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A NEW HALF-DISCRETE MULHOLLAND-TYPE INEQUALITY WITH PARAMETERS

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ABSTRACT. By means of weight functions and Hadamard's inequality, a new half-discrete Mulholland-type inequality with a best constant factor is given. A best extension with parameters, some equivalent forms, the operator expressions as well as some particular cases are also considered.

1. INTRODUCTION

Assuming that $f, g \in L^2(R_+), ||f|| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}} > 0, ||g|| > 0$, we have the following Hilbert's integral inequality (cf. [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi ||f|| ||g||, \tag{1.1}$$

where the constant factor π is best possible. If $a = \{a_n\}_{n=1}^\infty, b = \{b_n\}_{n=1}^\infty \in l^2, ||a|| = \{\sum_{n=1}^\infty a_n^2\}^{\frac{1}{2}} > 0, ||b|| > 0$, then we have the following analogous discrete Hilbert's inequality

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi ||a|| ||b||, \tag{1.2}$$

with the same best constant factor π. Inequalities (1.1) and (1.2) are important in analysis and its applications (cf. [10, 11, 12]). On the other-hand, we have the following Mulholland's inequality with the same best constant factor (cf. [3, 20]):

$$\sum_{m=2}^\infty \sum_{n=2}^\infty \frac{a_m b_n}{\ln mn} < \pi \left\{ \sum_{m=2}^\infty m a_m^2 \sum_{n=2}^\infty n b_n^2 \right\}^{\frac{1}{2}}. \tag{1.3}$$

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In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [14] gave an extension of (1.1). Refinement the results from [14], Yang [15] gave some best extensions of (1.1) and (1.2): If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1 - 1}dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$,

$$f(\geq 0) \in L_{p,\phi}(R_+) = \{f|||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\},$$

$g(\geq 0) \in L_{q,\psi}(R_+)$, and $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.4}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x > 0 (y > 0)$, then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a|||a||_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, and $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.5}$$

where the constant $k(\lambda_1)$ is still best value. Clearly, for $p = q = 2, \lambda = 1, k_1(x,y) = \frac{1}{x+y}, \lambda_1 = \lambda_2 = \frac{1}{2}$, (1.4) reduces to (1.1), while (1.5) reduces to (1.2).

Some other results about Hilbert-type inequalities can be found in [9]-[16]. On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [3]. But they did not prove that the constant factors are best possible. In 2005, Yang [18] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is best possible. Very recently, Yang [19, 20] gave the following half-discrete Hilbert's inequality with best constant factor:

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < \pi ||f|| ||a||. \tag{1.6}$$

In this paper, by means of weight functions and Hadamard's inequality, a new half-discrete Mulholland-type inequality similar to (1.3) and (1.6) with a best constant factor is given as follows:

$$\begin{aligned} &\int_{\frac{3}{2}}^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{1 + \ln(x - \frac{1}{2})\ln(n - \frac{1}{2})} dx \\ < \ &\pi \left\{ \int_{\frac{3}{2}}^\infty (x - \frac{1}{2}) f^2(x) dx \sum_{n=2}^\infty (n - \frac{1}{2}) a_n^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{1.7}$$

Moreover, a best extension of (1.7) with multi-parameters, some equivalent forms, the operator expressions as well as some particular inequalities are considered.

2. Some lemmas

Lemma 2.1. *If* $0 < \lambda \leq 2, \alpha \in \mathbf{R}, \beta \leq \frac{1}{2}$, *and* $\omega(n)$ *and* $\varpi(x)$ *are weight functions given by*

$$\omega(n) \quad : \quad = \int_{1+\alpha}^{\infty} \frac{\ln^{\frac{\lambda}{2}}(n-\beta)\ln^{\frac{\lambda}{2}-1}(x-\alpha)dx}{(x-\alpha)[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}, n \in \mathbf{N}\backslash\{1\}, \tag{2.1}$$

$$\varpi(x) \quad : \quad = \sum_{n=2}^{\infty} \frac{\ln^{\frac{\lambda}{2}}(x-\alpha)\ln^{\frac{\lambda}{2}-1}(n-\beta)}{(n-\beta)[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}, x > 1+\alpha, \tag{2.2}$$

then we have

$$\varpi(x) < \omega(n) = B(\frac{\lambda}{2}, \frac{\lambda}{2}). \tag{2.3}$$

Proof. Substituting $t = \ln(x-\alpha)\ln(n-\beta)$ in (2.1), and by simple calculation, we have

$$\omega(n) = \int_0^{\infty} \frac{1}{(1+t)^{\lambda}} t^{\frac{\lambda}{2}-1} dt = B(\frac{\lambda}{2}, \frac{\lambda}{2}).$$

