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EXISTENCE RESULTS ON RANDOM IMPULSIVE SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH DELAYS

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ABSTRACT. This article presents the result on existence of mild solutions for random impulsive semilinear functional differential inclusions under sufficient conditions. The results are obtained by using the Martelli fixed point theorem and the fixed point theorem due to Covitz and Nadler.

1. INTRODUCTION

Impulsive differential inclusions are suitable mathematical model to simulate the evolution of large classes of real processes. These processes are subjected to short temporary perturbations. The duration of these perturbations is negligible compared to the duration of whole process. These perturbations occurs in the form of impulses (see [2, 4, 6, 1, 3, 5] and the references therein).

When the impulses are exists at random, the solutions of the equation behave as a stochastic process. It is quite different from deterministic impulsive differential equations and stochastic differential equations (SDEs). Iwankievicz et al [7], investigated dynamic response of non-linear systems to poisson distributed random impulses. Tatsuyuki et al [8] presented a mathematical model of random impulse to depict drift motion of granules in chara cells due to myosin-actin interaction. In [9], Sanz-Serna et al first brought dissipative differential equations with random impulses and used Markov chains to simulate such systems. Wu and Meng [10] first gave the general random impulsive ordinary differential equations and investigated boundedness of solutions to these models by Liapunov's

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direct method. Shujin Wu et al [11, 12, 13], studied some qualitative properties of random impulses. In [14], the author studied the existence and exponential stability for a random impulsive semilinear functional differential equations through the fixed point technique under non-uniqueness. The existence, uniqueness and stability results were discussed in [15] through Banach fixed point method for the system of differential equations with random impulsive effect. The author [16], studied the existence results for the random impulsive neutral functional differential equations with delays. In [17], the author studied existence results of random impulsive neutral non-autonomous differential inclusions with delays via Dhage's fixed point theorem.

Motivated by the above mentioned works, the main purpose of this paper is to study of random impulsive semilinear functional differential inclusions (RIFDIns). We utilize the technique from [18, 19, 20].

The paper is organized as follows. In section 2, we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In section 3, we study the existence of RIFDIns in the convex case of the multivalued function using a fixed point theorem for condensing map due to Martelli. In the end of the section, we study the existence results of the problem by using Wintner growth condition. We study the existence of RIFDIns in the non-convex case of the multivalued function using the fixed point theorem due to Covitz and Nadler in section 4. Finally in section 5, we present an example to illustrate the application for the results in section 3.

2. Preliminaries

Let X be a real separable Hilbert space and Ω a nonempty set. Assume that τ_k is a random variable defined from Ω to $D_k \stackrel{def.}{=} (0, d_k)$ for $k = 1, 2, \cdots$, where $0 < d_k < +\infty$. Furthermore, assume that τ_i and τ_j are independent with each other as $i \neq j$ for $i, j = 1, 2, \cdots$. For the sake of simplicity, we denote

$$\Re^+ = [0, +\infty); \ \Re_\tau = [\tau, +\infty).$$

We consider the semilinear functional differential inclusions with random impulses of the form

$$\left\{ \begin{array}{lcl} x'(t) & \in & Ax(t) + F(t, x_t), \ \ t \neq \xi_k, \quad \tau \leq t \leq T, \\ x(\xi_k) & = & b_k(\tau_k)x(\xi_k^-), \ \ k = 1, 2, \cdots, \\ x_{t_0} & = & \varphi, \end{array} \right. \tag{2.1}$$

where A is the infinitesimal generator of strongly continuous semigroup of bounded linear operators $S(t) = \{S(t), t \geq 0\}$ with $D(A) \subset X$.

Now we make the system (2.1) precise: The functional $F : \Re_\tau \times \mathcal{C} \to \mathcal{P}(X)$, $\mathcal{P}(X)$ is the family of all nonempty measurable subsets of X; $\mathcal{C} = \mathcal{C}([-r, 0], X)$ is the set of piecewise continuous functions mapping $[-r, 0]$ into X with some given $r > 0$; x_t is a function when t is fixed, defined by $x_t(s) = x(t + s)$ for all $s \in [-r, 0]$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$, here $t_0 \in \Re_\tau$ is arbitrary given real number. Obviously, $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots$; $b_k : D_k \to \Re$ for each $k = 1, 2, \cdots$; $x(\xi_k^-) = \lim_{t \uparrow \xi_k} x(t)$ according to their paths with the norm

$\|x\|_t = \sup_{t-r\leq s\leq t} |x(s)|$ for each t satisfying $\tau \leq t \leq T$ and $\tau, T \in \Re^+$ are given numbers, $\|\cdot\|$ is any given norm in X; φ is a function defined from $[-r, 0]$ to X.

Denote $\{B_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{B_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $L_p = L_p(\Omega, P, \{\mathcal{F}_t\})$ be the space of all p^{th} integrable random variables with values in X, that are measurable with respect to $\{\mathcal{F}_t, t \geq t_0\}$. For the simplification, denote the Banach space $\mathcal{B}_{\mathcal{T}}\big([t_0 - r, T], L_p\big)$, the family of all $\mathcal{F}_t$-measurable, $\mathcal{C}$-valued random variables ψ with the norm

$$\|\psi\|_{\mathcal{B}_{\mathcal{T}}} = \Big(\sup_{t\in[t_0,T]} E\|\psi\|_t^p\Big)^{\frac{1}{p}}.$$

Let $L_p^0(\Omega, \mathcal{B}_{\mathcal{T}})$ denote the family of all $\mathcal{F}_0$ - measurable, $\mathcal{B}_{\mathcal{T}}$ - valued random variable φ.

We use the following notations: $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$. In a Hilbert space X, a multivalued map $G : X \to \mathcal{P}(X)$ is a convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(V) = \cup_{x\in V} G(x)$ is bounded in X, for all $V \in \mathcal{P}_{bd}(X)$ that is,

$$\sup_{x\in V}\{\sup\{\|y\| : y \in G(x)\}\} < \infty.$$

G is called upper semi continuous (u.s.c.) on X, if for each $x_0 \in X$, the set $G(x_0)$ is non-empty, closed subset of X, and if for each open set V of X containing $G(x_0)$ there exists an open neighborhood N of x_0 such that $G(N) \subseteq V$.

