

ON CERTAIN PROJECTIONS OF C^* -MATRIX ALGEBRAS

A. AL-RAWASHDEH

Communicated by T. Loring

ABSTRACT. In 1955, H. Dye defined certain projections of a C^* -matrix algebra by

$$\begin{aligned} P_{i,j}(a) &= (1 + aa^*)^{-1} \otimes E_{i,i} + (1 + aa^*)^{-1} a \otimes E_{i,j} \\ &+ a^*(1 + aa^*)^{-1} \otimes E_{j,i} + a^*(1 + aa^*)^{-1} a \otimes E_{j,j}, \end{aligned}$$

which was used to show that in the case of factors not of type I_{2n} , the unitary group determines the algebraic type of that factor. We study these projections and we show that in $M_2(\mathbb{C})$, the set of such projections includes all the projections. For infinite C^* -algebra A , having a system of matrix units, we have $A \simeq M_n(A)$. M. Leen proved that in a simple, purely infinite C^* -algebra A , the $*$ -symmetries generate $\mathcal{U}_0(A)$. Assuming $K_1(A)$ is trivial, we revise Leen's proof and we use the same construction to show that any unitary close to the unity can be written as a product of eleven $*$ -symmetries, eight of such are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$. In simple, unital purely infinite C^* -algebras having trivial K_1 -group, we prove that all $P_{i,j}(\omega)$ have trivial K_0 -class. Consequently, we prove that every unitary of \mathcal{O}_n can be written as a finite product of $*$ -symmetries, of which a multiple of eight are conjugate as group elements.

1. INTRODUCTION AND PRELIMINARIES

Let A be a unital C^* -algebra. The set of projections and the group of unitaries of A are denoted by $\mathcal{P}(A)$ and $\mathcal{U}(A)$, respectively. Recall that the C^* -matrix algebra over A which is denoted by $M_n(A)$ is the algebra of all $n \times n$ matrices $(a_{i,j})$ over A , with the usual addition, scalar multiplication, and multiplication of matrices and the involution (adjoint) is $(a_{i,j})^* = (a_{j,i}^*)$. As in Dye's viewpoint of $M_n(A)$, let $S_n(A)$ denote the direct sum of n copies of A , considered as a left

Date: Received: 22 February 2012; Revised: 20 May 2012; Accepted: 25 May 2012.

2010 Mathematics Subject Classification. Primary 46L05; Secondary 46L80.

Key words and phrases. C^* -algebras, matrix projection, K_0 -class.

This paper was presented in the 3rd Conference of Settat on Operator Algebras and Applications, 1–5 November 2011, Morocco.

A -module. Addition of n -tuples $\bar{x} = (x_1, x_2, \dots, x_n)$ in $S_n(A)$ is componentwise and $a \in A$ acts on \bar{x} by $a(\bar{x}) = (ax_1, ax_2, \dots, ax_n)$. Then $S_n(A)$ is a Hilbert C^* -algebra module, with the inner product defined by

$$\langle \bar{x}, \bar{y} \rangle = \sum_{i=1}^n x_i y_i^*.$$

By an A -endomorphism T of $S_n(A)$, we mean an additive mapping on $S_n(A)$ which commutes with left multiplication: $a(\bar{x}T) = (a\bar{x})T$. In a familiar way, assign to any T a uniquely determined matrix (t_{ij}) over A ($1 \leq i, j \leq n$) so that $\bar{x}T = (\sum_i x_i t_{i1}, \dots, \sum_i x_i t_{in})$.

If p is a projection in $\mathbb{M}_n(A)$, then p is a mapping on $S_n(A)$ having its range as a sub-module of $S_n(A)$. Then two projections are orthogonal means their sub-module ranges are so. The C^* -algebra $\mathbb{M}_n(A)$ contains numerous projections. For each $a \in A$ and each pair of indices i, j ($i \neq j, 1 \leq i, j \leq n$), H. Dye in [7] defined the projection $P_{i,j}(a)$ in $\mathbb{M}_n(A)$, whose range consists of all left multiples of the vector with 1 in the i^{th} -place, a in the j^{th} -place and zeros elsewhere. As a matrix, it has the form

$$P_{i,j}(a) = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & (1 + aa^*)^{-1} & \cdots & (1 + aa^*)^{-1}a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & a^*(1 + aa^*)^{-1} & \cdots & a^*(1 + aa^*)^{-1}a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Recall that (see [7], p.74) a system of matrix units of a unital C^* -algebra A is a subset $\{e_{i,j}^r\}, 1 \leq i, j \leq n$ and $1 \leq r \leq m$ of A , such that

$$e_{i,j}^r e_{k,l}^r = e_{i,k}^r, e_{i,j}^r e_{k,l}^s = 0 \text{ if } r \neq s \text{ or } j \neq k, (e_{i,j}^r)^* = e_{j,i}^r, \sum_{i,r} e_{i,i}^r = 1$$