For fixed $x > 1+\alpha$, in view of the conditions, it is easy to see that

$$h(x,y) := \frac{\ln^{\frac{\lambda}{2}-1}(y-\beta)}{(y-\beta)[1+\ln(x-\alpha)\ln(y-\beta)]^{\lambda}}$$

is decreasing and strictly convex with $h'_y(x,y) < 0$ and $h''_{y^2}(x,y) > 0$, for $y \in (\frac{3}{2}, \infty)$. Hence by (2.2) and Hadamard's inequality (cf. [7]), we find

$$\varpi(x) < \ln^{\frac{\lambda}{2}}(x-\alpha) \int_{\frac{3}{2}}^{\infty} \frac{\ln^{\frac{\lambda}{2}-1}(y-\beta)dy}{(y-\beta)[1+\ln(x-\alpha)\ln(y-\beta)]^{\lambda}}$$

$$\overset{t=\ln(x-\alpha)\ln(y-\beta)}{=} \int_{\ln(x-\alpha)\ln(\frac{3}{2}-\beta)}^{\infty} \frac{t^{\frac{\lambda}{2}-1}}{(1+t)^{\lambda}} dt \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}),$$

and (2.3) follows. □

Lemma 2.2. *Let the assumptions of Lemma 1 be fulfilled and additionally, let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n \geq 0, n \in \mathbf{N}\backslash\{1\}, f(x)$ *is a non-negative measurable function in* $(1+\alpha, \infty)$. *Then we have the following inequalities:*

$$J := \left\{ \sum_{n=2}^{\infty} \frac{\ln^{\frac{p\lambda}{2}-1}(n-\beta)}{n-\beta} \left[\int_{1+\alpha}^{\infty} \frac{f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} \right]^p \right\}^{\frac{1}{p}}$$

$$\leq \left[B(\frac{\lambda}{2}, \frac{\lambda}{2}) \right]^{\frac{1}{q}} \left\{ \int_{1+\alpha}^{\infty} \varpi(x)(x-\alpha)^{p-1} \ln^{p(1-\frac{\lambda}{2})-1}(x-\alpha) f^p(x) dx \right\}^{\frac{1}{p}}, \tag{2.4}$$

$$L_1 \quad : \quad = \left\{ \int_{1+\alpha}^{\infty} \frac{(x-\alpha)^{\frac{q\lambda}{2}-1}}{[\varpi(x)]^{q-1}} \left[\sum_{n=2}^{\infty} \frac{a_n}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} \right]^q dx \right\}^{\frac{1}{q}}$$

$$\leq \quad \left\{ B(\frac{\lambda}{2}, \frac{\lambda}{2}) \sum_{n=2}^{\infty} (n-\beta)^{q-1} \ln^{q(1-\frac{\lambda}{2})-1}(n-\beta) a_n^q \right\}^{\frac{1}{q}}. \tag{2.5}$$

Proof. Setting $k(x,n) := \frac{1}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}$, by Hölder's inequality (cf. [7]) and (2.3), it follows

$$\left[\int_{1+\alpha}^{\infty}\frac{f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]^{p}$$
$$= \left\{\int_{1+\alpha}^{\infty}k(x,n)\left[\frac{\ln^{(1-\frac{\lambda}{2})/q}(x-\alpha)}{\ln^{(1-\frac{\lambda}{2})/p}(n-\beta)}\frac{(x-\alpha)^{\frac{1}{q}}f(x)}{(n-\beta)^{\frac{1}{p}}}\right]\right.$$
$$\left.\times\left[\frac{\ln^{(1-\frac{\lambda}{2})/p}(n-\beta)}{\ln^{(1-\frac{\lambda}{2})/q}(x-\alpha)}\frac{(n-\beta)^{\frac{1}{p}}}{(x-\alpha)^{\frac{1}{q}}}\right]dx\right\}^{p}$$
$$\leq \int_{1+\alpha}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}f^{p}(x)dx$$
$$\times\left\{\int_{1+\alpha}^{\infty}k(x,n)\frac{(n-\beta)^{q-1}\ln^{(1-\frac{\lambda}{2})(q-1)}(n-\beta)}{(x-\alpha)\ln^{1-\frac{\lambda}{2}}(x-\alpha)}dx\right\}^{p-1}$$
$$= \left\{\omega(n)\frac{\ln^{q(1-\frac{\lambda}{2})-1}(n-\beta)}{(n-\beta)^{1-q}}\right\}^{p-1}$$
$$\times\int_{1+\alpha}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}f^{p}(x)dx$$
$$= \frac{\left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^{p-1}(n-\beta)}{\ln^{\frac{p\lambda}{2}-1}(n-\beta)}\int_{1+\alpha}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)f^{p}(x)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}dx.$$