G is said to be completely continuous if $G(V)$ is relatively compact, for every $V \in \mathcal{P}_{bd}(X)$.

If the multivalued map G is completely continuous with nonempty compact value, then G is u.s.c. if and only if G is closed graph, $\big($ie., $x^{(n)} \to x^*, y^{(n)} \to y^*, y^{(n)} \in G(x^{(n)})$ imply $y^* \in G(x^*)\big)$.

An upper semicontinuous map $G : X \to X$ is said to be condensing if for any bounded subset $V \subseteq X$ with $\alpha(G(V)) < \alpha(V)$, where α denotes the Kuratowski measure of noncompactness [21].

Remark 2.1. ([22]). A completely continuous multivalued map is the easiest example of a condensing map.

G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $Fix\ G$.

Define the function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \Re^+$ by

$$H(A, B) = \max\big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\big\},$$

where $d(A, b) = \inf\big\{\|a - b\|^p, a \in A\big\}$, $d(a, B) = \inf\big\{\|a - b\|^p, b \in B\big\}$. The function H is called a Hausdorff metric on $\mathcal{P}_{bd,cl}(X)$.

The multivalued map $\mathcal{Z} : [\tau, T] \to \mathcal{P}_{bd,cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : [\tau, T] \to \Re^+$ defined by

$$Y(t) = d(x, \mathcal{Z}(t)) = \inf\{\|x - z\|^p : z \in \mathcal{Z}(t)\} \text{ is measurable.}$$

Definition 2.2. A multivalued operator $\mathcal{Z} : [\tau, T] \to \mathcal{P}_{cl}(X)$ is called
(a) Contraction if and only if there exists $\eta > 0$ such that

$$H(\mathcal{Z}(x), \mathcal{Z}(y)) \le \eta \; \|x - y\|^p, \text{ for each } x, y \in X, \text{ with } \eta < 1,$$

(b) $\mathcal{Z}$ has a point if there exists $x \in X$ such that $x \in \mathcal{Z}(x)$.

For more details on multivalued maps see [22, 23, 24]. Our existence results are based on the following fixed point theorems of Martelli [25] and Covitz and Nadler [26].

Theorem 2.3. ([25]). *Let X be a Hilbert space and $\mathcal{Z} : X \to \mathcal{P}_{bd,cl,cv}(X)$ a u.s.c. and condensing map. If the set*

$$U = \big\{u \in X \mid \lambda u \in \mathcal{Z}(x) \textit{ for some } \lambda > 1\big\}$$

is bounded, then $\mathcal{Z}$ has a fixed point.

Theorem 2.4. ([26]). *Let X be a Banach space, If $\mathcal{Z} : X \to \mathcal{P}_{cl}(X)$ is a contraction then $Fix\ \mathcal{Z} \neq \phi$.*

Definition 2.5. The multivalued map $F : [\tau, T] \times \mathcal{C} \to \mathcal{P}(X)$ is said to be L_p-Carathèodory if:
(i) $t \longmapsto F(t, u)$ is measurable for each $u \in \mathcal{C}$;
(ii) $u \longmapsto F(t, u)$ is upper semicontinuous for almost all $t \in [\tau, T]$;
(iii) for each $a > 0$, there exists $h_a \in L_p([\tau, T], \Re^+)$ such that

$$\|F(t, u)\|^p := \sup \Big\{E\|f\|^p : f \in F(t, u)\Big\} \le h_a(t),$$

for all $\|u\|^p_{\mathcal{B}_\mathcal{T}} \le a$ and for a.e. $t \in [\tau, T]$. For each $x \in L_p(X)$ define the set of selections of F by

$$S_{F,x} = \big\{\ f \in L_p(X) : f(t) \in F(t, x_t) \text{ for a.e, } t \in [\tau, T]\big\}.$$

Lemma 2.6. ([27]). *Let I be a compact interval and X be a Hilbert space. Let F be an L_p- Carathèodory multivalued map with $S_{F,x} \neq \phi$ and Γ be a linear continuous mapping from $L_p(I, X) \to C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \to \mathcal{P}_{bd,cl,cv}(C(I, X)),\ x \longmapsto (\Gamma \circ S_F)(x) = \Gamma(S_{F,x}),$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Definition 2.7. A semigroup $\{S(t),\ t \ge 0\}$ is said to be exponentially bounded if there exist constants $M \ge 1$ and $\gamma \in \Re$ such that

$$\|S(t)\| \le Me^{\gamma t}, \text{ for } t \ge 0.$$

Definition 2.8. A stochastic process $\{x(t) \in \mathcal{B}_\mathcal{T},\ t_0 - r \le t \le T\}$ is called a mild solution to system (2.1) in $(\Omega, P, \{\mathcal{F}_t\})$, if
(i) $x(t) \in X$ is $\mathcal{F}_t-$adapted;
(ii) $x(t_0 + s) = \varphi(s) \in L^0_p(\Omega, \mathcal{B}_\mathcal{T})$ when $s \in [-r, 0]$, and

$$\begin{aligned} x(t) \;&=\; \sum_{k=0}^{+\infty} \Bigg[\prod_{i=1}^{k} b_i(\tau_i) S(t - t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t - s) f(s) ds \\ &\quad + \int_{\xi_k}^{t} S(t - s) f(s) ds\Bigg] I_{[\xi_k, \xi_{k+1})}(t), \quad f \in S_{F,x},\ a.s\ t \in [t_0, T], \end{aligned}$$

where $\prod_{j=m}^{n}(\cdot) = 1$ as $m > n$, $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k)b_{k-1}(\tau_{k-1})\cdots b_i(\tau_i)$, and $I_A(\cdot)$ is the index function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{if } t \notin A. \end{cases}$$

3. Existence result: Convex case

In this section, we discuss the existence of mild solutions of the system (2.1). We assume that the multivalued F has compact and convex values. We need the following hypotheses.