and for every $i, e_{i,i}^r \in \mathcal{P}(A)$. For the C^* -complex matrix algebra $\mathbb{M}_n(\mathbb{C})$, let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard system of matrix units of the algebra, that is $E_{i,j}$ is the $n \times n$ matrix over \mathbb{C} with 1 at the place $i \times j$ and zeros elsewhere. It is also known that $\mathbb{M}_n(A)$ is $*$ -isomorphic to $A \otimes \mathbb{M}_n(\mathbb{C})$ (see [11]). We will see that having a system of matrix units is a necessary condition in order that a C^* -algebra A is $*$ -isomorphic to a C^* -matrix algebra $\mathbb{M}_n(B)$. Using the notion of a system of matrix units, we write

$$P_{i,j}(a) = (1 + aa^*)^{-1} \otimes E_{i,i} + (1 + aa^*)^{-1}a \otimes E_{i,j} + a^*(1 + aa^*)^{-1} \otimes E_{j,i} + a^*(1 + aa^*)^{-1}a \otimes E_{j,j} \in \mathcal{P}(\mathbb{M}_n(A)).$$

If $a = 0$, then $P_{i,j}(a)$ is the i^{th} diagonal matrix unit of $\mathbb{M}_n(A)$, which is $1 \otimes E_{i,i}$, or simply E_i .

Also in [10], M. Stone called the projection $P_{i,j}(a)$ by the characteristics matrix of a .

H. Dye used these projections as a main tool to prove that an isomorphism between the discrete unitary groups of von Neumann factors not of type I_n , is implemented by a $*$ -isomorphism between the factors themselves [[7], Theorem 2]. Indeed, let us recall main parts of his proof. Let A and B be two unital C^* -algebras and let $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ be an isomorphism. As φ preserves self-adjoint unitaries, it induces a natural bijection $\theta_\varphi : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ between the sets of projections of A and B given by

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p), \quad p \in \mathcal{P}(A).$$

This mapping is called a projection orthoisomorphism, if it preserves orthogonality, i.e. $pq = 0$ iff $\theta(p)\theta(q) = 0$.

Now, let θ be an orthoisomorphism from $\mathcal{P}(\mathbb{M}_n(A))$ onto $\mathcal{P}(\mathbb{M}_n(B))$. In [[7], Lemma 8] when A and B are von Neumann algebras, Dye proved that for any unitary $u \in \mathcal{U}(A)$, $\theta(P_{i,j}(u)) = P_{i,j}(v)$, for some unitary $v \in \mathcal{U}(B)$. A similar result is proved in the case of simple, unital C^* -algebras by the author in [1]. Afterwards, Dye in [[7], Lemma 6], proved that there exists a $*$ -isomorphism (or $*$ -antiisomorphism) from $\mathbb{M}_n(A)$ onto $\mathbb{M}_n(B)$ which coincides with θ on the projections $P_{i,j}(a)$. In fact, he proved that θ induces the $*$ -isomorphism ϕ from A onto B defined by the relation $\theta(P_{i,j}(a)) = P_{i,j}(\phi(a))$.

In this paper, we study the projections $P_{i,j}(a)$ of a C^* -matrix algebra $\mathbb{M}_n(A)$, for some C^* -algebra A , and we deduce main results concerning such projections. The paper is organized as follows: In Section 2, we show that every projection in $\mathbb{M}_2(\mathbb{C})$ is of the form $P_{1,2}(a)$, for $a \in \mathbb{C}$. In Section 3, we show that some infinite C^* -algebra A is isomorphic to its matrix algebra $\mathbb{M}_n(A)$, such as the Cuntz algebra \mathcal{O}_n , so the projections $P_{i,j}(a)$ can be considered as projections of A .

In a simple, unital purely infinite C^* -algebra A , M. Leen proved that self-adjoint unitaries (also called $*$ -symmetries, or involutions) generate the connected component $\mathcal{U}_0(A)$ of the unitary group $\mathcal{U}(A)$. In Section 4, assuming in addition that $K_1(A)$ is trivial, we revise Leen's proof, we fix certain projections and then following the same construction, we show that every unitary which is close to the unity, can be written as a product of eleven $*$ -symmetries, eight of which are of the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(A)$.

