



A MOMENT PROBLEM ON SOME TYPES OF HYPERGROUPS

LÁSZLÓ SZÉKELYHIDI^{1*} AND LÁSZLÓ VAJDAY²

Communicated by C. P. Niculescu

ABSTRACT. The classical moment problem is formulated for commutative hypergroups and the uniqueness is proved for polynomial hypergroups in a single variable and for Sturm–Liouville hypergroups.

1. INTRODUCTION

The classical moment problem published in 1894 by Thomas Jan Stieltjes (see [8]) is the following: Given a sequence s_0, s_1, \dots of real numbers. Find necessary and sufficient conditions for the existence of a measure μ on $[0, \infty[$ so that

$$s_n = \int_0^\infty x^n d\mu(x)$$

holds for $n = 0, 1, \dots$. Another form of the moment problem, also called "Hausdorff's moment problem" or the "little moment problem," may be stated as follows: Given a sequence of numbers $(s_n)_{n=0}^\infty$, under what conditions is it possible to determine a function α of bounded variation in the interval $]0, 1[$ such that

$$s_n = \int_0^1 x^n d\alpha(x)$$

for $n = 0, 1, \dots$. Such a sequence is called a *moment sequence*, and Felix Hausdorff (see [2], [3]) was the first to obtain necessary and sufficient conditions for a sequence to be a moment sequence. In both cases the question of uniqueness of μ , resp. α arise. For a detailed discussion on classical moment problems see e.g. [1].

Date: Received: 26 February 2012; Accepted: 21 March 2012.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 20N20; Secondary 30E05, 44A60.

Key words and phrases. Sturm–Liouville hypergroup, moment problem, polynomial hypergroups.

Let K be a commutative hypergroup and N a nonnegative integer. We say that the continuous functions $\varphi_k : K \rightarrow \mathbb{C}$ ($k = 0, 1, \dots$) form a *generalized moment function sequence* if the equations

$$\varphi_k(x * y) = \sum_{j=0}^k \binom{k}{j} \varphi_j(x) \varphi_{k-j}(y) \quad (1.1)$$

hold for $k = 0, 1, \dots$ and for each x, y in K (see [5]). In this case the function φ_k is called a *generalized moment function of order k* . In particular, generalized moment functions of order 0 are exactly the exponentials on K . For more about generalized moment function sequences see [9, 5, 4, 6, 7].

Let μ be a compactly supported measure on K and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence. Then for each natural number n the complex number

$$m_n = \int_K \varphi_n d\mu$$

is called the *n -th generalized moment of μ* with respect to the given generalized moment function sequence. In this case the sequence $(m_n)_{n=0}^{\infty}$ is called the *generalized moment sequence of the measure μ* with respect to the given generalized moment function sequence.

In this setting we can formulate the problem of existence: Let the generalized moment function sequence $(\varphi_k)_{k=0}^{\infty}$ and the sequence of complex numbers $(m_n)_{n=0}^{\infty}$ be given. Under what conditions is there a compactly supported measure μ on K such that $(m_n)_{n=0}^{\infty}$ is the generalized moment sequence of the measure μ with respect to the given generalized moment function sequence? The other basic question is about the uniqueness: if the compactly supported measures μ and ν have the same generalized moment sequences with respect to the given generalized moment function sequence, then does it follow $\mu = \nu$?

In this paper we study the problem of uniqueness and solve it in the case of polynomial hypergroups in a single variable and in the case of Sturm–Liouville hypergroups.

2. THE CASE OF POLYNOMIAL HYPERGROUPS

In this section we shall use the results in [5] on the representation of generalized moment functions on polynomial hypergroups in the following form.

Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K . Then there exists a sequence $(c_k)_{k=0}^{\infty}$ such that for each natural number N we have

$$\varphi_k(n) = (P_n \circ f)^{(k)}(0) \quad (k = 0, 1, \dots, N),$$

where

$$f(t) = \sum_{j=0}^N c_j \frac{t^j}{j!}$$

for each t in \mathbb{R} .

Theorem 2.1. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ a finitely supported measure on \mathbb{N} and let $(\varphi_k)_{k=0}^{\infty}$ be a generalized moment function sequence on K . If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{N}} \varphi_k(n) d\mu(n) = 0$$

for $k = 0, 1, 2, \dots$, then $\mu = 0$.