Then by the Lebesgue term by term integration theorem (cf. [8]), we have

$$J \leq \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^{\frac{1}{q}}\left\{\sum_{n=2}^{\infty}\int_{1+\alpha}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}f^{p}(x)dx\right\}^{\frac{1}{p}}$$
$$= \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^{\frac{1}{q}}\left\{\int_{1+\alpha}^{\infty}\sum_{n=2}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}f^{p}(x)dx\right\}^{\frac{1}{p}}$$
$$= \left[B(\frac{\lambda}{2},\frac{\lambda}{2})\right]^{\frac{1}{q}}\left\{\int_{1+\alpha}^{\infty}\varpi(x)(x-\alpha)^{p-1}\ln^{p(1-\frac{\lambda}{2})-1}(x-\alpha)f^{p}(x)dx\right\}^{\frac{1}{p}},$$

hence, (2.4) follows. By Hölder's inequality again, we have

$$\left[\sum_{n=2}^{\infty}k(x,n)a_{n}\right]^{q} = \left\{\sum_{n=2}^{\infty}k(x,n)\left[\frac{(x-\alpha)^{\frac{1}{q}}\ln^{(1-\frac{\lambda}{2})/q}(x-\alpha)}{(n-\beta)^{\frac{1}{p}}\ln^{(1-\frac{\lambda}{2})/p}(n-\beta)}\right]\right.$$

$$\times\left[\frac{\ln^{(1-\frac{\lambda}{2})/p}(n-\beta)}{\ln^{(1-\frac{\lambda}{2})/q}(x-\alpha)}\frac{(n-\beta)^{\frac{1}{p}}a_n}{(x-\alpha)^{\frac{1}{q}}}\right]\Bigg\}^q$$

$$\leq \left\{\sum_{n=2}^{\infty}k(x,n)\frac{(x-\alpha)^{p-1}\ln^{(1-\frac{\lambda}{2})(p-1)}(x-\alpha)}{(n-\beta)\ln^{1-\frac{\lambda}{2}}(n-\beta)}\right\}^{q-1}$$

$$\times\sum_{n=2}^{\infty}k(x,n)\frac{(n-\beta)^{q-1}\ln^{(1-\frac{\lambda}{2})(q-1)}(n-\beta)}{(x-\alpha)\ln^{1-\frac{\lambda}{2}}(x-\alpha)}a_n^q$$

$$= \frac{[\varpi(x)]^{q-1}(x-\alpha)}{\ln^{\frac{q\lambda}{2}-1}(x-\alpha)}\sum_{n=2}^{\infty}k(x,n)\frac{\ln^{\frac{\lambda}{2}-1}(x-\alpha)}{(x-\alpha)(n-\beta)^{1-q}}\ln^{(1-\frac{\lambda}{2})(q-1)}(n-\beta)a_n^q.$$

By Lebesgue term by term integration theorem, we have

$$L_1 \leq \left\{\int_{1+\alpha}^{\infty}\sum_{n=2}^{\infty}k(x,n)\frac{\ln^{\frac{\lambda}{2}-1}(x-\alpha)}{(x-\alpha)}(n-\beta)^{q-1}\ln^{(1-\frac{\lambda}{2})(q-1)}(n-\beta)a_n^q dx\right\}^{\frac{1}{q}}$$

$$= \left\{\sum_{n=2}^{\infty}\left[\int_{1+\alpha}^{\infty}k(x,n)\frac{\ln^{\frac{\lambda}{2}}(n-\beta)\ln^{\frac{\lambda}{2}-1}(x-\alpha)dx}{x-\alpha}\right]\frac{\ln^{q(1-\frac{\lambda}{2})-1}(n-\beta)}{(n-\beta)^{1-q}}a_n^q\right\}^{\frac{1}{q}}$$

$$= \left\{\sum_{n=2}^{\infty}\omega(n)(n-\beta)^{q-1}\ln^{q(1-\frac{\lambda}{2})-1}(n-\beta)a_n^q\right\}^{\frac{1}{q}},$$

