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HARDY–HILBERT TYPE INEQUALITIES FOR HILBERT SPACE OPERATORS

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ABSTRACT. Some Hardy–Hilbert type inequalities for Hilbert space operators are established. Several particular cases of interest are given as well.

1. INTRODUCTION AND PRELIMINARIES

One of the applicable inequalities in analysis and differential equations is the Hardy inequality which says that if $p > 1$ and $\{a_n\}_{n=1}^{\infty}$ are positive real numbers such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.1}$$

The inequality is sharp, i.e., the constant $\left(\frac{p}{p-1}\right)^p$ is the smallest number such that the inequality holds. A continuous form of inequality (1.1) is as follows:

If $p > 1$ and f is a non-negative p-integrable function on $(0, \infty)$, then

$$\int_0^{\infty} \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^{\infty} f(x)^p dx. \tag{1.2}$$

This inequality has been studied by many mathematicians [1, 3, 6]. A weighted version of inequality (1.2) was given in [2]. A developed inequality, the so-called Hardy–Hilbert inequality reads as follows:

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If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n, b_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=1}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1.3}$$

An integral form of inequality (1.3) can be stated as the following:
If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f, g \geq 0$ with $0 < \int_0^{\infty} f(x)^p dx < \infty$ and $0 < \int_0^{\infty} g(x)^q dx < \infty$, then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f(x)^p dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g(x)^q dx\right)^{\frac{1}{q}}.$$

There are many refinements and reformulations of the above inequality. Yang [7] proved the following generalization of (1.3):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^s} < L_1 \left(\sum_{m=1}^{\infty} m^{1-s} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{1-s} b_n^q\right)^{\frac{1}{q}}$$

and

$$\sum_{n=1}^{\infty} n^{(s-1)(p-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{(n+m)^s}\right)^p < L_1 \sum_{m=1}^{\infty} m^{1-s} a_m^p$$

in which $2 - \min\{p, q\} < s \leq 2$ and $L_1 = B(\frac{p+s-2}{p}, \frac{q+s-2}{q})$, where $B(u, v)$ is the β-function. Also Yang [8] presented some reverse Hardy integral inequalities.

Hansen [4] gave an operator version of inequality (1.1) in the C^*-algebra $\mathbb{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, in the case when $1 \leq p \leq 2$:

Theorem.[4] Let $1 < p \leq 2$ be a real number and let f from $(0, \infty)$ to the set $\mathbb{B}(\mathscr{H})_+$ of all positive operators in $\mathbb{B}(\mathscr{H})$, be a weakly measurable map such that the integral

$$\int_0^{\infty} f(x)^p dx$$

defines a bounded linear operator on $\mathscr{H}$. Then the inequality

$$\int_0^{\infty} \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p dx$$

holds, and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. In the same paper Hansen proved a similar trace inequality in the case where $p > 1$.

An operator version of inequality (1.3) was also given in [5]
In this paper, we give some inequalities analogue to (1.3) for operators in the real space $\mathbb{B}(\mathscr{H})_h$ of all self-adjoint operators on $\mathscr{H}$.

2. Main results

We start this section with an analogous inequality to (1.3) for operators acting on a Hilbert space $\mathscr{H}$.

Theorem 2.1. *Let f, g be continuous functions defined on an interval J and $f, g \geq 0$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}\frac{1}{2}\langle f(A)g(A)x, x\rangle + \frac{1}{3}\langle f(A)x, x\rangle\langle g(B)y, y\rangle \\ + \frac{1}{3}\langle f(A)y, y\rangle\langle g(B)x, x\rangle + \frac{1}{4}\langle f(B)g(B)y, y\rangle \\ \leq \frac{\pi}{\sin(\pi/p)}\left\langle (f(A)^p + f(B)^p)^{\frac{1}{p}} (g(A)^q + g(B)^q)^{\frac{1}{q}} x, x\right\rangle\end{aligned}$$

for all operators $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J and all unit vectors $x, y \in \mathscr{H}$.