Hypotheses:

(H_1) : $A : D(A) \subset X \to X$ is the infinitesimal generator of strongly continuous semigroup $S(t)$ in X.

(H_2) : The condition $\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\}$ is uniformly bounded, that is, there is $C > 0$ such that

$$\max_{i,k}\left\{\prod_{j=i}^{k}\|b_j(\tau_j)\|\right\} \le C \quad \text{for all } \tau_j \in D_j,\ j = 1, 2, \cdots .$$

(H_3) : $F : [\tau, T] \times \mathcal{C} \to \mathcal{P}(X)$ is a compact convex L_p- Carathèodory multivalued function.

(H_4) : There exists a continuous nondecreasing function $\psi : \Re^+ \to (0, \infty)$, $r \in L_1([\tau, T], \Re^+)$ such that

$$\|F(t, x_t)\|^p = \sup\Big\{\|f\|^p : f \in F(t, x_t)\Big\} \le r(t)\psi(\|x\|_t^p),\ a.e\ t \in [\tau, T] \text{ for all } x \in \mathcal{C}.$$

Theorem 3.1. *Assume that $(H_1)-(H_4)$ hold. Then the problem (2.1) has atleast one mild solution on $[-r, T]$, provided*

$$Q_2\int_{t_0}^{T} e^{-\gamma s} r(s)ds \quad < \quad \int_{Q_1}^{\infty}\frac{du}{\psi(u)}, \tag{3.1}$$

where $Q_1 = 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p$, $Q_2 = 2^{p-1}M^p e^{p\gamma T}\max\big\{1, C^p\big\}(T-t_0)^{p-1}$ and $M^p e^{p\gamma(T-t_0)}C^p \ge \frac{1}{2^{p-1}}$.

Proof. Transform the problem (2.1) into a fixed point problem. Consider the multivalued operator $\mathcal{Z} : \mathcal{B}_{\mathcal{T}} \to \mathcal{P}(\mathcal{B}_{\mathcal{T}})$ defined by

$$\begin{aligned} \mathcal{Z}(x) &= h \in \mathcal{B}_{\mathcal{T}}\text{: } h(t) \\ &= \begin{cases} \varphi(t - t_0), & t \in [t_0 - r, t_0], \\ \displaystyle\sum_{k=0}^{+\infty}\Big(\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds & \\ \displaystyle + \int_{\xi_k}^{t} S(t-s)f(s)ds\Big)I_{[\xi_k,\xi_{k+1})}(t), \qquad f \in S_{F,x}, a.s & t \in [t_0, T]. \end{cases} \end{aligned}$$

We shall show that the operator $\mathcal{Z}$ satisfies all the conditions of Theorem 2.1. We give the proof in the following steps.
Step (1): $\mathcal{Z}(x)$ is convex for each $y \in \mathcal{B}_{\mathcal{T}}$. Since F has convex values it follows that $S_{F,x}$ is convex; so that $f_1, f_2 \in S_{F,x}$ then $\kappa f_1 + (1-\kappa) f_2 \in S_{F,x}$, which implies clearly that $\mathcal{Z}(x)$ is convex.
Step (2): $\mathcal{Z}$ is bounded on bounded sets of $\mathcal{B}_{\mathcal{T}}$. Let $\mathbb{B}_a = \{x \in \mathcal{B}_{\mathcal{T}} : \|x\|^p \leq a\}$, $a > 0$, be a bounded subset in $\mathcal{B}_{\mathcal{T}}$. We show that $\mathcal{Z}(\mathbb{B}_a)$ is a bounded subset of $\mathcal{B}_{\mathcal{T}}$. For each $x \in \mathbb{B}_a$ let $h \in \mathcal{Z}(x)$. Then there exists $f \in S_{F,x}$ such that for each $t \in [t_0, T]$, we have

$$\begin{aligned} h(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) f(s) ds \\ &\quad + \int_{\xi_k}^{t} S(t-s) f(s) ds \Big] I_{[\xi_k, \xi_{k+1})}(t) \end{aligned}$$

$$\begin{aligned} \|h(t)\|^p &\leq \Big[\sum_{k=0}^{+\infty} \big[\| \prod_{i=1}^{k} b_i(\tau_i) \| \|S(t-t_0)\| \|\varphi(0)\| \\ &\quad + \sum_{i=1}^{k} \| \prod_{j=i}^{k} b_j(\tau_j) \| \int_{\xi_{i-1}}^{\xi_i} \|S(t-s) f(s)\| ds \\ &\quad + \int_{\xi_k}^{t} \|S(t-s) f(s)\| ds \big] I_{[\xi_k, \xi_{k+1})}(t) \Big]^p \\ &\leq 2^{p-1} M^p e^{p\gamma(t-t_0)} \max_k \Big\{ \prod_{i=1}^{k} \|b_i(\tau_i)\|^p \Big\} \|\varphi(0)\|^p \\ &\quad + 2^{p-1} M^p \Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \Big\} \Big]^p \cdot \Big(\int_{t_0}^{t} e^{\gamma(t-s)} \|f(s)\| \, ds \Big)^p \\ &\leq 2^{p-1} M^p e^{p\gamma(T-t_0)} C^p \|\varphi(0)\|^p \\ &\quad + 2^{p-1} M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} \int_{t_0}^{t} e^{p\gamma(t-s)} \|f(s)\|^p \, ds. \end{aligned}$$

Thus,

$$\begin{aligned} E\|h\|_t^p &\leq 2^{p-1} M^p e^{p\gamma(T-t_0)} C^p E\|\varphi(0)\|^p \\ &\quad + 2^{p-1} M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} e^{p\gamma(T-t_0)} \int_{t_0}^{t} e^{-p\gamma(s-t_0)} r(s) \psi(E\|x\|_s^p) ds \\ &\leq 2^{p-1} M^p e^{p\gamma(T-t_0)} C^p E\|\varphi(0)\|^p + 2^{p-1} M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} e^{p\gamma(T-t_0)} \\ &\quad \times \sup_{t \in [t_0, T]} \psi(E\|x\|_t^2) \Big(\int_{t_0}^{t} e^{-p\gamma(s-t_0)} r(s) ds \Big). \end{aligned}$$