and in view of (2.3), inequality (2.5) follows. □

3. Main results

We introduce the functions

$$\Phi(x) \ : \ = (x-\alpha)^{p-1}\ln^{p(1-\frac{\lambda}{2})-1}(x-\alpha)(x\in(1+\alpha,\infty)),$$
$$\Psi(n) \ : \ = (n-\beta)^{q-1}\ln^{q(1-\frac{\lambda}{2})-1}(n-\beta)(n\in\mathbf{N}\backslash\{1\}),$$

wherefrom $[\Phi(x)]^{1-q}=\frac{1}{x-\alpha}\ln^{\frac{q\lambda}{2}-1}(x-\alpha)$, and $[\Psi(n)]^{1-p}=\frac{1}{n-\beta}\ln^{\frac{p\lambda}{2}-1}(n-\beta)$.

Theorem 3.1. *If* $0<\lambda\leq 2, \alpha\in\mathbf{R}, \beta\leq\frac{1}{2}$, $p>1, \frac{1}{p}+\frac{1}{q}=1, f(x), a_n\geq 0$, $f\in L_{p,\Phi}(1+\alpha,\infty), a=\{a_n\}_{n=2}^{\infty}\in l_{q,\Psi}$, $||f||_{p,\Phi}>0$ *and* $||a||_{q,\Psi}>0$, *then we have the following equivalent inequalities:*

$$I \ : \ = \sum_{n=2}^{\infty}\int_{1+\alpha}^{\infty}\frac{a_n f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}$$
$$= \int_{1+\alpha}^{\infty}\sum_{n=2}^{\infty}\frac{a_n f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi}, \quad (3.1)$$

$$J = \left\{\sum_{n=2}^{\infty}[\Psi(n)]^{1-p}\left[\int_{1+\alpha}^{\infty}\frac{f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]^{p}\right\}^{\frac{1}{p}}$$
$$< B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}, \tag{3.2}$$

$$L := \left\{\int_{1+\alpha}^{\infty}[\Phi(x)]^{1-q}\left[\sum_{n=2}^{\infty}\frac{a_n}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]^{q}dx\right\}^{\frac{1}{q}}$$
$$< B(\frac{\lambda}{2},\frac{\lambda}{2})||a||_{q,\Psi}, \tag{3.3}$$

where the constant $B(\frac{\lambda}{2},\frac{\lambda}{2})$ is the best possible in the above inequalities.

Proof. The two expressions for I in (3.1) follow from Lebesgue's term by term integration theorem. By (2.4) and (2.3), we have (3.2). By Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty}\left[\Psi^{\frac{-1}{q}}(n)\int_{1+\alpha}^{\infty}\frac{f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right][\Psi^{\frac{1}{q}}(n)a_n] \le J||a||_{q,\Psi}. \tag{3.4}$$

Then by (3.2), we have (3.1). On the other-hand, assume that (3.1) is valid. Setting

$$a_n := [\Psi(n)]^{1-p}\left[\int_{1+\alpha}^{\infty}\frac{f(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]^{p-1}, n \in \mathbf{N}\backslash\{1\},$$

where $J^{p-1} = ||a||_{q,\Psi}$. By (2.4), we find $J < \infty$. If $J = 0$, then (3.2) is trivially valid; if $J > 0$, then by (3.1), we have

$$||a||_{q,\Psi}^{q} = J^{q(p-1)} = J^{p} = I < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi},$$

therefore $||a||_{q,\Psi}^{q-1} = J < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}$, that is, (3.2) is equivalent to (3.1). On the other-hand, by (2.3) we have $[\varpi(x)]^{1-q} > [B(\frac{\lambda}{2},\frac{\lambda}{2})]^{1-q}$. Then in view of (2.5), we have (3.3). By Hölder's inequality, we find

$$I = \int_{1+\alpha}^{\infty}[\Phi^{\frac{1}{p}}(x)f(x)]\left[\Phi^{\frac{-1}{p}}(x)\sum_{n=2}^{\infty}\frac{a_n}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]dx \le ||f||_{p,\Phi}L. \tag{3.5}$$

Then by (3.3), we have (3.1). On the other-hand, assume that (3.1) is valid. Setting

$$f(x) := [\Phi(x)]^{1-q}\left[\sum_{n=2}^{\infty}\frac{a_n}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}}\right]^{q-1}, x \in (1+\alpha,\infty),$$

then $L^{q-1} = ||f||_{p,\Phi}$. By (2.5), we find $L < \infty$. If $L = 0$, then (3.3) is trivially valid; if $L > 0$, then by (3.1), we have

$$||f||_{p,\Phi}^{p} = L^{p(q-1)} = I < B(\frac{\lambda}{2},\frac{\lambda}{2})||f||_{p,\Phi}||a||_{q,\Psi},$$

therefore $||f||_{p,\Phi}^{p-1} = L < B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\Psi}$, that is, (3.3) is equivalent to (3.1). Hence, (3.1), (3.2) and (3.3) are equivalent.

