fixed point in $C$.*

**Lemma 2.5.** (Jung, [8]) *Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $E$ which satisfies Opial's condition and suppose $T : C \to E$ is nonexpansive. Then $I - T$ is demiclosed at zero, i.e., $x_n \rightharpoonup x,\ x_n - Tx_n \to 0$ implies that $x = Tx$.*

**Lemma 2.6.** *Let $E$ be a real Banach space, $f : E \to E$ a contraction with coefficient $0 < \alpha < 1$, and $A : E \to E$ a $\kappa$-Lipschitzian and $\eta$-strongly accretive operator with $\kappa > 0,\ \eta \in (0, 1)$. Then for $\gamma \in (0, \frac{\mu\eta}{\alpha})$,*

$$\langle(\mu A - \gamma f)x - (\mu A - \gamma f)y, j(x - y)\rangle \ge (\mu\eta - \gamma\alpha)||x - y||^2,\ \forall x, y \in E.$$

*That is, $\mu A - \gamma f$ is strongly accretive with coefficient $\mu\eta - \gamma\alpha$.*

## 3. Main Results

We begin with the following lemma.

**Lemma 3.1.** *Let $E$ be a $q$-uniformly smooth real Banach space with constant $d_q, q > 1$. Let $f : E \to E$ be a contraction mapping with constant of contraction $\alpha \in (0,1)$. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $A : E \to E$ be an $\eta$-strongly accretive mapping which is also $\kappa$-Lipschitzian. Let $\mu \in \Big(0, \min\Big\{1, (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{q-1}}\Big\}\Big)$ and $\tau := \mu\big(\eta - \frac{\mu^{q-1}d_q\kappa^q}{q}\big)$. For each $t \in (0,1)$ and $\gamma \in (0, \frac{\tau}{\alpha})$ define a map $T_t : E \to E$ by*

$$T_t x = t\gamma f(x) + (I - t\mu A)Tx,\ x \in E.$$

*Then, $T_t$ is a strict contraction. Furthermore*

$$||T_t x - T_t y|| \leq [1 - t(\tau - \gamma\alpha)]||x - y||.$$

*Proof.* Without loss of generality, assume $\eta < \frac{1}{q}$. Then, as $\mu < (\frac{q\eta}{d_q\kappa^q})^{\frac{1}{(q-1)}}$, we have $0 < q\eta - \mu^{q-1}d_q\kappa^q$. Furthermore, from $\eta < \frac{1}{q}$ we have $q\eta - \mu^{q-1}d_q\kappa^q < 1$ so that $0 < q\eta - \mu^{q-1}d_q\kappa^q < 1$. Also as $\mu < 1$ and $t \in (0,1)$ we obtained that $0 < t\mu(q\eta - \mu^{q-1}d_q\kappa^q) < 1$.

For each $t \in (0,1)$, define $S_t x = (I - t\mu A)Tx,\ x \in E$, then for $x, y \in K$

$$\begin{aligned}
||S_t x - S_t y||^q &= ||(I - t\mu A)Tx - (I - \mu A)Ty||^q \\
&= ||(Tx - Ty) - t\mu(A(Tx) - A(Ty))||^q \\
&\leq ||Tx - Ty||^q - qt\mu\langle A(Tx) - A(Ty), j_q(Tx - Ty)\rangle \\
&\quad + t^q\mu^q d_q||A(Tx) - A(Ty)||^q \\
&\leq ||Tx - Ty||^q - qt\mu\eta||Tx - Ty||^q \\
&\quad + t^q\mu^q\kappa^q d_q||Tx - Ty||^q \\
&\leq [1 - t\mu(q\eta - t^{q-1}\mu^{q-1}\kappa^q d_q)]||x - y||^q \\
&\leq \Big[1 - qt\mu\big(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q}\big)\Big]||x - y||^q \\
&\leq \Big[1 - t\mu\big(\eta - \frac{\mu^{q-1}\kappa^q d_q}{q}\big)\Big]^q||x - y||^q \\
&= (1 - t\tau)^q||x - y||^q. \qquad (3.1)
\end{aligned}$$