Proof. First we remark that

$$\int_{\mathbb{N}} \varphi_k(n) d\mu(n) = \sum_{n=0}^N \varphi_k(n) \mu_n.$$

Hence, by assumption, we have the following system of equations

$$\sum_{n=0}^N \varphi_k(n) \mu_n = 0 \tag{2.1}$$

for $k = 0, 1, 2, \dots, N$.

On the other hand, by the result quoted from [5] we have that

$$\varphi_k(n) = (P_n \circ f)^{(k)}(0) \tag{2.2}$$

for $k = 0, 1, 2, \dots, N$, $n = 0, 1, 2, \dots, N$, where

$$f(t) = \sum_{i=0}^N c_i \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (2.2) we have for $k = 1$

$$\varphi_1(n) = P'_n(\lambda) c_1,$$

which implies $c_1 \neq 0$.

Let n be a fixed nonnegative integer and we let for $k = 0, 1, 2, \dots, N$ and for each t in \mathbb{R} :

$$F_k(t) = (P_n \circ f)^{(k)}(t).$$

We show that for $k = 0, 1, 2, \dots, N$ and for each t in \mathbb{R}

$$F_k(t) = \sum_{j=0}^k p_{k,j}(t) P_n^{(j)}(f(t)), \tag{2.3}$$

where $p_{k,j}$ is a polynomial and $p_{k,k}(t) = f'(t)^k$.

We prove (2.3) by induction on k . For $k = 0$ the statement is trivial with $p_{0,0}(t) = 1$. Supposing that (2.3) is proved we show it for $k + 1$ instead of k . We have

$$F_{k+1}(t) = F'_k(t) = \sum_{j=0}^k p'_{k,j}(t) P_n^{(j)}(f(t)) + \sum_{j=0}^k p_{k,j}(t) P_n^{(j+1)}(f(t)) f'(t)$$

and this is the form (2.3) with $k+1$ for k . Moreover, $p_{k+1,k+1}(t) = p_{k,k}(t) \cdot f'(t) = f'(t)^{k+1}$. Then, using (2.2), we have

$$\varphi_k(n) = \sum_{j=0}^k c_{k,j} P_n^{(j)}(\lambda) \quad k = 0, 1, 2, \dots, N,$$

where $c_{k,k} = f'(0)^k \neq 0$, $c_{0,0} = 1$. By (2.1) it follows

$$\sum_{n=0}^N \sum_{j=0}^k c_{k,j} P_n^{(j)}(\lambda) \mu_n = 0 \quad (2.4)$$

for $k = 0, 1, 2, \dots, N$. For $k = 0$ this means

$$\sum_{n=0}^N P_n(\lambda) \mu_n = 0. \quad (2.5)$$

Now let $k = 1$ in (2.4), then we have by (2.5)

$$\begin{aligned} \sum_{n=0}^N c_{1,0} P_n(\lambda) \mu_n + c_{1,1} P_n'(\lambda) \mu_n &= c_{1,0} \sum_{n=0}^N P_n(\lambda) \mu_n + c_{1,1} \sum_{n=0}^N P_n'(\lambda) \mu_n = \\ &= c_{1,1} \sum_{n=0}^N P_n'(\lambda) \mu_n = 0. \end{aligned}$$

As $c_{1,1} \neq 0$, then it follows:

$$\sum_{n=0}^N P_n'(\lambda) \mu_n = 0.$$

Continuing this process we get the system of equations

$$\sum_{n=0}^N P_n^{(k)}(\lambda) \mu_n = 0, \quad (2.6)$$

for $k = 0, 1, 2, \dots, N$. Observe, that the degree of P_n is exactly n , hence we can rewrite (2.6) in the form

$$\sum_{n=k}^N P_n^{(k)}(\lambda) \mu_n = 0,$$

for $k = 0, 1, 2, \dots, N$. This is a homogeneous system of linear equations for the unknowns μ_n , $n = 0, 1, \dots, N$. The fundamental matrix of this system is an $N \times N$ upper triangular matrix with the nonzero numbers $P_n^{(k)}(\lambda)$ in the main diagonal, hence this matrix is regular, which means that the system has only trivial solution: $\mu_n = 0$ for $n = 0, 1, 2, \dots, N$. This means $\mu = 0$ and the proof is complete. \square

This result obviously implies the following uniqueness theorem.