Proof. Let a_1, a_2, b_1, b_2 be positive scalars. Using (1.3) we have

$$\frac{a_1 b_1}{2} + \frac{a_1 b_2}{3} + \frac{a_2 b_1}{3} + \frac{a_2 b_2}{4} \leq \frac{\pi}{\sin(\pi/p)} (a_1^p + a_2^p)^{\frac{1}{p}} (b_1^q + b_2^q)^{\frac{1}{q}} . \tag{2.1}$$

Let $t, s \in J$. Noticing that $f(t) \geq 0$ and $g(t) \geq 0$ for all $t \in J$ and putting $a_1 = f(t)$, $a_2 = f(s)$, $b_1 = g(t)$ and $b_2 = g(s)$ in (2.1) we obtain

$$\begin{aligned}\frac{f(t)g(t)}{2} + \frac{f(t)g(s)}{3} + \frac{f(s)g(t)}{3} + \frac{f(s)g(s)}{4} \\ \leq \frac{\pi}{\sin(\pi/p)} (f(t)^p + f(s)^p)^{\frac{1}{p}} (g(t)^q + g(s)^q)^{\frac{1}{q}}\end{aligned} \tag{2.2}$$

for all $s, t \in J$. Using the functional calculus for A to inequality (2.2) we get

$$\begin{aligned}\frac{f(A)g(A)}{2} + \frac{f(A)g(s)}{3} + \frac{f(s)g(A)}{3} + \frac{f(s)g(s)}{4} \\ \leq \frac{\pi}{\sin(\pi/p)} (f(A)^p + f(s)^p)^{\frac{1}{p}} (g(A)^q + g(s)^q)^{\frac{1}{q}} ,\end{aligned}$$

whence

$$\begin{aligned}\frac{1}{2}\langle f(A)g(A)x, x\rangle + \frac{1}{3}g(s)\langle f(A)x, x\rangle + \frac{1}{3}f(s)\langle g(A)x, x\rangle + \frac{f(s)g(s)}{4} \\ \leq \frac{\pi}{\sin(\pi/p)}\left\langle (f(A)^p + f(s)^p)^{\frac{1}{p}} (g(A)^q + g(s)^q)^{\frac{1}{q}} x, x\right\rangle\end{aligned}$$

for any unit vector $x \in \mathscr{H}$ and any $s \in J$. Applying the functional calculus once more to the self-adjoint operator B we get

$$\begin{aligned}\frac{1}{2}\langle f(A)g(A)x, x\rangle + \frac{1}{3}g(B)\langle f(A)x, x\rangle + \frac{1}{3}f(B)\langle g(A)x, x\rangle + \frac{f(B)g(B)}{4} \\ \leq \frac{\pi}{\sin(\pi/p)}\left\langle (f(A)^p + f(B)^p)^{\frac{1}{p}} (g(A)^q + g(B)^q)^{\frac{1}{q}} x, x\right\rangle .\end{aligned} \tag{2.3}$$

If $y \in \mathscr{H}$ is a unit vector, then it follows from inequality (2.3) that

$$\begin{aligned}\frac{1}{2}\langle f(A)g(A)x,x\rangle &+ \frac{1}{3}\langle g(B)y,y\rangle\langle f(A)x,x\rangle\\ &+ \frac{1}{3}\langle f(B)y,y\rangle\langle g(A)x,x\rangle + \frac{1}{4}\langle f(B)g(B)y,y\rangle\\ &\le \frac{\pi}{\sin(\pi/p)}\left\langle (f(A)^p + f(B)^p)^{\frac{1}{p}}\,(g(A)^q + g(B)^q)^{\frac{1}{q}}\,x,x\right\rangle.\end{aligned}$$

□

Replacing B by A and y by x in Theorem 2.1 we get:

Corollary 2.2. *If f,g are continuous functions defined on an interval J and $f,g \ge 0$, then*

$$\langle f(A)x,x\rangle\langle g(A)x,x\rangle \le \frac{3}{2}\left(2\pi - \frac{3}{4}\right)\langle f(A)g(A)x,x\rangle \tag{2.4}$$

for any self-adjoint operator A and any unit vector $x \in \mathscr{H}$.