It then follows from (3.1) that,

$$||S_t x - S_t y|| \leq (1 - t\tau)||x - y||.$$

Using the fact that $T_t x = t\gamma f(x) + S_t x,\ x \in E$, we obtain for all $x, y \in E$ that

$$\begin{aligned}
||T_t x - T_t y|| &= ||t\gamma(f(x) - f(y)) + (S_t x - S_t y)|| \\
&\leq t\gamma||f(x) - f(y)|| + ||S_t x - S_t y|| \\
&\leq t\gamma\alpha||x - y|| + (1 - t\tau)||x - y|| \\
&= [1 - t(\tau - \gamma\alpha)]||x - y||.
\end{aligned}$$

Therefore

$$||T_t x - T_t y|| \leq [1 - t(\tau - \gamma\alpha)]||x - y||,$$

which implies that $T_t$ is a strict contraction. Therefore, by Banach contraction mapping principle, there exists a unique fixed point $x_t$ of $T_t$ in $E$. That is,

$$x_t = t\gamma f(x_t) + (I - t\mu A)Tx_t. \tag{3.2}$$

□

**Proposition 3.2.** *Let $\{x_t\}$ be defined by (3.2), then*

(i) *$\{x_t\}$ is bounded for $t \in (0, \frac{1}{\tau})$.*
(ii) *$\lim\limits_{t\to 0}||x_t - Tx_t|| = 0$.*

*Proof.* (i) For any $p \in F(T)$, we have

$$\begin{aligned} ||x_t - p|| &= ||(I - t\mu A)Tx_t - (I - t\mu A)p + t(\gamma f(x_t) - \mu A(p))|| \\ &\leq (1 - t\tau)||x_t - p|| + t\gamma\alpha||x_t - p|| + t||\gamma f(p) - \mu A(p)|| \\ &= [1 - t(\tau - \gamma\alpha)]||x_t - p|| + t||\gamma f(p) - \mu A(p)||. \end{aligned}$$

Therefore,

$$||x_t - p|| \leq \frac{1}{\tau - \gamma\alpha}||\gamma f(p) - \mu A(p)||.$$

Hence, $\{x_t\}$ is bounded. Furthermore $\{f(x_t)\}$ and $\{A(Tx_t)\}$ are also bounded.
(ii) From (3.2), we have

$$||x_t - Tx_t|| = t||\gamma f(x_t) - \mu A(Tx_t)|| \to 0 \text{ as } t \to 0. \tag{3.3}$$

□

Next, we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\}$ be a sequence in (0,1) such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.3), we obtain that

$$||x_n - Tx_n|| \to 0 \text{ as } n \to \infty.$$

**Theorem 3.3.** *Assume that $\{x_t\}$ is defined by (3.2), then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x}$ of $T$ which solves the variational inequality problem:*

$$\langle(\mu A - \gamma f)\tilde{x}, j(\tilde{x} - z)\rangle \leq 0, \ z \in F(T). \tag{3.4}$$

*Proof.* By Lemma 2.6, $(\mu A - \gamma f)$ is strongly accretive, so the variational inequality (3.4) has a unique solution in $F(T)$. Below we use $x^* \in F(T)$ to denote the unique solution of (3.4).

We next prove that $x_t \to x^*$ $(t \to 0)$. Now, define a map $\phi : E \to \mathbb{R}$ by

$$\phi(x) := \mu_n||x_n - x||^2, \ \forall x \in E,$$

where $\mu_n$ is a Banach limit for each $n$. Then, $\phi(x) \to \infty$ as $||x|| \to \infty$, $\phi$ is continuous and convex, so as $E$ is reflexive, it follows that there exits $y^* \in E$ such that $\phi(y^*) = \min\limits_{u\in E}\phi(u)$. Hence, the set

$$K^* := \{x \in E : \phi(x) = \min_{u\in E}\phi(u)\} \neq \emptyset.$$

We now show that $T$ has a fixed point in $K^*$. We shall make use of Lemma 2.4. If $x$ is in $K^*$ and $y := \omega - \lim_j T^{m_j}x$, then from the weak lower semi-continuity of $\phi$ (since $\phi$ is lower semi-continuous and convex) and $\lim\limits_{n\to\infty}||x_n - Tx_n|| = 0$, we have (since $\lim\limits_{n\to\infty}||x_n - Tx_n|| = 0$ implies $\lim\limits_{n\to\infty}||x_n - T^m x_n|| = 0,\ m \geq 1$, this is easily proved by induction),