Theorem 2.2. *Let $K = (\mathbb{N}, P_n)$ be a polynomial hypergroup, μ, ν finitely supported measures on \mathbb{N} and let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K . If $\varphi_1 \neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu = \nu$.*

3. THE CASE OF STURM–LIOUVILLE HYPERGROUPS

Following the previous ideas in this section we extend consider the same problem on Sturm–Liouville hypergroups. We shall use the results in [6] on the representation of generalized moment functions on polynomial hypergroups in the following form.

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, and let Φ the exponential family of the hypergroups K (see [6]). This means that for each z in \mathbb{C} and x in \mathbb{R}_+ the function Φ satisfies

$$\partial_1^2 \Phi(x, z) + \frac{A'(x)}{A(x)} \partial_1 \Phi(x, z) = z \Phi(x, z),$$

further $\Phi(0, z) = 1$ and $\partial_1 \Phi(0, z) = 0$. It follows, that the function $z \mapsto \Phi(x, z)$ is entire for each x in \mathbb{R}_0 . Let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K . Then there exists a sequence $(c_k)_{k=0}^\infty$ such that for each natural number N we have

$$\varphi_k(x) = \frac{d^k}{dt^k} \Phi(x, f(t))(0)$$

for $k = 0, 1, 2, \dots, N$, x in \mathbb{R}_0 , t in \mathbb{C} , where

$$f(t) = \sum_{i=0}^N c_i \frac{t^i}{i!}$$

is a polynomial.

Theorem 3.1. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, μ a compactly supported measure on \mathbb{R}_0 and let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K . If $\varphi_1 \neq 0$ and*

$$\int_{\mathbb{R}_0} \varphi_k(x) d\mu(x) = 0$$

for $k = 0, 1, 2, \dots$, then $\mu = 0$.

Proof. We show that if

$$\int_{\mathbb{R}_0} \varphi_k(x) d\mu(x) = 0 \tag{3.1}$$

for $k = 0, 1, 2, \dots$, then $\mu = 0$.

Let N be a fixed positive integer. By the result quoted above from [6] we have

$$\varphi_k(x) = \frac{d^k}{dt^k} \Phi(x, f(t))(0) \tag{3.2}$$

for $k = 0, 1, 2, \dots, N$ x in \mathbb{R}_0 , t in \mathbb{C} , where

$$f(t) = \sum_{i=0}^N c_i \frac{t^i}{i!}$$

is a polynomial. Let $\lambda = f(0)$. From (3.2) we have for $k = 1$

$$\varphi_1(x) = \frac{d}{dt}\Phi((x, f(t)))(0) = \partial_2\Phi(x, \lambda) c_1,$$

which implies $c_1 \neq 0$.

Let x be a fixed nonnegative real number and we let for $k = 0, 1, 2, \dots, N$ and for each t in \mathbb{R} :

$$F_k(t) = \frac{d^k}{dt^k}\Phi(x, f(t)).$$

We show that for $k = 0, 1, 2, \dots, N$ and for each t in \mathbb{R}

$$F_k(t) = \sum_{j=0}^k p_{k,j}(t) \partial_2^{(j)}\Phi(x, f(t)), \quad (3.3)$$

where $p_{k,j}$ is a polynomial, and $p_{k,k}(t) = f'(t)^k$.

We prove (3.3) by induction on k . For $k = 0$ the statement is trivial. Supposing that (3.3) is proved we show it for $k + 1$ instead of k . We have

$$F_{k+1}(t) = F'_k(t) = \sum_{j=0}^k p'_{k,j}(t) \partial_2^{(j)}\Phi(x, f(t)) + \sum_{j=0}^k p_{k,j}(t) \partial_2^{(j+1)}\Phi(x, f(t)) f'(t)$$

and this is the form (3.3) with $k + 1$ for k . Moreover, $p_{k+1,k+1}(t) = p_{k,k}(t) \cdot f'(t) = f'(t)^k$.

Then, using (3.2), we have

$$\varphi_k(x) = \sum_{j=0}^k c_{k,j} \partial_2^{(j)}\Phi(x, \lambda) \quad k = 0, 1, 2, \dots, N,$$

where $c_{k,k} \neq 0$, $c_{0,0} = 1$.