Hence for each $h \in \mathcal{Z}(\mathbb{B}_a)$, we get

$$\begin{aligned}\|h\|^p_{\mathcal{B}_\mathcal{T}} &\leq 2^{p-1}M^p e^{p\gamma(T-t_0)}C^pE\|\varphi(0)\|^p \\ &\quad +2^{p-1}M^p\max\big\{1,C^p\big\}(T-t_0)^{p-1}e^{p\gamma(T-t_0)} \\ &\qquad \times\max_{x\in S_a}\sup_{t\in[t_0,T]}\psi(E\|x\|^2_t)\sup_{t\in[t_0,T]}\Big(\int_{t_0}^t e^{-p\gamma(s-t_0)}r(s)ds\Big)=\bar{\ell}.\end{aligned}$$

Then, for each $h \in \mathcal{Z}(x)$, we have $\|h\|^p_{\mathcal{B}_\mathcal{T}} \leq \bar{\ell}$.

Step (3): $\mathcal{Z}$ sends bounded sets into equi-continuous sets of $\mathcal{B}_\mathcal{T}$. Let $t_1, t_2 \in [t_0, T]$, $t_0 < t_1 < t_2 \leq T$ and $\mathbb{B}_a = \{x \in \mathcal{B}_\mathcal{T} : \|x\|^p \leq a\}$ be a bounded set in $\mathcal{B}_\mathcal{T}$. Now for each $x \in \mathbb{B}_a$, $h \in \mathcal{Z}(x)$, there exists a function $f \in S_{F,x}$ such that for each $t \in [t_0, T]$, we have

$$\begin{aligned}&h(t_1)-h(t_2)\\ &=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^k b_i(\tau_i)S(t_1-t_0)\varphi(0)+\sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t_1-s)f(s)ds\\ &\quad+\int_{\xi_k}^{t_1}S(t_1-s)f(s)ds\Bigg]I_{[\xi_k,\xi_{k+1})}(t_1)-\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^k b_i(\tau_i)S(t_2-t_0)\varphi(0)\\ &\quad+\sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t_2-s)f(s)ds+\int_{\xi_k}^{t_2}S(t_2-s)f(s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t_2)\\ &=\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^k b_i(\tau_i)S(t_1-t_0)\varphi(0)+\sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t_1-s)f(s)ds\\ &\quad+\int_{\xi_k}^{t_1}S(t_1-s)f(s)ds\Big]\Big(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\Big)\\ &\quad+\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^k b_i(\tau_i)\big(S(t_1-t_0)-S(t_2-t_0)\big)\varphi(0)\\ &\quad+\sum_{i=1}^k\prod_{j=i}^k b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\Big(S(t_1-s)-S(t_2-s)\Big)f(s)ds\\ &\quad+\int_{\xi_k}^{t_1}\Big(S(t_1-s)-S(t_2-s)\Big)f(s)ds+\int_{t_1}^{t_2}S(t_2-s)f(s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t_2).\end{aligned}$$

Then,

$$E\|h(t_1)-h(t_2)\|^p \leq 2^{p-1}E\|I_1\|^p+2^{p-1}E\|I_2\|^p, \tag{3.2}$$

where

$$\begin{aligned} I_1 &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t_1-t_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t_1-s)f(s)ds \\ &\quad +\int_{\xi_k}^{t_1} S(t_1-s)f(s)ds\Big]\Big(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\Big), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)\big(S(t_1-t_0)-S(t_2-t_0)\big)\varphi(0) \\ &\quad +\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}\Big(S(t_1-s)-S(t_2-s)\Big)f(s)ds \\ &\quad +\int_{\xi_k}^{t_1}\Big(S(t_1-s)-S(t_2-s)\Big)f(s)ds+\int_{t_1}^{t_2} S(t_2-s)f(s)ds\Big]I_{[\xi_k,\xi_{k+1})}(t_2). \end{aligned}$$

Furthermore,

$$\begin{aligned} E\|I_1\|^p &\le 2^{p-1}M^p e^{p\gamma(t_1-t_0)}C^p E\|\varphi(0)\|^p\ E\Big(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\Big) \\ &\quad +2^{p-1}\max\{1,C^p\}(t_1-t_0)^{p-1}E\int_{t_0}^{t_1}\|S(t_1-s)\|^p\|f(s)\|^p ds \\ &\quad \times E\Big(I_{[\xi_k,\xi_{k+1})}(t_1)-I_{[\xi_k,\xi_{k+1})}(t_2)\Big) \ \to\ 0 \quad \text{as} \quad t_2\to t_1, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} E\|I_2\|^p &\le 3^{p-1}C^p\|S(t_1-t_0)-S(t_2-t_0)\|^p E\|\varphi(0)\|^p \\ &\quad +3^{p-1}\max\{1,C^p\}(t_1-t_0)^{p-1}\int_{t_0}^{t_1}\|S(t_1-s)-S(t_2-s)\|^p E\|f(s)\|^p ds \\ &\quad +3^{p-1}(t_2-t_1)^{p-1}\int_{t_1}^{t_2}\|S(t_2-s)\|^p E\|f(s)\|^p ds\ \to\ 0 \text{ as } t_2\to t_1. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), it follows that the right hand side of (3.2) tends to zero as $t_2 \to t_1$. Since the compactness of $S(t-t_0)$ for $t > t_0$ implies the continuity in the uniform operator topology.

Step (4): $\mathcal{Z}$ maps bounded sets into relatively compact sets in $\mathcal{B}_{\mathcal{T}}$.