Consequently, since every unitary in the connected component of the unity can be written as a finite product of unitaries that are close to the unity (see [11], § 4.2), we have the following result:

Theorem 1.1. *Let A be a simple, unital purely infinite C^* -algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of A , with $e_{1,1} \sim 1$. Then every unitary of A can be written as a finite product of $*$ -symmetries, of which a multiple of eight have the form $1 - 2P_{i,j}(\omega)$, for some $\omega \in \mathcal{U}(A)$.*

Finally in Section 5, we compute the K_0 -class of such certain projections, and we prove that in simple, unital purely infinite C^* -algebras (assuming $K_1 = 0$), all projections of the form $P_{i,j}(u)$, $u \in \mathcal{U}(A)$ have trivial K_0 -class. As a good

application for \mathcal{O}_n , we have that every unitary can be written as a finite product of $*$ -symmetries, of which a multiple of eight have the form $1 - 2P_{i,j}(\omega)$, $\omega \in \mathcal{U}(\mathcal{O}_n)$. Hence using [2] (Lemma 2.1), all such involutions of the form $1 - 2P_{i,j}(\omega)$ are in fact conjugate, as group elements of $\mathcal{U}(\mathcal{O}_n)$.

2. THE 2×2 -COMPLEX ALGEBRA CASE

Let A be a unital C^* -algebra, and let $\mathcal{P}_{i,j}^n(A)$ denote the family of all projections in $\mathbb{M}_n(A)$ of the form $P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Also, let $\mathcal{U}_{i,j}^n(A)$ denote the set of all self-adjoint unitaries in $\mathbb{M}_n(A)$ of the form $1 - 2P_{i,j}(a)$, $1 \leq i, j \leq n$, $a \in A$. Notice that $\mathcal{P}_{i,j}^n(A)$ contains non-trivial projections. In this small section, we show that in the case of $\mathbb{M}_2(\mathbb{C})$, the set $\mathcal{P}_{i,j}^2(\mathbb{C})$ includes all the non-trivial projections $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$, i.e. every non-trivial projection is of the form $P_{i,j}(a)$, for some complex number a .

Proposition 2.1. *If $p \in \mathcal{P}(\mathbb{M}_2(\mathbb{C})) \setminus \{0, 1\}$, then $p \in \mathcal{P}_{i,j}^2(\mathbb{C})$.*

Proof. Let $p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a non-trivial projection in $\mathcal{P}(\mathbb{M}_2(\mathbb{C}))$. Then a and d are real numbers. If $b = 0$, then p is either the diagonal matrix unit $E_{1,1}$ or $E_{2,2}$. Otherwise, we have $a + b = 1$, $a = a^2 + |b|^2$ and $d = d^2 + |b|^2$, therefore $|b|^2 \leq \frac{1}{4}$. By straightforward computations, one can deduce that p is of the form

$$P_{1,2} \left(\frac{2b}{1 + \sqrt{1 - 4|b|^2}} \right), \text{ or } P_{1,2} \left(\frac{2b}{1 - \sqrt{1 - 4|b|^2}} \right).$$

□

Remark 2.2. The projections in $\mathcal{P}_{i,j}^n(\mathbb{C})$ are all of rank one by definition, this implies that in the case of $\mathbb{M}_3(\mathbb{C})$, the set $\mathcal{P}_{i,j}^3(\mathbb{C})$ does not cover all the non-trivial projections. Indeed, there are projections in $\mathcal{P}(\mathbb{M}_3(\mathbb{C}))$ of rank one which do not belong to $\mathcal{P}_{i,j}^3(\mathbb{C})$, since every projection in this latest family projects into a subspace of \mathbb{C}^3 which lies entirely in one coordinate plan.

3. SOME RESULTS FOR INFINITE C^* -ALGEBRAS

Let A be a unital C^* -algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$, for some $n \geq 3$. Recall that $e_{1,1}Ae_{1,1}$ is a C^* -algebra (corner algebra) which has $e_{1,1}$ as a unit. This system of matrix units implements a $*$ -isomorphism between A and $\mathbb{M}_n(e_{1,1}Ae_{1,1})$. Indeed, let us define the mapping

$$\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \rightarrow A$$

by

$$\eta_1((a_{i,j})^n) = \sum_{i,j=1}^n e_{i,1}a_{i,j}e_{1,j}.$$

Moreover if $e_{1,1}$ is equivalent to 1 (i.e. A is assumed to be an infinite C^* -algebra), then there exists a partial isometry v of A such that $v^*v = e_{1,1}$ and $vv^* = 1$, and this defines the $*$ -isomorphism $\Delta_v : A \rightarrow e_{1,1}Ae_{1,1}$ by $\Delta_v(x) = v^*xv$. The

isomorphism Δ_v can be used to decompose a projection as a sum of orthogonal equivalent projections.