By (3.1) it follows

$$\sum_{j=0}^k c_{k,j} \int_{\mathbb{R}_0} \partial_2^{(j)}\Phi(x, \lambda) d\mu(x) = 0$$

for $k = 0, 1, 2, \dots$. For $k = 0$ this gives

$$\int_{\mathbb{R}_0} \Phi(x, \lambda) d\mu(x) = 0. \quad (3.4)$$

For $k = 1$ we have

$$c_{1,0} \int_{\mathbb{R}_0} \Phi(x, \lambda) d\mu(x) + c_{1,1} \int_{\mathbb{R}_0} \partial_2\Phi(x, \lambda) d\mu(x) = 0. \quad (3.5)$$

By (3.4) and $c_{1,1} \neq 0$ this implies

$$\int_{\mathbb{R}_0} \partial_2\Phi(x, \lambda) d\mu(x) = 0.$$

Continuing this process we arrive at

$$\int_{\mathbb{R}_0} \partial_2^{(k)}\Phi(x, \lambda) d\mu(x) = 0 \quad (3.6)$$

for $k = 0, 1, 2, \dots, N$. As N is arbitrary, we actually have that (3.6) holds for $k = 0, 1, \dots$.

We recall that the function $\hat{\mu} : \mathbb{C} \rightarrow \mathbb{C}$ defined for each complex z by

$$\hat{\mu}(z) = \int_{\mathbb{R}_0} \Phi(x, z) d\mu(x) \quad (3.7)$$

is the Fourier–Laplace transform of the measure μ . As μ is compactly supported, $\hat{\mu}$ is an entire function. On the other hand, as the integration in (3.7) is performed on the compact support of μ and $z \mapsto \Phi(x, z)$ is an entire function, hence the differentiation and the integration in (3.5) can be interchanged. This means that we have

$$\hat{\mu}^{(k)}(z) \frac{d^k}{dz^k} \int_{\mathbb{R}_0} \Phi(x, z) d\mu(x)$$

holds for $k = 0, 1, 2, \dots$, and for all z in \mathbb{C} . In particular, for $z = \lambda$

$$\hat{\mu}^{(k)}(\lambda) \frac{d^k}{dz^k} \int_{\mathbb{R}_0} \Phi(x, \lambda) d\mu(x) = 0.$$

As $\hat{\mu}$ is an entire function, it follows $\hat{\mu} = 0$. Then $\mu = 0$ and our statement is proved. \square

Similarly as above, we have the corresponding uniqueness result.

Theorem 3.2. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, μ, ν compactly supported measures on \mathbb{R}_0 and let $(\varphi_k)_{k=0}^\infty$ be a generalized moment function sequence on K . If $\varphi_1 \neq 0$ and the generalized moment sequences of μ and ν with respect to the given generalized moment function sequence are the same, then $\mu = \nu$.*

Acknowledgement. The research was partly supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T-031995. and T-043080.

REFERENCES

1. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, (translated from the Russian by N. Kemmer), Hafner Publishing Co., New York, 1965.
2. F. Hausdorff, *Summationsmethoden und Momentfolgen. I.*, Math. Z. **9** (1921), 74–109.
3. F. Hausdorff, *Summationsmethoden und Momentfolgen. II.*, Math. Z. **9** (1921), 280–299.
4. Á. Orosz and L. Székelyhidi, *Moment Functions on Polynomial Hypergroups in Several Variables*, Publ. Math. Debrecen **65** (2004), no. 3-4, 429–438.
5. Á. Orosz and L. Székelyhidi, *Moment Functions on Polynomial Hypergroups*, Arch. Math. **85** (2005), 141–150.
6. Á. Orosz and L. Székelyhidi, *Moment functions on Sturm–Liouville–hypergroups*, Ann. Univ. Sci Budapest, Sect. Comp., **29** (2008), 141–156.
7. L. Székelyhidi, *Functional Equations on Sturm–Liouville Hypergroups*, Math. Pannonica, **17** (2006), no.2, 169–182.
8. T.J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, **8** (1894), 1–122; **9** (1894), 5–47.
9. H. Zeuner, *Moment functions and laws of large numbers on hypergroups*, Math. Z. **211** (1992), 369–407.

¹ INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, HUNGARY.
E-mail address: lszekelyhidi@gmail.com

² INSTITUTE OF MATHEMATICS, UNIVERSITY OF DEBRECEN, HUNGARY.
E-mail address: vlacika@gmail.com