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{f}(x) = \frac{1}{x-\alpha}\ln^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}(x-\alpha), x \in (1+\alpha, e+\alpha); \widetilde{f}(x) = 0, x \in [e+\alpha, \infty)$, and $\widetilde{a}_n = \frac{1}{n-\beta}\ln^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}(n-\beta), n \in \mathbf{N}\backslash\{1\}$, if there exists a positive number $k(\le B(\frac{\lambda}{2}, \frac{\lambda}{2}))$, such that (3.1) is valid as we replace $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ with k, then in particular, it follows that

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=2}^{\infty}\int_{1+\alpha}^{\infty}\frac{\widetilde{a}_n\widetilde{f}(x)dx}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} < k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\
&= k\left\{\int_{1+\alpha}^{e+\alpha}\frac{dx}{(x-\alpha)\ln^{-\varepsilon+1}(x-\alpha)}\right\}^{\frac{1}{p}} \\
&\quad \times\left\{\frac{1}{(2-\beta)\ln^{\varepsilon+1}(2-\beta)} + \sum_{n=3}^{\infty}\frac{1}{(n-\beta)\ln^{\varepsilon+1}(n-\beta)}\right\}^{\frac{1}{q}} \\
&< k(\frac{1}{\varepsilon})^{\frac{1}{p}}\left\{\frac{1}{(2-\beta)\ln^{\varepsilon+1}(2-\beta)} + \int_{2}^{\infty}\frac{dy}{(y-\beta)\ln^{\varepsilon+1}(y-\beta)}\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{\frac{\varepsilon}{(2-\beta)\ln^{\varepsilon+1}(2-\beta)} + \frac{1}{\ln^{\varepsilon}(2-\beta)}\right\}^{\frac{1}{q}}, \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=2}^{\infty}\frac{\ln^{\frac{\lambda}{2}-\frac{\varepsilon}{q}-1}(n-\beta)}{n-\beta}\int_{1+\alpha}^{e+\alpha}\frac{\ln^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}(x-\alpha)dx}{(x-\alpha)[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} \\
&\overset{t=\ln(x-\alpha)\ln(n-\beta)}{=} \sum_{n=2}^{\infty}\frac{1}{(n-\beta)\ln^{\varepsilon+1}(n-\beta)}\int_{0}^{\ln(n-\beta)}\frac{t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}}{(1+t)^{\lambda}}dt \\
&= B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right)\sum_{n=2}^{\infty}\frac{1}{(n-\beta)\ln^{\varepsilon+1}(n-\beta)} - A(\varepsilon) \\
&> B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right)\int_{2}^{\infty}\frac{dy}{(y-\beta)\ln^{\varepsilon+1}(y-\beta)} - A(\varepsilon) \\
&= \frac{1}{\varepsilon\ln^{\varepsilon}(2-\beta)}B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right) - A(\varepsilon), \\
A(\varepsilon) &:= \sum_{n=2}^{\infty}\frac{1}{(n-\beta)\ln^{\varepsilon+1}(n-\beta)}\int_{\ln(n-\beta)}^{\infty}\frac{1}{(t+1)^{\lambda}}t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}dt. \qquad (3.7)
\end{aligned}$$

We find

$$\begin{aligned}
0 &< A(\varepsilon) \le \sum_{n=2}^{\infty}\frac{1}{(n-\beta)\ln^{\varepsilon+1}(n-\beta)}\int_{\ln(n-\beta)}^{\infty}\frac{1}{t^{\lambda}}t^{\frac{\lambda}{2}+\frac{\varepsilon}{p}-1}dt \\
&= \frac{1}{\frac{\lambda}{2}-\frac{\varepsilon}{p}}\sum_{n=2}^{\infty}\frac{1}{(n-\beta)\ln^{\frac{\lambda}{2}+\frac{\varepsilon}{q}+1}(n-\beta)} < \infty,
\end{aligned}$$