Proposition 3.1. *Let A be a unital C^* -algebra having a system of matrix units $\{e_{i,j}\}_{i=1}^n$. If p is equivalent to the unity, then p can be written as a sum of orthogonal equivalent subprojections.*

Proof. As p equivalent to 1, we consider the isomorphism Δ_v , then apply it to the equality $1 = \sum_{i=1}^n e_{i,i}$, to get $p = \sum_{i=1}^n v^* e_{i,i} v$. Then $p_i = v^* e_{i,i} v$, for all $1 \leq i \leq n$, are equivalent subprojections of p . \square

Recall that, for two unital C^* -algebras A and B , if $\alpha : A \rightarrow B$ is a $*$ -isomorphism, then α induces the $*$ -isomorphism $\hat{\alpha} : \mathbb{M}_n(A) \rightarrow \mathbb{M}_n(B)$, which is defined by $(a_{i,j}) \mapsto (\alpha(a_{i,j}))$. Then we have the following result.

Proposition 3.2. *Let A be an infinite unital C^* -algebra having a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$. If $e_{1,1}$ is equivalent to 1, then $\mathbb{M}_n(A)$ is $*$ -isomorphic to A .*

Proof. Let $\Delta_v : A \rightarrow e_{1,1} A e_{1,1}$ and $\eta_1 : \mathbb{M}_n(e_{1,1} A e_{1,1}) \rightarrow A$ be defined as above. Then the mapping $\eta = \eta_1 \circ \widehat{\Delta}_v$ is a $*$ -isomorphism from $\mathbb{M}_n(A)$ onto A . Moreover,

$$\eta(a_{i,j})^n = \sum_{i,j}^n e_{i,1} v^* a_{i,j} v e_{1,j}, \text{ and}$$

$$\eta^{-1}(x) = (v e_{1,i} x e_{j,1} v^*)_{i,j}^n.$$

\square

As a main example of purely infinite C^* -algebras, let us recall the Cuntz algebra \mathcal{O}_n ; $n \geq 2$, is the universal C^* -algebra which is generated by isometries s_1, s_2, \dots, s_n , such that $\sum_{i=1}^n s_i s_i^* = 1$ with $s_i^* s_j = 0$, when $i \neq j$ and $s_i^* s_i = 1$ (for more details, see [5], [[6], p.149]). Let

$$e_{i,j} = s_i s_j^*, \quad 1 \leq i, j \leq n.$$

Then $\{e_{i,j}\}_{i,j=1}^n$ forms a system of matrix units for \mathcal{O}_n . As s_1^* partial isometry between $e_{1,1}$ and the unity, then Proposition 3.2 shows that the mapping

$$\eta : \mathbb{M}_n(\mathcal{O}_n) \rightarrow \mathcal{O}_n, \quad (a_{i,j})_{i,j} \mapsto \sum_{i,j=1}^n s_i a_{i,j} s_j^*$$

is a $*$ -isomorphism. Moreover, for $x \in \mathcal{O}_n$, $\eta^{-1}(x) = (s_i^* x s_j)_{i,j} \in \mathbb{M}_n(\mathcal{O}_n)$. Therefore, we have proved the following result, which is in fact known, but for sake of completeness:

Proposition 3.3. *The Cuntz algebra \mathcal{O}_n is isomorphic to the C^* -algebra $\mathbb{M}_n(\mathcal{O}_n)$.*

Then for $a \in \mathcal{O}_n$, $P_{i,j}(a)$ are considered as projections of \mathcal{O}_n by applying the mapping η . Therefore,

$$P_{i,j}(a) = s_i (1 + a a^*)^{-1} s_i^* + s_i (1 + a a^*)^{-1} a s_j^* + s_j a^* (1 + a a^*)^{-1} s_i^* + s_j a^* (1 + a a^*)^{-1} a s_j^*.$$

4. UNITARY FACTORS IN PURELY INFINITE C^* -ALGEBRAS

Recall that in a unital C^* -algebra A , every self-adjoint unitary u can be written as $u = 1 - 2p$, for some projection $p \in \mathcal{P}(A)$, let us say " the self-adjoint unitary u is associated to the projection p ". In this section, we assume that A is purely infinite simple C^* -algebra, and we study the factorizations of unitaries of A . In order to prove our main theorem (Theorem 4.2), let us first recall the following result of M. Leen.

Theorem 4.1 ([9], Theorem 3.8). *Let A be a simple, unital purely infinite C^* -algebra. Then the $*$ -symmetries (self-adjoint unitaries) generate the connected component of the unity $\mathcal{U}_0(A)$.*

Now, consider a system of matrix units $\{e_{i,j}\}_{i,j=1}^n$ of A , with $e_{1,1} \sim 1$. Let us recall the $*$ -isomorphisms $\eta_1 : \mathbb{M}_n(e_{1,1}Ae_{1,1}) \rightarrow A$, and $\eta = \eta_1 \circ \widehat{\Delta}_v$ from $\mathbb{M}_n(A)$ onto A . In this section we revise Leens' proof of Theorem 3.5 in [9] and we fix some projections, then by following the same construction, we prove the following main theorem, which shows that every unitary of A which lies within a neighborhood of the unity can be factorized as a product of eleven self-adjoint unitaries moreover, eight of such factors are associated to the projections $P_{i,j}(\mu)$, for some $\mu \in \mathcal{U}(A)$.