Let ϵ a real number satisfying $\epsilon \in (0, t-t_0)$, for $t \in [t_0, T]$. For $x \in \mathbb{B}_a$ we define a function h_ϵ by

$$\begin{aligned} h_\epsilon(t) &= \sum_{k=0}^{+\infty}\left[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds\right. \\ &\quad \left.+\int_{\xi_k}^{t-\epsilon} S(t-s)f(s)ds\right] I_{[\xi_k,\xi_{k+1})}(t),\ t\in(t_0,t-\epsilon) \end{aligned}$$

where $f \in S_{F,x}$. Since $S(t-t_0)$ is a compact operator, the set $\mathbb{H}_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in \mathcal{Z}(x)\}$ is relatively compact in $\mathcal{B}_{\mathcal{T}}$ for every $\epsilon \in (0, t-t_0)$. Moreover, for every $h \in \mathcal{Z}(x)$ we have

$$\begin{aligned} h(t) - h_\epsilon(t) \\ &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds \\ &+ \int_{\xi_k}^{t} S(t-s)f(s)ds \Big] I_{[\xi_k,\xi_{k+1})}(t) - \sum_{k=0}^{+\infty} [\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) \\ &+ \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds + \int_{\xi_k}^{t-\epsilon} S(t-s)f(s)ds] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

By using $(H_1) - (H_4)$, we obtain

$$E\|h - h_\epsilon\|_t^p \ \leq \ M^p \max\{1, C^p\}(T-t_0)^{p-1} \int_{t-\epsilon}^{t} e^{p\gamma(t-s)}\psi(a)r(s)ds$$

Therefore, there are relatively compact sets arbitrarily close to the set $\{h(t) : h \in \mathcal{Z}(\mathbb{B}_a)\}$. Hence the set $\{h(t) : h \in \mathcal{Z}(\mathbb{B}_a)\}$ is also relatively compact in $\mathcal{B}_{\mathcal{T}}$.

As the consequence of Step 1-4, together with Ascoli-Arzela theorem, we can conclude that $\mathcal{Z}$ is a compact multivalued map, and therefore, a condensing map.

Step (5): $\mathcal{Z}$ has a closed graph

Let $x^{(n)} \to x^*$ and $h^{(n)} \in \mathcal{Z}(x^{(n)})$ with $h^{(n)} \to h^*$. We shall show that $h^* \in \mathcal{Z}(x^*)$. There exists $f^{(n)} \in S_{F,x^{(n)}}$, such that

$$\begin{aligned} h^{(n)}(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f^{(n)}(s)ds \\ &+ \int_{\xi_k}^{t} S(t-s)f^{(n)}(s)ds \Big] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

We must prove that there exists $f^* \in S_{F,x^*}$, such that

$$\begin{aligned} h^*(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f^*(s)ds \\ &+ \int_{\xi_k}^{t} S(t-s)f^*(s)ds \Big] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

Consider the linear continuous operator $\Gamma : L_p(X) \to \mathcal{B}_{\mathcal{T}}$ defined by

$$\Gamma(f)(t) = \sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds + \int_{\xi_k}^{t} S(t-s)f(s)ds \Big] I_{[\xi_k,\xi_{k+1})}(t).$$

Then we have

$$\|\Big(h^{(n)}(t)-\sum_{k=0}^{+\infty}\big[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)\varphi(0)\big]I_{[\xi_k,\xi_{k+1})}(t)\Big)$$
$$-\Big(h^{*}(t)-\sum_{k=0}^{+\infty}\big[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)\varphi(0)\big]I_{[\xi_k,\xi_{k+1})}(t)\Big)\ \|^p\to 0,\ \text{ as } n\to\infty.$$

From Lemma (2.3), it follows that $\Gamma\circ S_F$ is a closed graph operator and from the definition of Γ one has

$$h^{(n)}(t)-\sum_{k=0}^{+\infty}\big[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)\varphi(0)\big]I_{[\xi_k,\xi_{k+1})}(t)\in\Gamma\circ S_{F,x^{(n)}}.$$

As $x^{(n)}\to x^*$ and $h^{(n)}\to h^*$, there is a $f^*\in S_{F,x^*}$ such that

$$h^{*}(t)-\sum_{k=0}^{+\infty}\big[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)\varphi(0)\big]I_{[\xi_k,\xi_{k+1})}(t)$$
$$=\sum_{k=0}^{+\infty}\big[\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)f^*(s)ds+\int_{\xi_k}^{t}S(t-s)f^*(s)ds\big]I_{[\xi_k,\xi_{k+1})}(t)$$

for all $t\in[t_0,T]$. Hence $h^*\in\mathcal{Z}(x^*)$, which shows that the graph $\mathcal{Z}$ is closed.
Step (6): A priori bounds

Now it remains to show that the set

$$U=\{x\in\mathcal{B}_{\mathcal{T}}:\lambda\ x\in\mathcal{Z}(x),\ \text{ for some }\lambda>1\}\text{ is bounded.}$$

Let $x\in U$, then for some $\lambda>1$, $\lambda\ x\in\mathcal{Z}(x)$ and there exists $f\in S_{F,x}$ such that

$$\begin{aligned}x(t)\ &=\ \lambda^{-1}\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\tau_i)S(t-t_0)\varphi(0)+\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}S(t-s)f(s)ds\\&\quad+\int_{\xi_k}^{t}S(t-s)f(s)ds\Bigg]\ I_{[\xi_k,\xi_{k+1})}(t),\ \ t\in[t_0,T].\end{aligned}$$

Thus, by $(H_1)-(H_4)$, for each $t \in [t_0, T]$, we have

$$\begin{aligned}
\|x(t)\|^p \leq & \Big(\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\tau_i)\|\|S(t-t_0)\|\|\varphi(0)\| \\
& + \sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|S(t-s)\|\|f(s)\|ds \\
& + \int_{\xi_k}^{t}\|S(t-s)\|\|f(s)\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big)^p \\
\leq & 2^{p-1}M^p e^{p\gamma(t-t_0)}\max_k\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|^p\Big\}\|\varphi(0)\|^p \\
& +2^{p-1}M^p\Big[\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\Big]^p\cdot\Big(\int_{t_0}^{t}e^{\gamma(t-s)}\|f(s)\|ds\Big)^p \\
E\|x\|_t^p \leq & 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi(0)\|^p \\
& +2^{p-1}M^p\max\big\{1,C^p\big\}(T-t_0)^{p-1}E\int_{t_0}^{t}e^{p\gamma(t-s)}\|f(s)\|^p ds.
\end{aligned}$$

