Ann. Funct. Anal. 4 (2013), no. 1, 163–174
Annals of Functional Analysis
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

WEIGHTED COMPOSITION OPERATORS FROM CAUCHY INTEGRAL TRANSFORMS TO LOGARITHMIC WEIGHTED-TYPE SPACES

AJAY K. SHARMA

Communicated by K. Gurlebeck

Abstract. We characterize boundedness and compactness of weighted composition operators from the space of Cauchy integral transforms to logarithmic weighted-type spaces. We also manage to compute norm of weighted composition operators acting between these spaces.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e. $A(\mathbb{D}) = 1$), $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, H^∞ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$ and $\mathfrak{M}$ the space of all complex Borel measures on $\partial\mathbb{D}$. Let

$$\eta_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a, z \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0. Let ν be a positive continuous function on $\mathbb{D}$ (*weight*). A weight ν is called *typical* if it is radial, i.e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$ and $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. A positive continuous function ν on the interval $[0, 1)$ is called normal if there are $\delta \in [0, 1)$ and τ and t, $0 < \tau < t$ such that

$$\frac{\nu(r)}{(1-r)^\tau} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to1} \frac{\nu(r)}{(1-r)^\tau} = 0;$$

Date: Received: 18 August 2012; Accepted: 6 November 2012.
2010 *Mathematics Subject Classification.* Primary 47B33, 46E10; Secondary 30D55.
Key words and phrases. Weighted composition operator, Cauchy integral transforms, logarithmic weighted-type space, little logarithmic weighted-type space.

$$\frac{\nu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\nu(r)}{(1-r)^t}=\infty.$$

If we say that a function $\nu : \mathbb{D} \to [0, \infty)$ is a normal weight function, then we also assume that it is radial. We denote by $LA_{\ln}(\nu)$ the logarithmic weighted-type space of functions $f \in H(\mathbb{D})$ for which

$$||f||_{LA_{\ln}(\nu)} = \sup_{z\in\mathbb{D}} \nu(|z|)|f(z)| \ln\frac{2}{1-|z|^2} < \infty.$$

Likewise we write $LA_{\ln,0}(\nu)$ for little logarithmic weighted-type space of holomorphic functions f on $\mathbb{D}$ for which

$$\lim_{|z|\to 1} \nu(|z|)|f(z)| \ln\frac{2}{1-|z|^2} = 0.$$

With the norm $\|\cdot\|_{LA_{\ln}(\nu)}$, the space $LA_{\ln}(\nu)$ is a Banach space and the little logarithmic weighted space $LA_{\ln,0}(\nu)$ is a closed subspace of $LA_{\ln}(\nu)$.

A function $f \in H(\mathbb{D})$ is in the space of Cauchy integral transforms $\mathcal{K}$, if it admits a representation of the form

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z} \tag{1.1}$$

where $\mu \in \mathfrak{M}$. The space $\mathcal{K}$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{K}} = \inf\left\{\|\mu\| \;:\; f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}\right\},$$

where $\|\mu\|$ denotes the total variation of measure μ. It is clear that the Banach space $\mathcal{K}$ is the quotient of Banach space $\mathfrak{M}$ of Borel measures by the subspace of measures whose Cauchy transforms vanish. By the E. and M. Riesz theorem it follows that the Borel measure μ has a vanishing Cauchy transform if and only if it has the form $d\mu = f dm$, where $f \in \overline{H_0^1}$, the subspace of L^1 consisting of functions with mean value 0 whose conjugate belongs to the Hardy space H^1, and dm is the normalized Lebesgue measure on $\partial\mathbb{D}$. Hence $\mathcal{K}$ is isometrically isomorphic to $\mathfrak{M}/\overline{H_0^1}$. Since $\mathfrak{M}$ has a decomposition $\mathfrak{M} = L^1 \oplus \mathfrak{M}_s$, where $\mathfrak{M}_s$ is the space of all Borel measures which are singular with respect to Lebesgue measure, and $\overline{H_0^1} \subset L^1$, it follows that $\mathcal{K}$ is isometrically isomorphic to $L^1/\overline{H_0^1} \oplus \mathfrak{M}_s$. Consequently, $\mathcal{K}$ has an analogous decomposition $\mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s$, where $\mathcal{K}_a$ is isometrically isomorphic to $L^1/\overline{H_0^1}$ and $\mathcal{K}_s$ is isometrically isomorphic to $\mathfrak{M}_s$. It is known that