$$\begin{aligned}
\phi(y) &\leq \liminf_{j\to\infty}\phi\Big(T^{m_j}x\Big) \leq \limsup_{m\to\infty}\phi\Big(T^m x\Big)\\
&= \limsup_{m\to\infty}\Big(\mu_n||x_n - T^m x||^2\Big)\\
&= \limsup_{m\to\infty}\Big(\mu_n||x_n - T^m x_n + T^m x_n - T^m x||^2\Big)\\
&\leq \limsup_{m\to\infty}\Big(\mu_n||T^m x_n - T^m x||^2\Big) \leq \limsup_{m\to\infty}\Big(\mu_n||x_n - x||^2\Big) = \phi(x)\\
&= \min_{u\in E}\phi(u).
\end{aligned}$$

So, $y \in K^*$. By Lemma 2.4, $T$ has a fixed point in $K^*$ and so $K^* \cap F(T) \neq \emptyset$. Now let $y \in K^* \cap F(T)$. Then, it follows that $\phi(y) \leq \phi(y + t(\gamma f - \mu A)y)$ and using Lemma 2.1, we obtain that

$$||x_n - y - t(\gamma f - \mu A)y||^2 \leq ||x_n - y||^2 - 2t\langle(\gamma f - \mu A)y, j(x_n - y - t(\gamma f - \mu A)y)\rangle.$$

This implies that $\mu_n\langle(\gamma f - \mu A)y, j(x_n - y - t(\gamma f - \mu A)y)\rangle \leq 0$. Moreover, $\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle = \mu_n\langle(\gamma f - \mu A)y, j(x_n - y) - j(x_n - y + t(\mu A - \gamma f)y)\rangle + \mu_n\langle(\gamma f - \mu A)y, j(x_n - y + t(\mu A - \gamma f)y)\rangle \leq \mu_n\langle(\gamma f - \mu A)y, j(x_n - y) - j(x_n - y + t(\mu A - \gamma f)y)\rangle$.

Since $j$ is norm-to-norm uniformly continuous on bounded subsets of $E$, we obtain as $t \to 0$ that

$$\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle \leq 0.$$

Now, using (3.2), we have

$$\begin{aligned}
||x_n - y||^2 &= t_n\langle\gamma f(x_n) - \mu Ay, j(x_n - y)\rangle + \langle(I - t_n\mu A)(Tx_n - y), j(x_n - y)\rangle\\
&= t_n\langle\gamma f(x_n) - \mu Ay, j(x_n - y)\rangle + \langle(I - \mu A)Tx_n - (I - \mu A)y, j(x_n - y)\rangle\\
&\leq [1 - t_n(\tau - \gamma\alpha)]||x_n - y||^2 + t_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle.
\end{aligned}$$

So,

$$||x_n - y||^2 \leq \frac{1}{\tau - \gamma\alpha}\langle(\gamma f - \mu A)y, j(x_n - y)\rangle.$$

Again, taking Banach limit, we obtain

$$\mu_n||x_n - y||^2 \leq \frac{1}{\tau - \gamma\alpha}\mu_n\langle(\gamma f - \mu A)y, j(x_n - y)\rangle \leq 0,$$

which implies that $\mu_n||x_n - y||^2 = 0$. Hence, there exists a subsequence of $\{x_n\}_{n=1}^\infty$ which we still denoted by $\{x_n\}_{n=1}^\infty$ such that $\lim\limits_{n\to\infty} x_n = y$. We now show that $y$ solves the variational inequality (3.4). Since

$$x_t = t\gamma f(x_t) + (I - t\mu A)Tx_t,$$

we can derive that

$$(\mu A-\gamma f)(x_t)=-\frac{1}{t}(I-T)x_t+\mu(Ax_t-ATx_t).$$

It follows that for $z\in F(T)$,

$$\begin{aligned}\langle(\mu A-\gamma f)(x_t),j(x_t-z)\rangle &= -\frac{1}{t}\langle(I-T)x_t-(I-T)z,j(x_t-z)\rangle\\ &\quad+\mu\langle(Ax_t-ATx_t),j(x_t-z)\rangle\\ &\leq \mu\langle(Ax_t-ATx_t),j(x_t-z)\rangle. \qquad (3.5)\end{aligned}$$