Noting that the last term of the right hand side of the above inequality increases in t and choose $M^p e^{p\gamma(T-t_0)}C^p \geq \frac{1}{2^{p-1}}$, we obtain that

$$\begin{aligned}
E\|x\|_t^p \leq & 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p \\
& + 2^{p-1}M^p e^{p\gamma(T-t_0)}\max\big\{1,C^p\big\}(T-t_0)^{p-1}\int_{t_0}^{t}e^{-p\gamma(s-t_0)}r(s)\psi(E\|x\|_s^p)ds.
\end{aligned}$$

The function μ defined by

$$\mu(t) = \sup_{t_0\leq s\leq t}E\|x\|_s^p\ ,\ t\in[t_0,T]. \tag{3.5}$$

Then, for any $[t_0, T]$, it follows that

$$\begin{aligned}
\mu(t) \leq & 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p \\
& +2^{p-1}M^p e^{p\gamma(T-t_0)}\max\big\{1,C^p\big\}(T-t_0)^{p-1}\int_{t_0}^{t}e^{-p\gamma(s-t_0)}r(s)\psi(\mu(s))ds.
\end{aligned} \tag{3.6}$$

Denoting the right hand side of the above inequality (3.6) as $v(t)$, we obtain that

$$\begin{aligned}
\mu(t) &\leq v(t),\ t\in[t_0,T], \\
v(t_0) &= 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p = Q_1,
\end{aligned}$$

and

$$\begin{aligned}
v'(t) &= 2^{p-1}M^p e^{p\gamma(T-t_0)}\max\big\{1,C^p\big\}(T-t_0)^{p-1}e^{-p\gamma(t-t_0)}r(t)\psi(\mu(t)) \\
&\leq 2^{p-1}M^p e^{p\gamma(T-t_0)}\max\big\{1,C^p\big\}(T-t_0)^{p-1}e^{-p\gamma(t-t_0)}r(t)\psi(v(t)).
\end{aligned}$$

Then

$$\frac{v'(t)}{\psi(v(t))} \leq 2^{p-1}M^p e^{p\gamma(T-t_0)}\max\big\{1,C^p\big\}(T-t_0)^{p-1}e^{-p\gamma(t-t_0)}r(t). \tag{3.7}$$

Integrating (3.7) from t_0 to t and by making use of change of variables, we obtain

$$\begin{aligned}\int_{v(t_0)}^{v(t)} \frac{du}{\psi(u)} &\leq 2^{p-1}M^p e^{p\gamma(T-t_0)} \max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^{t} e^{-p\gamma(s-t_0)}r(s)ds \\ &\leq Q_2\int_{t_0}^{T} e^{-p\gamma s}r(s)ds \\ &< \int_{Q_1}^{\infty}\frac{du}{\psi(u)}.\end{aligned}$$

Hence by (3.1) there exists a constant β_1 such that $\mu(t) \leq v(t) \leq \beta_1$, for all $t \in [t_0, T]$.

Since for every $t \in [t_0, T]$, $E\|x\|^p \leq \mu(t)$ we have

$$\|x\|^p_{\mathcal{B}_\mathcal{T}} = \sup_{t_0 \leq t \leq T}\{E\|x\|^p_t\} \leq \beta_1,$$

where β_1 depends only on T and the function ψ and r. This shows that U is bounded. As a consequence of Theorem 2.1, we deduce that $\mathcal{Z}$ has a fixed point x defined on the interval $[-r, T]$, which is a solution of (2.1). □

We now present another existence result for the problem (2.1). The multivalued F is relaxed by using Wintner-type growth condition in the following Theorem.

Theorem 3.2. *Assume that (H_1)- (H_3) and the following condition holds (H_F) : There exists some function $\ell \in L_p([t_0, T], \Re^+)$ such that*

$$\begin{aligned}H(F(t, x_t), F(t, y_t)) &\leq \ell(t)\,\|x - y\|^p_t\,, \text{ for all } t \in [t_0, T],\ x, y \in \mathcal{C}, \\ H(0, F(t, 0)) &\leq \ell(t),\ \text{for a.e } t \in [t_0, T],\end{aligned}$$

where, $\int_{t_0}^{T} e^{-p\gamma s}\ell(s)ds < \infty$,

$$\begin{aligned}Q_3 &= 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p \\ &\quad +2^{p-1}M^p e^{p\gamma T}\max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^{T} e^{-p\gamma s}\ell(s)ds \\ Q_4 &= 2^{p-1}M^p e^{p\gamma T}\max\{1, C^p\}(T-t_0)^{p-1},\end{aligned}$$

then the problem (2.1) has at least one mild solution on $[-r, T]$.

Proof. Let the operator $\mathcal{Z}$ is defined as in Theorem 3.1. It can be shown, as in the proof of Theorem 3.1 that $\mathcal{Z}$ is completely continuous and upper semi-continuous. Now we prove that

$$U = \{x \in \mathcal{B}_\mathcal{T} : \lambda\, x \in \mathcal{Z}(x) \text{ for some } \lambda > 1\} \ \text{ is bounded.}$$

Let $x \in U$, then there exists $f \in S_{F,x}$, for $t \in [t_0, T]$,

$$\begin{aligned}x(t) &= \lambda^{-1}\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f(s)ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)f(s)ds\Bigg] I_{[\xi_k,\xi_{k+1})}(t),\ \ t \in [t_0, T].\end{aligned}$$

Thus, by $(H_1)-(H_3)$ and (H_F), we have

$$\begin{aligned} E\|x\|_t^p &\leq 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi(0)\|^p \\ &\quad +2^{p-1}M^p \max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^t e^{p\gamma(t-s)}E\|f(s)\|^p ds. \end{aligned}$$