$$H^1 \subset \mathcal{K} \subset \cap_{0<p<1} H^p,$$

where H^p is the Hardy space. For more about the space $\mathcal{K}$, see [3], [4], [5] and [9].

Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Define a linear operator

$$W_{\psi,\varphi} f(z) = \psi(z) f(\varphi(z))$$

for $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The operator $W_{\psi,\varphi}$ is called a weighted composition operator. We can regard this operator as a generalization of a multiplication operator M_ψ induced by ψ and a composition operator C_φ induced by φ, where $M_\psi f(z) = \psi(z)f(z)$ and $C_\varphi f(z) = f(\varphi(z))$. In fact, $W_{\psi,\varphi} = M_\psi C_\varphi$. For more

about these operators, see [6] and [15].
It is well known that every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on $\mathcal{K}$. In fact, Bourdon and Cima [3] proved that

$$\|C_\varphi\|_{\mathcal{K}\to\mathcal{K}} \leq \frac{2+2\sqrt{2}}{1-|\varphi(0)|}$$

which was improved to

$$\|C_\varphi\|_{\mathcal{K}\to\mathcal{K}} \leq \frac{1+2|\varphi(0)|}{1-|\varphi(0)|} \tag{1.2}$$

by Cima and Matheson [4]. Moreover, equality is attained for certain linear fractional maps.
Isometries in many Banach spaces of analytic functions are weighted composition operators for example see [7] and [8]. It is of interest to provide function-theoretic characterizations indicating when ψ and φ induce bounded or compact weighted composition operators on spaces of holomorphic functions. For some recent results in this area, see [1],[2], [10]-[14], [16]-[26] and the references therein. In this paper, we provide, in a concise way, a function theoretic characterizations indicating when ψ and φ induce bounded or compact weighted composition operators from the space of Cauchy integral transforms to logarithmic weighted-type spaces. We also manage to calculate the norm of the operator $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ which is one of the problems that recently attracted some attention. Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. We write $A \asymp B$ if there is a positive constant C such that $CA \leq B \leq A/C$.

2. Boundedness and Compactness of $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ and $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln,0}(\nu)$

In this section, we characterize the boundedness and compactness of $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ and $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln,0}(\nu)$.

Theorem 1. *Let $\nu:\mathbb{D}\to[0,\infty)$ be a normal weight function, $\psi\in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ is bounded if and only if*

$$M := \sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2} < \infty. \tag{2.1}$$

Moreover, if $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ is bounded then

$$\|W_{\psi,\varphi}\|_{\mathcal{K}\to LA_{\ln}(\nu)} = M. \tag{2.2}$$

Proof. First suppose that $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ is bounded. Consider the family of functions

$$f_\zeta(z) = \frac{1}{1-\overline{\zeta}z}, \quad \zeta\in\partial\mathbb{D}. \tag{2.3}$$

Then $\|f_\zeta\|_{\mathcal{K}} = 1$, for each $\zeta \in \partial\mathbb{D}$ (see, e.g., [3, p. 468]). Thus by the boundedness of $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ we have that

$$\|W_{\psi,\varphi} f_\zeta\|_{LA_{\ln}(\nu)} \le \|W_{\psi,\varphi}\|_{\mathcal{K}\to LA_{\ln}(\nu)},$$

for every $\zeta \in \partial\mathbb{D}$ and so

$$M := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} \le \|W_{\psi,\varphi}\|_{\mathcal{K}\to W_{\psi,\varphi} LA_{\ln}(\nu)}. \tag{2.4}$$