With $p = q = 2$ in Theorem 2.1 we obtain

Corollary 2.3. *If f,g are continuous functions defined on an interval J and $f,g \ge 0$, then*

$$\begin{aligned}\frac{1}{2}\langle f(A)g(A)x,x\rangle &+ \frac{1}{3}\langle f(A)x,x\rangle\langle g(B)y,y\rangle\\ &+ \frac{1}{3}\langle f(A)y,y\rangle\langle g(B)x,x\rangle + \frac{1}{4}\langle f(B)g(B)y,y\rangle\\ &\le \pi\left\langle \left(f(A)^2 + f(B)^2\right)^{\frac{1}{2}}\left(g(A)^2 + g(B)^2\right)^{\frac{1}{2}} x,x\right\rangle\end{aligned}$$

for all operators $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J and all unit vectors $x,y \in \mathscr{H}$.

Another version of inequality (1.3) is given in the next theorem.

Theorem 2.4. *Let f,g be continuous functions defined on an interval J and $f,g \ge 0$. If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}\frac{1}{2}\langle f(B)y,y\rangle\langle f(A)x,x\rangle &+ \frac{1}{3}\langle g(B)y,y\rangle\langle f(A)x,x\rangle\\ &+ \frac{1}{3}\langle f(B)y,y\rangle\langle g(A)x,x\rangle + \frac{1}{4}\langle g(B)y,y\rangle\langle g(A)x,x\rangle\\ &\le \frac{\pi}{\sin\pi/p}\left\langle (f(B)^q + g(B)^q)^{\frac{1}{q}}\,y,y\right\rangle\left\langle (f(A)^p + g(A)^p)^{\frac{1}{p}}\,x,x\right\rangle\end{aligned} \tag{2.5}$$

for all operators $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J and all unit vectors $x,y \in \mathscr{H}$.

Proof. Let $s,t\in J$. We use inequality (2.1) with $a_1=f(t)$, $a_2=g(t)$, $b_1=f(s)$ and $b_2=g(s)$ to get

$$\frac{f(t)f(s)}{2}+\frac{f(t)g(s)}{3}+\frac{g(t)f(s)}{3}+\frac{g(t)g(s)}{4} \le \frac{\pi}{\sin\pi/p}\left(f(t)^p+g(t)^p\right)^{\frac{1}{p}}\left(f(s)^q+g(s)^q\right)^{\frac{1}{q}}.$$

Applying the functional calculus for A to the above inequality we get

$$\frac{f(A)f(s)}{2}+\frac{f(A)g(s)}{3}+\frac{g(A)f(s)}{3}+\frac{g(A)g(s)}{4} \le \frac{\pi}{\sin\pi/p}\left(f(A)^p+g(A)^p\right)^{\frac{1}{p}}\left(f(s)^q+g(s)^q\right)^{\frac{1}{q}},$$

whence

$$\frac{f(s)}{2}\langle f(A)x,x\rangle+\frac{g(s)}{3}\langle f(A)x,x\rangle+\frac{f(s)}{3}\langle g(A)x,x\rangle+\frac{g(s)}{4}\langle g(A)x,x\rangle \le \frac{\pi}{\sin\pi/p}\left(f(s)^q+g(s)^q\right)^{\frac{1}{q}}\left\langle\left(f(A)^p+g(A)^p\right)^{\frac{1}{p}}x,x\right\rangle$$

for any unit vector $x\in\mathscr{H}$. Using the functional calculus for B to the last inequality we obtain

$$\begin{aligned}\frac{1}{2}\langle f(B)y,y\rangle\langle f(A)x,x\rangle&+\frac{1}{3}\langle g(B)y,y\rangle\langle f(A)x,x\rangle\\ &+\frac{1}{3}\langle f(B)y,y\rangle\langle g(A)x,x\rangle+\frac{1}{4}\langle g(B)y,y\rangle\langle g(A)x,x\rangle\\ &\le\frac{\pi}{\sin\pi/p}\left\langle\left(f(B)^q+g(B)^q\right)^{\frac{1}{q}}y,y\right\rangle\left\langle\left(f(A)^p+g(A)^p\right)^{\frac{1}{p}}x,x\right\rangle\end{aligned}$$

for any unit vector $y\in\mathscr{H}$. □

With $A=B$ and $x=y$, inequality (2.5) gives rise to

$$\frac{1}{2}\langle f(A)x,x\rangle^2+\frac{2}{3}\langle g(A)x,x\rangle\langle f(A)x,x\rangle+\frac{1}{4}\langle g(A)x,x\rangle^2 \le\frac{\pi}{\sin\pi/p}\left\langle\left(f(A)^q+g(A)^q\right)^{\frac{1}{q}}x,x\right\rangle\left\langle\left(f(A)^p+g(A)^p\right)^{\frac{1}{p}}x,x\right\rangle.$$