Theorem 4.2. *Let A be a simple, unital purely infinite C^* -algebra, such that $K_1(A) = 0$ and for $n \geq 3$, let $\{e_{i,j}\}_{i,j=1}^n$ be a system of matrix units of A , with $e_{1,1} \sim 1$. Then there exists $\epsilon > 0$ such that every unitary a of A with $\|a - 1\| < \epsilon$ can be written as a product of eleven self-adjoint unitaries, of which eight have the form:*

$$\begin{aligned} &1 - 2\eta(P_{1,2}(-\alpha)), \quad 1 - 2\eta(P_{1,2}(-1)) \\ &1 - 2\eta(P_{1,3}(-\alpha)), \quad 1 - 2\eta(P_{1,3}(-1)) \\ &1 - 2\eta(P_{1,2}(-\gamma)), \quad 1 - 2\eta(P_{1,2}(-1)) \\ &1 - 2\eta(P_{1,3}(-\gamma)), \quad 1 - 2\eta(P_{1,3}(-1)), \end{aligned}$$

for some $\alpha, \gamma \in \mathcal{U}(A)$.

Consequently, as the Cuntz algebra is simple, unital purely infinite C^* -algebra with $K_1(\mathcal{O}_n) = 0$ (see [4]) and using Proposition 3.3, we have the following result.

Corollary 4.3. *Let n be given, there is a positive number ϵ such that if $u \in \mathcal{U}(\mathcal{O}_n)$ with $\|u - 1\| < \epsilon$, then*

$$\begin{aligned} u = &z_1(1 - 2P_{1,2}(-\alpha))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\alpha))(1 - 2P_{1,3}(-1)) \\ &(1 - 2P_{1,2}(-\gamma))(1 - 2P_{1,2}(-1))(1 - 2P_{1,3}(-\gamma))(1 - 2P_{1,3}(-1))z_2z_3, \end{aligned}$$

for some self-adjoint unitaries z_1, z_2, z_3 and $\alpha, \gamma \in \mathcal{U}(\mathcal{O}_n)$.

Let us introduce the following lemma which is used by M. Leen in his proof, and we shall use it as well.

Lemma 4.4. *Let A be a simple, unital purely infinite C^* -algebra, and let ρ be a non-trivial projections of A . There is a positive number ϵ such that if $a \in \mathcal{U}_0(A)$ with $\|a - 1\| < \epsilon$, then there exist self-adjoint unitaries z_1, z_2, z_3 of A and $x \in \mathcal{U}_0(\rho A \rho)$ such that*

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1 - \rho \end{pmatrix}.$$

Proof. Mimic the first part of the proof of Theorem 3.5 in [9], with replacing symmetries by $*$ -symmetries and invertible by unitaries. \square

Proof of Theorem 4.2:

Proof. Since A is a simple, unital purely infinite C^* -algebra, using [4], we have $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$. As $K_1(A)$ is assumed to be trivial, we have $\mathcal{U}(A) = \mathcal{U}_0(A)$.

Let $p = e_{1,1}$, as $p \sim 1$, use Proposition 3.1 and the isomorphism Δ_u ($u^*u = e_{1,1}, uu^* = 1$) to find a projection $p_1 < p$ (precisely, $p_1 = u^*e_{1,1}u$) which is equivalent to p moreover, set the partial isometry $v = u^*e_{1,1}$, and put $\rho = p - p_1$, so ρ is a non-trivial projection. Therefore applying Lemma 4.4, there is a positive number ϵ such that if $a \in \mathcal{U}(A)$ with $\|a - 1\| < \epsilon$, then there exist self-adjoint unitaries z_1, z_2 and z_3 such that

$$z_1 a z_2 z_3 = \begin{pmatrix} x & 0 \\ 0 & 1 - \rho \end{pmatrix},$$

where $x \in \mathcal{U}(\rho A \rho)$.

Now, we shall use Leen's approach to exhibit the desired factorization of a . Choose $q = e_{2,2}$, $r = e_{3,3}$ and put $r_1 = p + q + r$, then we have $q \sim r < 1 - p - q$. Following Leen's notations, we choose $v_1 = e_{2,1}$, $v_2 = e_{3,2}$ and $v_3 = e_{1,3}$, so v_1, v_2 and v_3 are partial isometries such that

$$v_1^* v_1 = p, v_1 v_1^* = q, v_2^* v_2 = q, v_2 v_2^* = r, v_3^* v_3 = r, \text{ and } v_3 v_3^* = p.$$