Conversely, suppose that (2.1) holds. Let $f \in \mathcal{K}$. Then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{K}}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{1-\overline{\zeta}z}. \tag{2.5}$$

Replacing z in (2.5) by $\varphi(z)$, using a well-known inequality and multiplying such obtained inequality with $\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}$, we obtain

$$\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2} |f(\varphi(z))| \le \int_{\partial\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} d|\mu|(\zeta). \tag{2.6}$$

Thus

$$\begin{aligned}\nu(|z|) \ln \frac{2}{1-|z|^2} |W_{\psi,\varphi} f(z)| &\le \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} \int_{\partial\mathbb{D}} d|\mu|(\zeta) \\ &= \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} \|f\|_{\mathcal{K}}.\end{aligned}$$

Taking the supremum in the last inequality over all $z \in \mathbb{D}$ it follows that

$$||W_{\psi,\varphi} f||_{LA_{\ln}(\nu)} \le \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} \|f\|_{\mathcal{K}}. \tag{2.7}$$

This shows that $\|W_{\psi,\varphi} f\|_{LA_{\ln}(\nu)} \le M\|f\|_{\mathcal{K}}$, hence $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded. From (2.4) and (2.7), equality (2.2) follows. □

In the following corollary we give another necessary and sufficient condition for the boundedness of the operator $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$.

Corollary 1. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function, $\psi \in H(\mathbb{D})$, φ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded if and only if*

$$N := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty. \tag{2.8}$$

Moreover, $N \asymp M$.

Proof. First assume that the operator $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded. Using Theorem 1 and the identity

$$(1-|z|^2)|\eta_a'(z)| = 1-|\eta_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2},$$

we have

$$N \le M \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} dA(z) = MC < \infty. \tag{2.9}$$

Conversely, assume that (2.8) holds. Let $D(a,(1-|a|)/2) = \{z \in \mathbb{D} : |z-a| < (1-|a|)/2\}$. Since $\nu : \mathbb{D} \to [0,\infty)$ is a normal weight function, so

$$\nu(|a|) \ln \frac{2}{1-|a|^2} \asymp \nu(|z|) \ln \frac{2}{1-|z|^2}, \tag{2.10}$$

for $z \in D(a,(1-|a|)/2)$. Using (2.9), (2.10) and the subharmonicity of the function $|\psi|/|1-\overline{\zeta}\varphi|$, we have

$$\begin{aligned} N &\ge \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|} |\psi(z)| \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z) \\ &\ge C \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \frac{\nu(|a|)}{|1-\overline{\zeta}\varphi(a)|} |\psi(a)| \ln \frac{2}{1-|a|^2} = CM, \end{aligned}$$

so that (2.1) holds. Thus by Theorem 1, we have that the operator $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded and $M \asymp N$, as desired. □

Lemma 1. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $f \in LA_{\ln}(\nu)$ if and only if*

$$I := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|\nu(|z|) \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty. \tag{2.11}$$

Moreover, the following asymptotic relationship holds

$$\|f\|_{LA_{\ln}(\nu)} \asymp I.$$

Proof. Assume that (2.11) holds. Let $E(a,1/2) = \{z \in \mathbb{D} : |\eta_a(z)| < 1/2\}$. Then

$$\begin{aligned} |f(a)| = |f_a(\eta(0))| &\le 4 \int_{|z|<1/2} |f(\eta(z))| dA(z) \\ &= 4 \int_{E(a,1/2)} |f(z)||\eta_a'(z)|^2 dA(z) \end{aligned}$$