Noting that the last term of the right hand side of the above inequality increases in t and choose $M^p e^{p\gamma(T-t_0)}C^p \geq \frac{1}{2^{p-1}}$, we obtain that

$$\begin{aligned} E\|x\|_t^p &\leq 2^{p-1}M^p e^{p\gamma(T-t_0)}C^p E\|\varphi\|^p \\ &\quad +2^{p-1}M^p e^{p\gamma(T-t_0)} \max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^t e^{-p\gamma(s-t_0)}\ell(s)ds \\ &\quad +2^{p-1}M^p e^{p\gamma(T-t_0)} \max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^t e^{-p\gamma(s-t_0)}\ell(s)E\|x\|_s^p ds. \end{aligned}$$

Using the function $\mu(t)$ defined by (3.5), we obtain

$$\mu(t) \leq Q_3 + Q_4\int_{t_0}^t e^{-p\gamma s}\ell(s)\mu(s)ds.$$

Grownwall inequality, we get

$$\mu(t) \leq Q_3 \exp\big(Q_4\int_{t_0}^t e^{-\gamma s}l(s)ds\big), \text{ for all } t\in[t_0, T].$$

Therefore, there exists $\beta_2 > 0$ such that

$$\mu(t) \leq \beta_2 \text{ for all } t\in[t_0, T],$$

which implies that

$$\|x\|_{\mathcal{B}_T}^p \leq \beta_2.$$

This shows that the set U is bounded. As a consequence of Theorem 2.1, we deduce that $\mathcal{Z}$ has a fixed point which is a mild solution of (2.1). □

4. Existence results : non- convex case

In this section, we consider problem (2.1) with a non-convex valued right hand side. We assume that the multivalued map F has compact values. Our result in this section is based on the fixed point theorem for contraction multivalued operators given by Covitz and Nadler.

We now make additional assumption:

(H_{Fcp}) : $F : [\tau, T]\times\mathcal{C} \to \mathcal{P}_{cp}(X)$ has the property that $F(\cdot, x) : [\tau, T] \to \mathcal{P}_{cp}(X)$ is measurable for each $x\in\mathcal{C}$.

Theorem 4.1. *Assume that hypotheses $(H_1)-(H_2), (H_F)$ and (H_{Fcp}) are satisfied, then the IVP (2.1) has at least one mild solution on $[-r, T]$, provided*

$$\eta = \Big[M^p \max\{1, C^p\}(T-t_0)^{p-1}\int_{t_0}^T e^{p\gamma(t-s)}\ell(s)ds\Big] < 1. \tag{4.1}$$

Proof. Transform problem (2.1) into a fixed point problem. Consider the multivalued operator $\mathcal{Z}$ defined in Theorem 3.1. We shall show that $\mathcal{Z}$ satisfies the assumptions of Theorem 2.2.

Step (1): $\mathcal{Z}(x) \in \mathcal{P}_{cl}(\mathcal{B}_{\mathcal{T}})$ for each $x \in \mathcal{B}_{\mathcal{T}}$.

Indeed, let $(x^{(n)})_{n\geq 0} \in \mathcal{Z}(x)$ such that $x^{(n)} \to \overline{x}$ in $\mathcal{B}_{\mathcal{T}}$. Then $\overline{x} \in \mathcal{B}_{\mathcal{T}}$ and there exists $f^n \in S_{F,x}$ such that, for every $t \in [t_0, T]$,

$$\begin{aligned} x^{(n)}(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f^{(n)}(s)ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)f^{(n)}(s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t), \end{aligned}$$

using the fact that F has compact values and from (H_F). We may pass to a subsequence if necessary to get that $f^{(n)}$ converges to f in $L_p([\tau, T], X)$ and hence $f \in S_{F,x}$. Then, for each $t \in [t_0, T]$,

$$\begin{aligned} x^{(n)}(t) &\to \overline{x}(t) \\ &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f^{(n)}(s)ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)f^{(n)}(s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

So $\overline{x} \in \mathcal{Z}(x)$.

Step (2): Contraction

Let $x^1, x^2 \in \mathcal{B}_{\mathcal{T}}$ and $h^1 \in \mathcal{Z}(x^1)$ then there exists $f^1(t) \in F(t, x^1_t)$ such that for $t \in [t_0, T]$,

$$\begin{aligned} h^1(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)S(t-t_0)\varphi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i} S(t-s)f^1(s)ds \\ &\quad + \int_{\xi_k}^{t} S(t-s)f^1(s)ds\Big] I_{[\xi_k,\xi_{k+1})}(t). \end{aligned}$$

From (H_F) it follows that

$$H(F(t, x^1_t), F(t, x^2_t)) \leq \ell(t)\left\|x^1 - x^2\right\|_t^p, \ t \in [t_0, T].$$

Hence, there is $y \in F(t, x^2_t)$ such that

$$\|f^1(t) - y\|^p \leq \ell(t)\|x^1 - y\|_t^p, \ t \in [t_0, T].$$

Consider $N : [t_0, T] \to \mathcal{P}_{cp}(X)$, given by

$$N(t) = \{y \in X : \|f^1(t) - y\|^p \leq \ell(t)\|x^1 - y\|_t^p\}.$$

Since the multivalued operator $V(t) = N(t) \cap F(t, x^2_t)$ is measurable, there exists $f^2(t)$ a measurable selection for V. So $f^2(t) \in F(t, x^2_t)$ and

$$\|f^1(t) - f^2(t)\|^p \leq \ell(t)\|x^1 - y\|_t^p, \text{ for each } t \in [t_0, T].$$