Since $T$ is nonexpansive, then, $I-T$ is accretive, which implies, $\langle(I-T)x_t-(I-T)z,j(x_t-z)\rangle\geq 0$. Now replacing $t$ in (3.5) with $t_n$ and letting $n\to\infty$, noticing that $(Ax_{t_n}-ATx_{t_n})\to(Ay-Ay)$ we obtain

$$\langle(\mu A-\gamma f)y,j(y-z)\rangle\leq 0,$$

since $z\in F(T)$ is arbitrary, we get $y=x^*$.
Assume now that there exists another subsequence $\{x_m\}_{m=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\lim\limits_{m\to\infty}x_m=u^*$. Then, since $\lim\limits_{n\to\infty}||x_n-Tx_n||=0$, we have that $u^*\in F(T)$. Repeating the argument above with $y$ replaced by $u^*$ we will get that $u^*$ solves the variational inequality (3.4), and so by uniqueness, we obtain $x^*=y=u^*$. This complete the proof. □

**Theorem 3.4.** *Let $E$ be a real $q$-uniformly smooth Banach space with whose duality map is weakly sequentially continuous. Let $T:E\to E$ be a nonexpansive mapping with $F(T)\neq\emptyset$. Let $A:E\to E$ be an $\eta$-strongly accretive map which is also $\kappa$-Lipschitzian. Let $f:E\to E$ be a contraction map with coefficient $0<\alpha<1$. Let $\{\alpha_n\}_{n=1}^\infty$ be a real sequence in [0,1] satisfying:*
$(C1)$ $\lim\alpha_n=0$,
$(C2)$ $\sum\alpha_n=\infty$ *and*
$(C3)$ $\sum|\alpha_{n+1}-\alpha_n|<\infty$.
*Let $\mu$, $\gamma$ and $\tau$ be as in Lemma 3.1. Define a sequence $\{y_n\}_{n=1}^\infty$ iteratively in $E$ by $y_0\in E$,*

$$y_{n+1}=\alpha_n\gamma f(y_n)+(I-\alpha_n\mu A)Ty_n. \qquad (3.6)$$

*Then, $\{y_n\}_{n=1}^\infty$ converges strongly to $x^*\in F(T)$ which is also a solution to the following variational inequality*

$$\langle(\gamma f-\mu A)x^*,j(y-x^*)\rangle\leq 0,\ \forall\ y\in F(T). \qquad (3.7)$$

*Proof.* Since the mapping $T:E\to E$ is nonexpansive, then from Theorem 3.3, the variational inequality (3.7) has a unique solution $x^*$ in $F(T)$. Furthermore, the sequence $\{y_n\}$ satisfies

$$||y_n-x^*||\leq\max\Big\{||y_0-x^*||,\frac{||\gamma f(x^*)-\mu Ax^*||}{\tau-\gamma\alpha}\Big\},\ \forall n\geq 0.$$

It is obvious that this is true for $n = 0$. Assume it is true for $n = k$ for some $k \in \mathbb{N}$, from the recursion formula (3.6), we have

$$\begin{aligned} ||y_{k+1} - x^*|| &= ||\alpha_k \gamma f(y_k) + (I - \alpha_k \mu A)Ty_k - x^*|| \\ &= ||\alpha_k(\gamma f(y_k) - \mu A x^*) + (I - \alpha_k \mu A)Ty_k - (I - \alpha_k \mu A)x^*|| \\ &\leq [1 - \alpha_k(\tau - \gamma\alpha)]||y_k - x^*|| + \alpha_k(\tau - \gamma\alpha)\frac{||\gamma f(x^*) - \mu A x^*||}{\tau - \gamma\alpha} \\ &\leq \max\left\{||y_k - x^*||, \frac{||\gamma f(x^*) - \mu A x^*||}{\tau - \gamma\alpha}\right\} \end{aligned}$$