Putting $p=q=2$ in the above inequality we obtain

$$\begin{aligned}\frac{1}{2}\langle f(A)x,x\rangle^2+\frac{2}{3}\langle g(A)x,x\rangle\langle f(A)x,x\rangle+\frac{1}{4}\langle g(A)x,x\rangle^2 &\le\pi\left\langle\left(f(A)^2+g(A)^2\right)^{\frac{1}{2}}x,x\right\rangle^2\\ &\le\pi\left\langle\left(f(A)^2+g(A)^2\right)x,x\right\rangle.\end{aligned}$$

In the case where the functions f and g are convex, we reach to the next result:

Theorem 2.5. *Let $f, g : J \to [0, \infty)$ be convex functions and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}\frac{1}{2}f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) &+ \frac{1}{3}f(\langle Ax,x\rangle)g(\langle By,y\rangle)\\ &+ \frac{1}{3}g(\langle Ax,x\rangle)f(\langle By,y\rangle) + \frac{1}{4}\langle f(B)g(B)y,y\rangle\\ \le \frac{\pi}{\sin(\pi/p)}&\left(\frac{1}{p}(\langle f(A)^p x,x\rangle + \langle f(B)^p y,y\rangle) + \frac{1}{q}(\langle g(A)^q x,x\rangle + \langle g(B)^q y,y\rangle)\right)\end{aligned}$$

for all $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J and all unit vectors x, y.

Proof. Put $t = \langle Ax, x\rangle$ in (2.2) to get

$$\begin{aligned}&\frac{1}{2}f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) + \frac{1}{3}f(\langle Ax,x\rangle)g(s) + \frac{1}{3}f(s)g(\langle Ax,x\rangle) + \frac{1}{4}f(s)g(s)\\ &\quad\le \frac{\pi}{\sin(\pi/p)}\left(f(\langle Ax,x\rangle)^p + f(s)^p\right)^{\frac{1}{p}}\left(g(\langle Ax,x\rangle)^q + g(s)^q\right)^{\frac{1}{q}}.\end{aligned}$$

A use of the functional calculus for B to the above inequality yields that

$$\begin{aligned}&\frac{1}{2}f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) + \frac{1}{3}f(\langle Ax,x\rangle)\langle g(B)y,y\rangle\\ &\qquad\qquad + \frac{1}{3}\langle f(B)y,y\rangle g(\langle Ax,x\rangle) + \frac{1}{4}\langle f(B)g(B)y,y\rangle\\ &\quad\le \frac{\pi}{\sin(\pi/p)}\left\langle\left(f(\langle Ax,x\rangle)^p + f(B)^p\right)^{\frac{1}{p}}\left(g(\langle Ax,x\rangle)^q + g(B)^q\right)^{\frac{1}{q}}y,y\right\rangle. \qquad (2.6)\end{aligned}$$

It follows from the convexity of f and g that $f(\langle By, y\rangle) \le \langle f(B)y, y\rangle$ and $g(\langle By, y\rangle) \le \langle g(B)y, y\rangle$. Therefore

$$\begin{aligned}&\frac{1}{2}f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) + \frac{1}{3}f(\langle Ax,x\rangle)\langle g(B)y,y\rangle)\\ &\qquad\qquad + \frac{1}{3}g(\langle Ax,x\rangle)\langle f(B)y,y\rangle + \frac{1}{4}\langle f(B)g(B)y,y\rangle\\ &\qquad\ge \frac{1}{2}f(\langle Ax,x\rangle)g(\langle Ax,x\rangle) + \frac{1}{3}f(\langle Ax,x\rangle)g(\langle By,y\rangle)\\ &\qquad\qquad + \frac{1}{3}g(\langle Ax,x\rangle)f(\langle By,y\rangle) + \frac{1}{4}\langle f(B)g(B)y,y\rangle. \qquad (2.7)\end{aligned}$$