Let $w = v_1 + v_2 + v v_3$. Then following the construction in Leen's proof, we get

$$\begin{aligned} z_1 a z_2 z_3 &= (1 - 2\eta_1(P_{1,2}(-\alpha_p)))(1 - 2\eta_1(P_{1,2}(-p))) \\ &\quad (1 - 2\eta_1(P_{1,3}(-\alpha_p)))(1 - 2\eta_1(P_{1,3}(-p))) \\ &\quad (1 - 2\eta_1(P_{1,2}(-\gamma_p)))(1 - 2\eta_1(P_{1,2}(-p))) \\ &\quad (1 - 2\eta_1(P_{1,3}(-\gamma_p)))(1 - 2\eta_1(P_{1,3}(-p))) \end{aligned}$$

where α_p and γ_p are in $\mathcal{U}(pAp)$. Notice that the factors in the right hand side are self-adjoint unitaries in A . Hence using the mapping η , we then get

$$\begin{aligned} a &= z_1 (1 - 2\eta(P_{1,2}(-\alpha)))(1 - 2\eta(P_{1,2}(-1))) \\ &\quad (1 - 2\eta(P_{1,3}(-\alpha)))(1 - 2\eta(P_{1,3}(-1))) \\ &\quad (1 - 2\eta(P_{1,2}(-\gamma)))(1 - 2\eta(P_{1,2}(-1))) \\ &\quad (1 - 2\eta(P_{1,3}(-\gamma)))(1 - 2\eta(P_{1,3}(-1))) z_3 z_2 \end{aligned}$$

where α and γ are unitaries in A , and this ends the proof. \square

Finally, let us finish this section by presenting the following open question:

Q. In the Cuntz algebra \mathcal{O}_n , do self-adjoint unitaries of the form $\{1 - 2P_{i,j}(a)\}$ generate the unitary group $\mathcal{U}(\mathcal{O}_n)$?

5. K -THEORY OF CERTAIN PROJECTIONS

In this section, we study the K_0 -class of the projections $P_{i,j}(u)$, where u is a unitary of some unital C^* -algebra A . In particular, if A is a simple purely infinite C^* -algebra, with $K_1(A) = 0$, or A is a von Neumann factor of type II_1 , or III , then for any unitary u of A , $P_{i,j}(u)$ has trivial K_0 -class. Afterwards, we present an application of Theorem 4.2, to the case of Cuntz algebras.

Proposition 5.1. *Let A be a unital C^* -algebra. If v is a unitary in A of finite order, then $[P_{i,j}(v)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary v in A , such that $v^m = 1$, for some positive integer m . For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(v \otimes E_{i,i} + v \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}),$$

then $W^* = \frac{1}{\sqrt{2}}(v^{m-1} \otimes E_{i,i} + E_{i,j} + v^{m-1} \otimes E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k})$, therefore $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$\begin{aligned} W^* P_{i,j}(v) W &= \frac{1}{4}(2v^{m-1} \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\ &= \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \quad (1 \text{ at the } i\text{-th place}) \\ &= E_{i,i}. \end{aligned}$$

This implies that the projection $P_{i,j}(v)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(v)] = [1]$ in $K_0(A)$, hence the proposition has been checked. \square

Proposition 5.2. *Let A be a unital C^* -algebra. If w_1, w_2 and v are unitaries of A such that v has order m , then $[P_{i,j}(w_1 v w_2)] = [1]$ in $K_0(A)$.*

Proof. As w_1 and w_2 are unitaries in A , then for all $i \neq j$, $W = w_1 \otimes E_{i,i} + w_2^* \otimes E_{j,j} + \sum_{k \notin \{i,j\}} E_{k,k} \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover, $W P_{i,j}(v) W^* = P_{i,j}(w_1 v w_2)$, therefore by Proposition 5.1 we have $[P_{i,j}(w_1 v w_2)] = [P_{i,j}(v)] = [1]$. \square

Proposition 5.3. *Let A be a unital C^* -algebra. If u and v are self-adjoint unitaries in A , then $[P_{i,j}(uv)] = [1]$ in $K_0(A)$.*

Proof. For $i \neq j$, let

$$W = \frac{1}{\sqrt{2}}(uv \otimes E_{i,i} + uv \otimes E_{i,j} + E_{j,i} - E_{j,j} + \sum_{k \notin \{i,j\}} \sqrt{2} \otimes E_{k,k}),$$

then $W \in \mathcal{U}(\mathbb{M}_n(A))$. Moreover,

$$\begin{aligned} W^*P_{i,j}(uv)W &= \frac{1}{4}(2uv \otimes E_{i,i} + 2 \otimes E_{i,j})(\sqrt{2}W) \\ &= E_{i,i}, \end{aligned}$$

and this implies that the projection $P_{i,j}(uv)$ is unitarily equivalent to $E_{i,i}$ in $\mathbb{M}_n(A)$, therefore we have that $[P_{i,j}(uv)] = [1]$ in $K_0(A)$, hence the proposition has been checked. \square

Combining the previous results, we have the following theorem concerning the K_0 -class of those projections $P_{i,j}(u)$ in $\mathcal{P}(\mathbb{M}_n(A))$, evaluated at any unitary u of A .