and the claim follows by induction. Thus, the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded and so are $\{f(y_n)\}_{n=1}^{\infty}$ and $\{Ty_n\}_{n=1}^{\infty}$. Also from (3.6), we have

$$\begin{aligned} ||y_{n+1} - y_n|| &= ||\alpha_n \gamma(f(y_n) - f(y_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(y_{n-1}) \\ &\quad +(I - \mu\alpha_n A)Ty_n - (I - \mu\alpha_n A)Ty_{n-1} + \mu(\alpha_n - \alpha_{n-1})ATy_{n-1}|| \\ &\leq (1 - \alpha_n(\tau - \gamma\alpha))||y_n - y_{n-1}|| + M|\alpha_n - \alpha_{n-1}|, \end{aligned}$$

for some $M > 0$. By Lemma 2.3, we have $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$. Furthermore, we obtain

$$\begin{aligned} ||y_n - Ty_n|| &\leq ||y_n - y_{n+1}|| + ||y_{n+1} - Ty_n|| \\ &= ||y_n - y_{n+1}|| + \alpha_n||\gamma f(y_n) - \mu ATy_n|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.8}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\underset{n\to\infty}{limsup}\langle(\gamma f - \mu A)x^*, j(y_n - x^*)\rangle = \lim_{j\to\infty}\langle(\gamma f - \mu A)x^*, j(y_{n_j} - x^*)\rangle.$$

Assume also $y_{n_j} \rightharpoonup z$ as $j \to \infty$, for some $z \in E$. Then, using this, (3.8) and the demiclosedness of $(I - T)$ at zero, we have $z \in F(T)$. Since $j$ is weakly sequentially continuous, we have

$$\begin{aligned} \underset{n\to\infty}{limsup}\langle(\gamma f - \mu A)x^*, j(y_n - x^*)\rangle &= \lim_{j\to\infty}\langle(\gamma f - \mu A)x^*, j(y_{n_j} - x^*)\rangle \\ &= \langle(\gamma f - \mu A)x^*, j(z - x^*)\rangle \leq 0. \end{aligned}$$

Finally, we show that $y_n \to x^*$. From the recursion formula (3.6), let

$$T_n y_n := \alpha_n \gamma f(y_n) + (I - \alpha_n \mu A)Ty_n,$$

and from Lemma 3.1, we have

$$\begin{aligned} ||y_{n+1} - x^*||^2 &= ||T_n y_n - T_n x^* + T_n x^* - x^*||^2 \\ &= ||T_n y_n - T_n x^* + \alpha_n(\gamma f - \mu A)x^*||^2 \\ &\leq ||T_n y_n - T_n x^*||^2 + 2\alpha_n\langle(\gamma f - \mu A)x^*, j(y_{n+1} - x^*)\rangle \\ &\leq [1 - \alpha_n(\tau - \gamma\alpha)]||y_n - x^*||^2 \\ &\quad +2\alpha_n(\tau - \gamma\alpha)\frac{\langle(\gamma f - \mu A)x^*, j(y_{n+1} - x^*)\rangle}{\tau - \gamma\alpha} \end{aligned}$$

and by Lemma 2.3 we have that $y_n \to x^*$ as $n \to \infty$. This completes the proof. □

We have the following corollaries.

**Corollary 3.5.** *Let $E = l_p$ space, $(1 < p < \infty)$ and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in E$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n.$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality*

$$\langle (\gamma f - \mu A)x^*, y - x^* \rangle \leq 0, \ \forall \ y \in F(T). \tag{3.9}$$

**Corollary 3.6.** (Tian [6]) *Let $E = H$ be a real Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu A)Tx_n.$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality* (3.9)

**Corollary 3.7.** (Marino and Xu [7]) *Let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n.$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$ which solves the variational inequality* (3.9).

**Corollary 3.8.** (Xu [12]) *Let $\{x_n\}_{n=1}^{\infty}$ be generated by $x_0 \in H$,*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \ n \geq 0.$$

*Assume that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] satisfying $(C1)-(C3)$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$.*

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