Since $\nu : \mathbb{D} \to [0,\infty)$ is a normal weight function, so

$$\nu(|a|) \ln \frac{2}{1-|a|^2} \asymp \nu(|z|) \ln \frac{2}{1-|z|^2}$$

for $z \in E(a,1/2)$. Thus

$$|f(a)| \le \frac{C}{\nu(|a|) \ln \frac{2}{1-|a|^2}} \int_{E(a,1/2)} |f(z)|\nu(|z|) \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z).$$

Hence

$$\begin{aligned} &\nu(|a|) \ln \frac{2}{1-|a|^2} |f(a)| \\ &\qquad \le C \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|\nu(|z|) \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z), \end{aligned}$$

which implies that if I is finite, then $f \in LA_{\ln}(\nu)$ and $\|f\|_{LA_{\ln}(\nu)} \leq CI$. Conversely, assume that $f \in LA_{\ln}(\nu)$, then, we get

$$I \leq \|f\|_{LA_{\ln}(\nu)} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \leq C\|f\|_{LA_{\ln}(\nu)} < \infty.$$

Hence $\|f\|_{LA_{\ln}(\nu)} \asymp I$, as desired. □

Using the fact that the family of functions

$$\left\{ f_\zeta = \frac{1}{1-\bar{\zeta}z} : \zeta \in \partial\mathbb{D} \right\}$$

satisfies $\|f_\zeta\|_{\mathcal{K}} = 1$, $\zeta \in \mathbb{D}$, by Corollary 1 and Lemma 1, we easily obtain the following result.

Corollary 2. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function, $\psi \in H(\mathbb{D})$ and φ a holomorphic-self map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded if and only if the family of functions*

$$\left\{ \frac{\psi}{1-\bar{\zeta}\varphi} : \zeta \in \partial\mathbb{D} \right\}$$

is norm-bounded in $LA_{\ln}(\nu)$.

By (1.1), it is easy to see that the unit ball of $\mathcal{K}$ is a normal family of holomorphic functions. A standard normal family argument then yields the proof of the following lemma (see, e.g. Proposition 3.11 of [6] or Lemma 3 in [13]).

Lemma 2. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function, $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is compact if and only if for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{K}} \leq 1$ converging to zero on compacts of $\mathbb{D}$, we have $\lim_{n\to\infty} \|W_{\psi,\varphi} f_n\|_{LA_{\ln}(\nu)} = 0$.*

Lemma 3. *Let $f \in B_{\mathcal{K}}$, the unit ball in $\mathcal{K}$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $f_t \in \mathcal{K}$ and $\sup_{0<t<1} \|f_t\|_{\mathcal{K}} \leq \|f\|_{\mathcal{K}}$.*

Proof. Let $f \in B_{\mathcal{K}}$ and $f_t(z) = f(tz)$, $0 < t < 1$. For $0 < t < 1$, let φ_t be defined on $\mathbb{D}$ as $\varphi_t(z) = tz$. Then φ_t is a holomorphic self-map of $\mathbb{D}$ and $\varphi_t(0) = 0$. Also $f_t(z) = f(tz) = (f \circ \varphi_t)(z) = C_{\varphi_t} f(z)$ for all $z \in \mathbb{D}$. Therefore, $f_t = C_{\varphi_t} f$. Since every self-map of $\mathbb{D}$ induces bounded composition operator on $\mathcal{K}$, we have that C_{φ_t} is bounded on $\mathcal{K}$. Moreover, by (1.2), we have

$$\|f_t\|_{\mathcal{K}} = \|C_{\varphi_t} f\|_{\mathcal{K}} \leq \frac{1+2|\varphi_t(0)|}{1-|\varphi_t(0)|} \|f\|_{\mathcal{K}} = \|f\|_{\mathcal{K}}.$$

Taking supermum over t, $0 < t < 1$, we get the desired result. □

Theorem 2. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function, $\psi \in H(\mathbb{D})$ and φ a holomorphic-self map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is compact if and only if*

$$M_1 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2} (1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty \tag{2.12}$$

and

$$\lim_{r\to 1}\sup_{\zeta\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z)=0 \quad (2.13)$$

for every $\zeta\in\partial\mathbb{D}$.