Let us define for each $t \in [t_0, T]$,

$$
\begin{aligned}
h^2(t) \;&=\; \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) S(t-t_0)\varphi(0) + \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s) f^2(s) ds \\
&\quad + \int_{\xi_k}^{t} S(t-s) f^2(s) ds \Big] I_{[\xi_k,\xi_{k+1})}(t).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
h^1(t) - h^2(t) &= \sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} S(t-s)\big[f^1(s) - f^2(s)\big] ds \\
&\quad + \int_{\xi_k}^{t} S(t-s)\big[f^1(s) - f^2(s)\big] ds \Big] I_{[\xi_k,\xi_{k+1})}(t) \\
\|h^1(t) - h^2(t)\|^p &\le \Big(\sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} \|b_j(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} \|S(t-s)\| \|f^1(s) - f^2(s)\| ds \\
&\quad + \int_{\xi_k}^{t} \|S(t-s)\| \|f^1(s) - f^2(s)\| ds \Big] I_{[\xi_k,\xi_{k+1})}(t) \Big)^p \\
&\le M^p \Big[\max_{i,k} \Big\{ 1, \prod_{j=i}^{k} \|b_j(\tau_j)\| \Big\} \Big]^p \cdot \Big(\int_{t_0}^{t} e^{\gamma(t-s)} \|f^1(s) - f^2(s)\| ds \Big)^p \\
E\|h^1 - h^2\|_t^p &\le M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} E \int_{t_0}^{t} e^{p\gamma(t-s)} \|f^1(s) - f^2(s)\|^p ds \\
&\le M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} \int_{t_0}^{t} e^{p\gamma(t-s)} \ell(s) E\|x^1 - x^2\|_s^p ds \\
&\le \Big[M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} \int_{t_0}^{T} e^{p\gamma(t-s)} \ell(s) ds \Big] E\|x^1 - x^2\|_t^p.
\end{aligned}
$$

Taking supremum over t, we get

$$
\|h^1 - h^2\|_{\mathcal{B}_{\mathcal{T}}}^p \;\le\; \Big[M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} \int_{t_0}^{T} e^{p\gamma(t-s)} \ell(s) ds \Big] \|x^1 - x^2\|_{\mathcal{B}_{\mathcal{T}}}^p.
$$

By the analogous relation, obtained by interchanging the role of x^1 and x^2, it follows that

$$
H(M(x^1), M(x^2)) \le \eta \; \|x^1 - x^2\|_{\mathcal{B}_{\mathcal{T}}}^p,
$$

where, $\eta = \Big[M^p \max\big\{1, C^p\big\} (T-t_0)^{p-1} \int_{t_0}^{T} e^{p\gamma(t-s)} \ell(s) ds \Big]$.
From (4.1), $0 < \eta < 1$ and hence $\mathcal{Z}$ is a contraction, and thus by Theorem 2.2, $\mathcal{Z}$ has a fixed point x, which is a mild solution of (2.1). □

5. An example

Consider the following partial differential inclusion with finite delay of the form

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} & \in & \dfrac{\partial^2 u(x,t)}{\partial x^2} + \displaystyle\int_{-r}^{t} \mu(t,x,\theta) G(u(t+\theta,x))d\theta, \\ & & \qquad 0 < x < \pi,\ t_0 \le t \le T,\ t \ne \xi_k, \\ u(x,\xi_k) & = & q(k)\ \tau_k\ u(x,\xi_k^-),\ t = \xi_k, \\ u(0,t) & = & u(\pi,t) =\ 0, \\ u(x,t) & = & \varphi(x,t),\ -r \le t \le 0,\ 0 \le x \le \pi, \end{cases} \tag{5.1}$$

Let $X = L_p[0,\pi]$ and the operator $A = \frac{\partial^2}{\partial x^2}$ with the domain

$$D(A) = \left\{ u \in X \,\middle|\, u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 u}{\partial x^2} \in X,\ u(0) = u(\pi) = 0 \right\}.$$

Then,

$$Au = \sum_{n=1}^{\infty} n^2 (u, u_n),\ u \in D(A),$$

where $u_n(x) = \sqrt{\frac{2}{n}} \sin(nx), n = 1, 2 \dots$ is the orthogonal set of eigenvectors in A. It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self adjoint and

$$\|S(t)\| \le e^{\gamma t},\ \text{for } t \ge 0, \text{ where } M = 1 \text{ and } \gamma \in \Re\ .$$

Thus $S(t)$ is exponentially bounded.

We assume that the following conditions hold:

(i) The function $\mu(\cdot) \ge 0$ is continuous in $[t_0, T] \times [0,\pi] \times [-r, 0]$ with

$$\int_{-r}^{0} \mu(t,x,\theta)d\theta = \mathfrak{K}(t,x) < \infty, \mathfrak{K}(t) = \Big(\int_0^{\pi} \mathfrak{K}^p(t,x)dx \Big)^{\frac{1}{p}} < \infty.$$

(ii) The multifunction $G(\cdot)$ is an L^p- Carathèodory multivalued function with compact and convex values and

$$0 \le \|G(u(\theta,x))\| \le \hat{\psi}\Big(\|u(\theta,\cdot)\|_{L^p} \Big),\ (\theta,x) \in [t_0,T] \times [0,\pi],$$

where $\hat{\psi}(\cdot) : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing.

(iii) $E\Big[\max\limits_{i,k} \Big\{ \prod\limits_{j=i}^{k} \|q(j)(\tau_j)\| \Big\} \Big] \le \hat{C} < \infty.$

Assuming that conditions (i) -(iii) are verified, then the problem (5.1) can be modeled as the abstract random impulsive functional differential inclusions of the form (2.1), with

$$F(t,x_t) = \int_{-r}^{t} \mu(t,x,\theta) G(u(t+\theta,x))d\theta \text{ and } \ b_k(\tau_k) \ = \ q(k)\tau_k.$$

The next results are consequence of Theorem 3.1.

Proposition 5.1.
Assume that the conditions $(i)-(iii)$ *hold. Then there exists at least one mild solution* u *of the system* (5.1) *provided that*

$$N_2\int_{t_0}^{T} e^{-\gamma s}\mathfrak{K}(s)ds \quad < \quad \int_{N_1}^{\infty}\frac{du}{\hat{\psi}(u)},$$

where, $N_1 = 2^{p-1}e^{p\gamma(T-t_0)}\hat{C}^p E\|\varphi\|^p$, $N_2 = 2^{p-1}e^{p\gamma T}\max\{1,\hat{C}^p\}(T-t_0)^{p-1}$ *and* $e^{p\gamma(T-t_0)}\hat{C}^p \geq \frac{1}{2^{p-1}}$ *is satisfied.*

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