Theorem 5.4. *Let A be a simple, unital purely infinite C^* -algebra, such that $K_1(A)$ is the trivial group. If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Consider a unitary u of A . As $K_1(A) = 0$, and we know by [[4], p.188] that $K_1(A) \simeq \mathcal{U}(A)/\mathcal{U}_0(A)$ then using M. Leen’s result (Theorem 4.1), we have that $u = \prod_{k=1}^n v_k$, where v_k is a self-adjoint unitary ($*$ -symmetry) of A . If $n = 1$, then the result holds by using Proposition 5.1. Proposition 5.3 proves the case $n = 2$. If $n \geq 3$, then the result is done by Proposition 5.2, hence the proof is completed. \square

Moreover, as M. Broise in [[3], Theorem 1] proved that in the case of von Neumann factors of either type II_1 or III , the unitaries are generated by the self-adjoint unitaries, then a similar result in the case of von Neumann factors can be deduced as follows:

Theorem 5.5. *Let A be a von Neumann factor of type II_1 or III . If $u \in \mathcal{U}(A)$, then $[P_{i,j}(u)] = [1]$ in $K_0(A)$.*

Proof. Let u be a unitary of A . By [[3], Theorem 1], u can be written as a finite product of self-adjoint unitaries of A , then mimic the proof of Theorem 5.4. \square

Consequently, we have the following results concerning the K_0 -class of some certain projections.

Corollary 5.6. *Let A be a unital C^* -algebra which is either:*

- (1) *simple, purely infinite, with $K_1(A) = 0$, or*
- (2) *von Neumann factor of type II_1 , or III .*

If v is a unitary of A , and p is the projection of $\mathbb{M}_n(A)$ defined by

$$p = \frac{1}{2} \otimes E_{1,1} + \frac{v}{2} \otimes E_{1,2} + \frac{v^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} + E_{3,3} + E_{4,4} \cdots + E_{m,m}$$

for some positive integer $m \leq n - 2$, then $[p] = (m - 1)[1]$, in $K_0(A)$.

Proof. As the projection p is the orthogonal sums of $P_{1,2}(v) + E_{3,3} + E_{4,4} \cdots + E_{m,m}$, then by either Theorem 5.4 or 5.5,

$$[p] = [1] + ([1] + \cdots + [1]) = (m - 1)[1].$$

\square

Corollary 5.7. *Let A be a unital C^* -algebra which is either:*

- (1) *simple, purely infinite, with $K_1(A) = 0$, or*
- (2) *von Neumann factor of type II_1 , or III .*

If $v_1, v_2 \cdots v_n$ are unitaries of A , and p is the projection of $\mathbb{M}_{2n}(A)$ defined by

$$\begin{aligned} p &= \frac{1}{2} \otimes E_{1,1} + \frac{v_1}{2} \otimes E_{1,2} + \frac{v_1^*}{2} \otimes E_{2,1} + \frac{1}{2} \otimes E_{2,2} \\ &+ \frac{1}{2} \otimes E_{3,3} + \frac{v_2}{2} \otimes E_{3,4} + \frac{v_2^*}{2} \otimes E_{4,3} + \frac{1}{2} \otimes E_{4,4} + \cdots \\ &+ \frac{1}{2} \otimes E_{2n-1,2n-1} + \frac{v_n}{2} \otimes E_{2n-1,2n} + \frac{v_n^*}{2} \otimes E_{2n,2n-1} + \frac{1}{2} \otimes E_{2n,2n}, \end{aligned}$$

then $[p] = n[1]$, in $K_0(A)$.

Proof. Using Theorem 5.4 (or Theorem 5.5), we have

$$[p] = [P_{1,2}(v_1)] + [P_{3,4}(v_2) + \cdots + [P_{2n-1,2n}(v_n)] = n[1].$$

□

Now let us prove the following lemma, which will be used in order to prove our main result in this section (Theorem 5.9), which is in fact a consequence application of Theorem 4.2, to the case of Cuntz algebras \mathcal{O}_n .