Proof. First suppose that (2.12) and (2.13) hold. Let $\{f_m\}_{m\in\mathbb{N}}$ be a bounded sequence in $\mathcal{K}$, say by L and converging to 0 uniformly on compacts of $\mathbb{D}$ as $m\to\infty$. By Lemma 2, we have to show that $\|W_{\psi,\varphi}f_m\|_{LA_{\ln}(\nu)}\to 0$ as $m\to\infty$. For each $m\in\mathbb{N}$, we can find $\mu_m\in\mathfrak{M}$ with $\|\mu_m\|=\|f_m\|_{\mathcal{K}}$ such that

$$f_m(z)=\int_{\partial\mathbb{D}}\frac{d\mu_m(\zeta)}{1-\bar{\zeta}z}.$$

By (2.13), we have for every $\epsilon>0$, there is an $r_1\in(0,1)$ such that for $r\in(r_1,1)$, we have

$$\sup_{\zeta\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z)<\epsilon.$$

By Lemma 1, applied to the function $\psi(f_m\circ\varphi)$ we have

$$\begin{aligned}\|W_{\psi,\varphi}f_m\|_{LA_{\ln}(\nu)}\asymp\sup_{a\in\mathbb{D}}\Big(\int_{|\varphi(z)|\le r}+\int_{|\varphi(z)|>r}\Big)|f_m(\varphi(z))|\\ \times\nu(z)|\psi(z)|\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z).\end{aligned}$$

Since the set $|w|\le r$ is compact we have $\sup_{|\varphi(z)|\le r}|f_m(\varphi(z))|<\epsilon$ for sufficiently large m, say $m\ge m_0$. Thus by Fubini's theorem, we have

$$\begin{aligned}\|W_{\psi,\varphi}f_m\|_{LA_{\ln}(\nu)}&\le C\sup_{|\varphi(z)|\le r}|f_m(\varphi(z))|\sup_{a\in\mathbb{D}}\int_{|\varphi(z)|\le r}|\psi(z)|\\ &\qquad\times\nu(z)\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z)\\ &\quad+\sup_{\zeta\in\partial\mathbb{D}}\sup_{a\in\mathbb{D}}\int_{\partial\mathbb{D}}\int_{|\varphi(z)|>r}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}\\ &\qquad\times|\psi(z)|\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z)d|\mu_m|(\zeta)\\ &\le C\Big(M_1+\int_{\partial\mathbb{D}}d|\mu_m|(\zeta)\Big)\epsilon\le C(M_1+\|f_m\|_{\mathcal{K}})\epsilon\\ &\le C(M_1+L)\epsilon\end{aligned}$$

for $m\ge m_0$. Since $\epsilon>0$ is arbitrary, $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ is compact.
Conversely, suppose that $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln}(\nu)$ is compact. By choosing $f(z)=1$ in $\mathcal{K}$, we have

$$M:=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\nu(|z|)|\psi(z)|\ln\frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z)<\infty.$$

Let $f_m(z)=z^m$, $m\in\mathbb{N}$. It is easy to see that $\{f_m\}_{m\in\mathbb{N}}$ is a norm bounded sequence in $\mathcal{K}$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Hence by

Lemma 2, it follows that $\|W_{\psi,\varphi}f_m\|_{LA_{\ln}(\nu)} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2m}\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < \epsilon. \tag{2.14}$$

From (2.14), we have for each $r \in (0,1)$

$$r^{2m} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < \epsilon.$$

Hence for $r \in (1/2^{1/(2m_0)}, 1)$, we have

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < 2\epsilon. \tag{2.15}$$

Let $f \in B_{\mathcal{K}}$ and $f_t(z) = f(tz)$, $0 < t < 1$, then by Lemma 3, we have that $\sup_{0<t<1} \|f_t\|_{\mathcal{K}} \leq \|f\|_{\mathcal{K}}$, $f_t \in \mathcal{K}$, $t \in (0,1)$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ implies that

$$\lim_{t\to 1} \|W_{\psi,\varphi}f_t - W_{\psi,\varphi}f\|_{LA_{ln}(\nu)} = 0.$$

Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t(\varphi(z)) - f(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < \epsilon. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) \\
&\quad \leq \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t(\varphi(z)) - f(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) \\
&\qquad + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) \\
&\quad \leq \epsilon(2 + ||f_t||_\infty).
\end{aligned}$$

Thus we conclude that for every $f \in B_{\mathcal{K}}$, there is an $r_0 \in (0,1)$ such that for $r \in (r_0, 1)$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < C\epsilon. \tag{2.17}$$

Since $W_{\psi,\varphi} : \mathcal{K} \to LA_{ln}(\nu)$ is compact, we have for every $\epsilon > 0$, there is a finite collection of functions $f_1, f_2, \cdots, f_k \in B_{\mathcal{K}}$ such that for each $f \in B_{\mathcal{K}}$, there is a $j \in \{1, 2, \cdots, k\}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(\varphi(z)) - f_j(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < \epsilon. \tag{2.18}$$

On the other hand from (2.17) it follows that if $\delta := \max_{1\leq j\leq k} \delta_j(\epsilon, f_j)$, then for $r \in (\delta, 1)$ and all $j \in \{1, 2, \cdots, k\}$ we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_j(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < \epsilon. \tag{2.19}$$

From (2.18) and (2.19) we have for $r \in (\delta, 1)$ and every $f \in B_{\mathcal{K}}$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f(\varphi(z))|\nu(|z|)|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < C\epsilon. \quad (2.20)$$

If we apply (2.20) to the function $f_\zeta(z) = 1/(1-\overline{\zeta}z)$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)| \ln \frac{2}{1-|z|^2}(1-|\eta_a(z)|^2)^2 d\lambda(z) < C\epsilon,$$

from which (2.13) follows as desired. □

Theorem 3. *Let $\nu : \mathbb{D} \to [0, \infty)$ be a normal weight function, $\psi \in H(\mathbb{D})$ and φ a holomorphic-self map of $\mathbb{D}$. Then $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$ is bounded if and only if*

$$M := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)| \ln \frac{2}{1-|z|^2} < \infty \quad (2.21)$$

and

$$\lim_{|z|\to 1} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)| \ln \frac{2}{1-|z|^2} = 0 \quad (2.22)$$

for every $\zeta \in \partial\mathbb{D}$.

Proof. First suppose that $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$ is bounded. Once again consider the family of test functions in (2.3). Then $\|f_\zeta\|_{\mathcal{K}} = 1$. Thus by the boundedness of $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$, we have $W_{\psi,\varphi} f_\zeta \in LA_{\ln,0}(\nu)$ for every $\zeta \in \partial\mathbb{D}$ and so

$$\lim_{|z|\to 1} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)| \ln \frac{2}{1-|z|^2} = 0$$

for every $\zeta \in \partial\mathbb{D}$. Again, if $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$ is bounded, then for every $f \in \mathcal{K}$, we have that $W_{\psi,\varphi} f \in LA_{\ln,0}(\nu) \subset LA_{\ln}(\nu)$. So by the closed graph theorem $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln}(\nu)$ is bounded. Thus by Theorem 1, we have (2.21). Conversely, suppose that (2.21) and (2.22) hold. By (2.22), the inner expression in the second term of (2.6) tends to zero for every $\zeta \in \partial\mathbb{D}$, as $|z| \to 1$. Also the inner expression in the second term of (2.6) is dominated by

$$M := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(|z|)}{|1-\overline{\zeta}\varphi(z)|}|\psi(z)| \ln \frac{2}{1-|z|^2}.$$

Thus by the bounded convergence theorem, the second term of (2.6) tend to zero as $|z| \to 1$, so

$$\lim_{|z|\to 1} \nu(|z|) \ln \frac{2}{1-|z|^2}|W_{\psi,\varphi} f(z)| = 0.$$

Thus we conclude that if $f \in \mathcal{K}$, then $W_{\psi,\varphi} f \in LA_{\ln,0}(\nu)$. Therefore, the boundedness of $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$ follows by the closed graph theorem. □

In order to prove the compactness of $W_{\psi,\varphi} : \mathcal{K} \to LA_{\ln,0}(\nu)$, we require the following lemma.