and so $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Hence by (3.6) and (3.7), it follows that

$$\frac{1}{\ln^{\varepsilon}(2-\beta)} B\left(\frac{\lambda}{2}+\frac{\varepsilon}{p}, \frac{\lambda}{2}-\frac{\varepsilon}{p}\right) - \varepsilon O(1) < k\left\{\frac{\varepsilon}{(2-\beta)\ln^{\varepsilon+1}(2-\beta)} + \frac{1}{\ln^{\varepsilon}(2-\beta)}\right\}^{\frac{1}{q}},$$

and $B(\frac{\lambda}{2}, \frac{\lambda}{2}) \leq k(\varepsilon \to 0^+)$. Hence $k = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best value of (3.1).

By the equivalence of the inequalities, in view of (3.4) and (3.5), the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ in (3.2) and (3.3) is the best possible. □

Remark 3.2. (i) Define the first type half-discrete Hilbert-type operator $T_1 : L_{p,\Phi}(1+\alpha, \infty) \to l_{p,\Psi^{1-p}}$ as follows: For $f \in L_{p,\Phi}(1+\alpha,\infty)$, we define $T_1 f \in l_{p,\Psi^{1-p}}$ by

$$T_1 f(n) = \int_{1+\alpha}^{\infty} \frac{1}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} f(x)dx, n \in \mathbf{N}\backslash\{1\}.$$

Then by (3.2), $||T_1 f||_{p.\Psi^{1-p}} \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})||f||_{p,\Phi}$ and so T_1 is a bounded operator with $||T_1|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Since by Theorem 3.1, the constant factor in (3.2) is best possible, we have $||T_1|| = B(\frac{\lambda}{2}, \frac{\lambda}{2})$.

(ii) Define the second type half-discrete Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(1+\alpha,\infty)$ as follows: For $a \in l_{q,\Psi}$, we define $T_2 a \in L_{q,\Phi^{1-q}}(1+\alpha,\infty)$ by

$$T_2 a(x) = \sum_{n=2}^{\infty} \frac{1}{[1+\ln(x-\alpha)\ln(n-\beta)]^{\lambda}} a_n, x \in (1+\alpha, \infty).$$

Then by (3.3), $||T_2 a||_{q.\Phi^{1-q}} \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})||a||_{q,\Psi}$ and so T_2 is a bounded operator with $||T_2|| \leq B(\frac{\lambda}{2}, \frac{\lambda}{2})$. Since by Theorem 3.1, the constant factor in (3.3) is best possible, we have $||T_2|| = B(\frac{\lambda}{2}, \frac{\lambda}{2})$.

Remark 3.3. For $p = q = 2, \lambda = 1$ in (3.1), (3.2) and (3.3), (i) if $\alpha = \beta = \frac{1}{2}$, then we have (1.7) and the following equivalent inequalities:

$$\sum_{n=2}^{\infty} \frac{1}{n-\frac{1}{2}} \left[\int_{\frac{3}{2}}^{\infty} \frac{f(x)dx}{1+\ln(x-\frac{1}{2})\ln(n-\frac{1}{2})}\right]^2 < \pi^2 \int_{\frac{3}{2}}^{\infty} (x-\frac{1}{2}) f^2(x)dx,$$

$$\int_{\frac{3}{2}}^{\infty} \frac{1}{x-\frac{1}{2}} \left[\sum_{n=2}^{\infty} \frac{a_n}{1+\ln(x-\frac{1}{2})\ln(n-\frac{1}{2})}\right]^2 dx < \pi^2 \sum_{n=2}^{\infty} (n-\frac{1}{2}) a_n^2;$$

(ii) if $\alpha = \beta = 0$, then we have the following equivalent inequalities

$$\int_1^{\infty} f(x) \sum_{n=2}^{\infty} \frac{a_n}{1+\ln x \ln n} dx < \pi \left\{\int_1^{\infty} x f^2(x)dx \sum_{n=2}^{\infty} n a_n^2\right\}^{\frac{1}{2}},$$

$$\sum_{n=2}^{\infty} \frac{1}{n} \left[\int_1^{\infty} \frac{f(x)}{1+\ln x \ln n} dx\right]^2 < \pi^2 \int_1^{\infty} x f^2(x)dx,$$

$$\int_1^\infty \frac{1}{x}\left[\sum_{n=2}^\infty \frac{a_n}{1+\ln x \ln n}\right]^2 dx < \pi^2 \sum_{n=2}^\infty n a_n^2.$$

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