Lemma 5.8. *Let A be a unital, simple purely infinite C^* -algebra, with $K_1(A) = 0$, and let $\{e_{i,j}\}^n$, with $e_{1,1} \sim 1$ be a system of matrix units of A . Then for any unitary $u \in \mathcal{U}(A)$ we have $[\eta(P_{i,j}(u))] = [1]$ in $K_0(A)$.*

Proof. As we have seen in the proof of Propositions 5.1, 5.2, 5.3 and Theorem 5.4, there exists a unitary $W \in \mathcal{U}(\mathbb{M}_n(A))$, such that $W^*P_{i,j}(u)W = E_{i,i}$. Therefore,

$$\eta(W)^*\eta(P_{i,j}(u))\eta(W) = \eta(E_{i,i}) = \eta_1 \hat{\Delta}_v(E_{i,i}) = \eta_1(e_{1,1} \otimes E_{i,i}) = e_{i,i}.$$

Then

$$\eta(P_{i,j}(u)) \sim_u e_{i,i} \sim e_{1,1} \sim 1,$$

hence $\eta(P_{i,j}(u))$ and 1 have the same class in $K_0(A)$. □

Finally, let us consider the case of the Cuntz algebra \mathcal{O}_n . Let u be a self-adjoint unitary (involution), so $u = 1 - 2p$, for some $p \in \mathcal{P}(\mathcal{O}_n)$. We recall the concept *type of involution* which is introduced by the author in [2], as follows: Since $K_0(\mathcal{O}_n) \simeq \mathbb{Z}_{n-1}$ (see [4]), then the type of u is defined to be the element $[p]$ in $K_0(\mathcal{O}_n)$. By ([2], Lemma 2.1), two involutions are conjugate as group elements in $\mathcal{U}(\mathcal{O}_n)$ if and only if they have the same type.

As a consequence of Theorem 4.2, and the results concerning the K_0 -group of the projections $P_{i,j}(u)$, which are deduced in this section, we have the following result.

Theorem 5.9. *Let n be given. There is a positive number ϵ such that every unitary of \mathcal{O}_n that lies within ϵ -neighborhood of 1 can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$ and consequently, all such eight involutions are conjugate group elements of $\mathcal{U}(\mathcal{O}_n)$.*

Proof. Using [4] and [5], the Cuntz algebra \mathcal{O}_n is a simple, unital purely infinite C^* -algebra with trivial K_1 -group. Then by Theorem 4.2, there exists $\epsilon > 0$ such that for every $u \in \mathcal{U}(\mathcal{O}_n)$ with $\|u - 1\| < \epsilon$, then u can be written as a product of eleven involutions, of which eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$. The type of the involution $(1 - 2\eta P_{i,j}(\omega))$ is $[\eta P_{i,j}(\omega)]$ and by Lemma 5.8 equals 1 in $K_0(\mathcal{O}_n)$. Hence, by [[2], Lemma 2.1], all these involutions are conjugate indeed, to the trivial involution -1 . \square

Consequently, and as every unitary (precisely in the connected component of unity) can be written as a finite product of unitaries that are close to the unity (see for example [11], § 4.2), we have the following:

Corollary 5.10. *Every unitary of \mathcal{O}_n can be written as a finite product of involutions, of which a multiple of eight have the form $(1 - 2\eta P_{i,j}(\omega))$, for some $\omega \in \mathcal{U}(\mathcal{O}_n)$ and consequently, all such multiple of eight involutions are conjugate group elements of $\mathcal{U}(\mathcal{O}_n)$.*

Acknowledgement. The author would like to thank the referee and the editor for their valuable comments and suggestions.

REFERENCES

1. A. Al-Rawashdeh, *The Unitary Group As An Invariant of a Simple Unital C^* -Algebra*, Ph.D Thesis, University of Ottawa, Canada (2003).
2. A. Al-Rawashdeh, *Normal generation of unitary groups of Cuntz algebras by involutions*, Acta Math. Univ. Comenianae **78** (2008), no. 1, 1–7.
3. M. Broise, *Commuteurs dans le groupe unitaire d'un facteur*, J. Math. Pures et appl. **46** (1967), 299–312.
4. J. Cuntz, *K-theory for certain C^* -algebras*, Ann. of Math. **113** (1981), 181–197.
5. J. Cuntz, *Simple C^* -algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
6. K.R. Davidson, *C^* -Algebras by Example*, Fields Institute Monographs, **6**, Amer. Math. Soc., Providences, RI, 1996.
7. H. Dye, *On the geometry of projections in certain operator Algebras*, Ann. of Math. **61** (1955), 73–89.
8. P. de la Harpe and V. Jones, *An Introduction to C^* -Algebras*, Université de Genève, 1995.
9. M. Leen, *Factorization in the invertible group of a C^* -algebra*, Canad. J. Math. **49** (1997), no. 6, 1188–1205.
10. M. Stone, *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. **4** (1936), 37–111.
11. N. Wegge-Olsen, *K-Theory and C^* -Algebras*, Oxford Science Publications, Oxford University Press, New York, 1993.

DEPARTMENT OF MATHEMATICAL SCIENCES, UAEU, 17551, AL-AIN, UNITED ARAB EMIRATES.

E-mail address: aalrawashdeh@uaeu.ac.ae