Lemma 4. *A subset F of $LA_{\ln,0}(\nu)$ is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in F}\nu(z)|f(z)|\ln\frac{2}{1-|z|^2}=0.$$

The proof is similar to that of Lemma 1 in [13], we omit the details.

Theorem 4. *Let $\nu:\mathbb{D}\to[0,\infty)$ be a normal weight function, $\psi\in H(\mathbb{D})$ and φ a holomorphic-self map of $\mathbb{D}$. Then $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln,0}(\nu)$ is compact if and only if*

$$\lim_{|z|\to 1}\sup_{\zeta\in\partial\mathbb{D}}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}=0. \tag{2.23}$$

Proof. By Lemma 4, a closed set F in $LA_{\ln,0}(\nu)$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1}\sup_{f\in F}\nu(|z|)|f(z)|\ln\frac{2}{1-|z|^2}=0.$$

Thus the set $\{W_{\psi,\varphi}f : f\in\mathcal{K}, \|f\|_{\mathcal{K}}\le 1\}$ has compact closure in $LA_{\ln,0}(\nu)$ if and only if

$$\lim_{|z|\to 1}\sup\{\nu(|z|)\ln\frac{2}{1-|z|^2}|W_{\psi,\varphi}f(z)| : f\in\mathcal{K}, \|f\|_{\mathcal{K}}\le 1\}=0. \tag{2.24}$$

Let $f\in LA_{\ln,0}(\nu)$ with $\|f\|_{LA_{\ln}(\nu)}\le 1$. Then there is a $\mu\in\mathfrak{M}$ such that $\|\mu\|=\|f\|_{\mathcal{K}}$ and

$$f(z)=\int_{\partial\mathbb{D}}\frac{d\mu(\zeta)}{1-\bar{\zeta}z}.$$

Then by (2.6), we have

$$\begin{aligned}\nu(|z|)\ln\frac{2}{1-|z|^2}|W_{\psi,\varphi}f(z)| &\le \int_{\partial\mathbb{D}}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}d|\mu|(\zeta)\\ &\le \|\mu\|\sup_{\zeta\in\partial\mathbb{D}}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}\\ &= \|f\|_{\mathcal{K}}\sup_{\zeta\in\partial\mathbb{D}}\frac{\nu(|z|)}{|1-\bar{\zeta}\varphi(z)|}|\psi(z)|\ln\frac{2}{1-|z|^2}.\end{aligned}$$

Hence by (2.23), we have

$$\lim_{|z|\to 1}\sup\{\nu(z)|W_{\psi,\varphi}f(z)|\ln\frac{2}{1-|z|^2} : f\in\mathcal{K}, \|f\|_{\mathcal{K}}\le 1\}=0.$$

Conversely, suppose that $W_{\psi,\varphi}:\mathcal{K}\to LA_{\ln,0}(\nu)$ is compact. Taking the test functions in (2.3) and using the fact that $\|f_\zeta\|_{\mathcal{K}}=1$ we obtain that (2.23) follows from (2.24). □

Acknowledgements. The author would like to thank Prof. S. Stević and the referee for their helpful comments and suggestions.