The convexity of f and g and the power functions t^r $(r \ge 1)$ follow that

$$\begin{aligned}f(\langle Ax,x\rangle)^p \le \langle f(A)x,x\rangle^p \le \langle f(A)^p x,x\rangle,\\ g(\langle Ax,x\rangle)^q \le \langle g(A)x,x\rangle^q \le \langle g(A)^q x,x\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&(f(\langle Ax,x\rangle)^p + f(B)^p)^{\frac{1}{p}}(g(\langle Ax,x\rangle)^q + g(B)^q)^{\frac{1}{q}}\\ &\qquad\le (\langle f(A)^p x,x\rangle + f(B)^p)^{\frac{1}{p}}(\langle g(A)^q x,x\rangle + g(B)^q)^{\frac{1}{q}}. \qquad (2.8)\end{aligned}$$

Since the operators $\langle f(A)^p x, x\rangle + f(B)^p$ and $\langle g(A)^p x, x\rangle + g(B)^q$ commute, we infer from the arithmetic-geometric mean inequality that

$$(\langle f(A)^p x, x\rangle + f(B)^p)^{\frac{1}{p}}(\langle g(A)^q x, x\rangle + g(B)^q)^{\frac{1}{q}} \leq \frac{1}{p}(\langle f(A)^p x, x\rangle + f(B)^p) + \frac{1}{q}(\langle g(A)^q x, x\rangle + g(B)^q). \tag{2.9}$$

Combining (2.8) and (2.9) we obtain

$$\begin{aligned}&\left\langle (f(\langle Ax, x\rangle)^p + f(B)^p)^{\frac{1}{p}}(g(\langle Ax, x\rangle)^q + g(B)^q)^{\frac{1}{q}} y, y\right\rangle \\ &\leq \frac{1}{p}(\langle f(A)^p x, x\rangle + \langle f(B)^p y, y\rangle) + \frac{1}{q}(\langle g(A)^q x, x\rangle + \langle g(B)^q y, y\rangle).\end{aligned} \tag{2.10}$$

The result now follows by combining (2.6), (2.7) and (2.10). □

An application of Corollary 2.5 with $A = B$ yields that:

Corollary 2.6. *Let $f, g : J \to [0, \infty)$ be convex functions and let If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$f(\langle Ax, x\rangle)g(\langle Ax, x\rangle) \leq \frac{12}{17}\frac{\pi}{\sin(\pi/p)}\left(\frac{2}{p}\langle f(A)^p x, x\rangle + \frac{2}{q}\langle g(A)^q x, x\rangle\right)$$

for any $A \in \mathbb{B}(\mathscr{H})_h$ and any unit vector $x \in \mathscr{H}$. In particular if $f = g$ we get

$$f(\langle Ax, x\rangle)^2 \leq \frac{12}{17}\frac{\pi}{\sin(p/\pi)}\left(\frac{2}{p}\langle f(A)^p x, x\rangle + \frac{2}{q}\langle f(A)^q x, x\rangle\right).$$

References

1. A. Balinsky, W.D. Evans and R.T. Lewis, *Hardy's inequality and curvature*, J. Funct. Anal. **262** (2012), no. 2, 648–666.
2. M.I. Aguilar Cañestro, P. Ortega Salvador and C. Rami'rez Torreblanca, *Weighted bilinear Hardy inequalities*, J. Math. Anal. Appl. **387** (2012), no. 1, 320–334.
3. P. Gao, *A note on Hardy-type inequalities*, Proc. Amer. Math. Soc. **133** (2005), no. 7, 1977–1984.
4. F. Hansen, *Non-commutative Hardy inequalities*, Bull. Lond. Math. Soc. **41** (2009), no. 6, 1009–1016.
5. F. Hansen, K. Krulić, J. Pečarić and L.-E. Persson, *Generalized noncommutative Hardy and Hardy-Hilbert type inequalities*, Internat. J. Math. **21** (2010), no. 10, 1283–1295.
6. A. Kufner, L. Maligranda and L.-E. Persson, *The prehistory of the Hardy inequality*, Amer. Math. Monthly **113** (2006), no. 8, 715–732.
7. B. Yang, *On an extension of Hardy–Hilberts integral inequality*, Chinese Quart. J. Math. **21** (2006), no. 1, 96–102.
8. B. Yang, *On a new reverse extended Hardy's integral inequality*, J. Math. Res. Exposition **31** (2011), no. 3, 474–478.

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