References

1. R.F. Allen and F. Collona, *Weighted composition operators from H^∞ to the Bloch space of a bounded homogeneous domain*, Integral Equations Operator Theory **66** (2010), no. 1, 21–40.
2. R.F. Allen and F. Collona, *Weighted composition operators on the Bloch space of a bounded homogeneous domain*, Oper. Theory Adv. Appl. **202** (2010), 11–37.
3. P. Bourdon and J.A. Cima, *On integrals of Cauchy-Stieltjes type*, Houston J. Math. **14** (1988), no. 4, 465–474.
4. J.A. Cima and A.L. Matheson, *Cauchy transforms and composition operators*, Illinois J. Math. **4** (1998), no. 1, 58–69.
5. J.A. Cima and T.H. MacGregor, *Cauchy transforms of measures and univalent functions*, Lecture Notes in Math. 1275, Spinger-Verlag 1987, 78–88.
6. C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press Boca Raton, New York, 1995.
7. F. Forelli, *The isometries of H^p*, Canad. J. Math. **16** (1964), 721–728.
8. F. Forelli, *A theorem on isometries and the application of it to the isometries of $H^p(S)$ for* $2 < p < \infty$, Canad. J. Math. **25** (1973), 284–289.
9. R. Hibschweiler and E. Nordgren, *Cauchy transforms of measures and weighted shift operators on the disc algebra*, Rocky Mountain J. Math. **26** (1996), no. 2, 627–654.
10. S. Li and S. Stević, *Generalized composition operators on Zygmund spaces and Bloch type spaces*, J. Math. Anal. Appl. **338** (2008), no. 2, 1282–1295.
11. S. Li and S. Stević, *Products of composition and integral type operators from H^∞ and the Bloch space*, Complex Var. Elliptic Equ. **53** (2008), no. 5, 463–474.
12. S. Li and S. Stević, *Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces*, J. Math. Anal. Appl. **345** (2008), no. 1, 40–50.
13. K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2679–2687.
14. S. Ohno, K. Stroethoff and R. Zhao, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), no. 1, 191–215.
15. J.H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, New York. 1993.
16. A.K. Sharma, *Volterra composition operators between Bergman-Nevanlinna and Bloch-type spaces*, Demonstratio Math. **42** (2009), no. 3, 607–618.
17. A.K. Sharma, *Products of multiplication, composition and differentiation between weighted Bergman-Nevanlinna and Bloch-type spaces*, Turkish. J. Math. **35** (2011), no. 2, 275–291.
18. A.K. Sharma and S. Ueki, *Compactness of composition operators acting on weighted Bergman Orlicz spaces*, Ann. Polon. Math. **103** (2011), no. 1, 1–13.
19. A.K. Sharma and S. Ueki, *Composition operators from Nevanlinna type spaces tp Bloch type spaces*, Banach J. Math. Anal. **6** (2012), no. 1, 112–123.
20. A. Sharma and A.K. Sharma, *Carleson measures and a class of generalized integration operators on the Bergman space*, Rocky Mountain J. Math. **41** (2011), no. 5, 711–724.
21. S. Stević, *Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball*, J. Inequal. Appl. **2007**, Article ID 28629, 9 pp.
22. S. Stević, *Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces*, Util. Math. **77** (2008) 167–172.
23. S. Stević, *Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball*, Abstr. Appl. Anal. **2010** Article ID 801264, 14 pp.
24. S. Stević and A.K. Sharma, *Essential norm of composition operators between weighted Hardy spaces*, Appl. Math. Comput. **217** (2011), no. 13, 6192–6197.
25. S. Stević and A.K. Sharma, *Composition operators from the space of Cauchy transforms to Bloch and the little Bloch-type spaces on the unit disk*, Appl. Math. Comput. **217** (2011), no. 24, 10187–10194.

26. S. Stević, A.K. Sharma and A. Bhat, *Products of multiplication composition and differentiation operators on weighted Bergman spaces*, Appl. Math. Comput. **217** (2011), no. 20, 8115–8125.

[1] School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Katra-182320, J& K, India.

E-mail address: aksju_76@yahoo.com; aksju76@gmail.com