

## GENERAL MULTIPLE OPIAL-TYPE INEQUALITIES FOR THE CANAVATI FRACTIONAL DERIVATIVES

M. ANDRIĆ<sup>1</sup>, J. PEČARIĆ<sup>2</sup> AND I. PERIĆ<sup>3</sup> \*

Communicated by S. Barza

ABSTRACT. In this paper we establish some general multiple Opial-type inequalities involving the Canavati fractional derivatives. In some cases the best possible constants are discussed.

### 1. INTRODUCTION AND PRELIMINARIES

In 1960, Opial [7] proved the following inequality:

Let  $f \in C^1[0, h]$  be such that  $f(0) = f(h) = 0$  and  $f(x) > 0$  for  $x \in (0, h)$ . Then

$$\int_0^h |f(x) f'(x)| dx \leq \frac{h}{4} \int_0^h [f'(x)]^2 dx, \quad (1.1)$$

where  $h/4$  is the best possible.

This inequality has been generalized and extended over the last 50 years in several directions, and used in many applications in differential equations (for more details see [1], [9]). The aim of our research is an Opial-type inequality for fractional derivatives, which has the general form

$$\int_a^b w_1(t) \left( \prod_{i=1}^N |D^{\beta_i} f(t)|^{r_i} \right)^p |D^\alpha f(t)|^q dt \leq C \left( \int_a^b w_2(t) |D^\alpha f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}},$$

where  $w_1$  and  $w_2$  are weight functions,  $r = \sum_{i=1}^N r_i$  and  $D^\gamma f$  denotes the Canavati fractional derivative of  $f$  of order  $\gamma$ .

First we survey some facts about the fractional integrals and derivatives needed in this paper. For more details see the monographs [8, Chapter 1] and [3].

Date: Received: 17 July 2012; Accepted: 3 November 2012.

\* Corresponding author.

2010 *Mathematics Subject Classification*. Primary 26A33; Secondary 26D15, 46N20.

*Key words and phrases*. Opial-type inequalities, Canavati fractional derivatives, weights.

By  $C^n[a, b]$  we denote the space of all functions on  $[a, b]$  which have continuous derivatives up to order  $n$ , and  $AC[a, b]$  is the space of all absolutely continuous functions on  $[a, b]$ . By  $AC^n[a, b]$  we denote the space of all functions  $f \in C^{n-1}[a, b]$  with  $f^{(n-1)} \in AC[a, b]$ .

By  $L_p[a, b]$ ,  $1 \leq p < \infty$ , we denote the space of all Lebesgue measurable functions  $f$  for which  $|f|^p$  is Lebesgue integrable on  $[a, b]$ , and by  $L_\infty[a, b]$  the set of all functions measurable and essentially bounded on  $[a, b]$ . Clearly,  $L_\infty[a, b] \subset L_p[a, b]$  for all  $p \geq 1$ .

Let  $x \in [a, b]$ ,  $\alpha > 0$ ,  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integral part of  $\alpha$  and  $\Gamma$  is the gamma function  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ . For  $f \in L_1[a, b]$  the *Riemann-Liouville fractional integrals*  $J_{a+}^\alpha f$  (left-sided) and  $J_{b-}^\alpha f$  (right-sided) of order  $\alpha$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt,$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt.$$

The subspaces  $C_{a+}^\alpha[a, b]$  and  $C_{b-}^\alpha[a, b]$  of  $C^{n-1}[a, b]$  are defined by

$$C_{a+}^\alpha[a, b] = \{f \in C^{n-1}[a, b] : J_{a+}^{n-\alpha} f^{(n-1)} \in C^1[a, b]\},$$

$$C_{b-}^\alpha[a, b] = \{f \in C^{n-1}[a, b] : J_{b-}^{n-\alpha} f^{(n-1)} \in C^1[a, b]\}.$$

For  $f \in C_{a+}^\alpha[a, b]$  and  $g \in C_{b-}^\alpha[a, b]$  the *Canavati fractional derivatives*  $D_{a+}^\alpha f$  (left-sided) and  $D_{b-}^\alpha g$  (right-sided) of order  $\alpha$  are defined by

$$D_{a+}^\alpha f(x) = \frac{d}{dx} J_{a+}^{n-\alpha} f^{(n-1)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t) dt,$$

$$D_{b-}^\alpha g(x) = (-1)^n \frac{d}{dx} J_{b-}^{n-\alpha} g^{(n-1)}(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{n-\alpha-1} g^{(n-1)}(t) dt.$$

In addition, we stipulate

$$D_{a+}^0 f := f =: J_{a+}^0 f,$$

$$D_{b-}^0 g := g =: J_{b-}^0 g.$$

If  $\alpha \in \mathbb{N}$  then  $D_{a+}^\alpha f = f^{(\alpha)}$  and  $D_{b-}^\alpha g = (-1)^\alpha g^{(\alpha)}$ , the ordinary  $\alpha$ -order derivatives.

The composition identity for the Canavati left-sided fractional derivatives comes from [5], and will be used in all presented Opial-type inequalities. Notice that we relaxed some conditions on parameters and a function, comparing to the analogous identity given in [3].

**Theorem 1.1.** [5, Theorem 2.1] *Let  $\alpha > \beta \geq 0$ ,  $n = [\alpha] + 1$ ,  $m = [\beta] + 1$ . Let  $f \in C_{a+}^\alpha[a, b]$  be such that  $f^{(i)}(a) = 0$  for  $i = m-1, \dots, n-2$ . Then  $f \in C_{a+}^\beta[a, b]$  and*

$$D_{a+}^\beta f(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_a^x (x-t)^{\alpha-\beta-1} D_{a+}^\alpha f(t) dt, \quad x \in [a, b]. \quad (1.2)$$

Our goal is to give general multiple Opial-type inequalities for the Canavati fractional derivatives. The starting point is next Opial-type inequality for the Riemann-Liouville<sup>1</sup> left-sided fractional derivatives  $\mathbf{D}_{a+}^{\alpha} f$  that comes from [6].

**Theorem 1.2.** [6, Theorem 4.1] *Let  $p_i, q, \beta_i, \alpha, r$  ( $i = 1, \dots, N$ ) be real numbers such that  $p_i \geq 0$ ,  $p := \sum_{i=1}^N p_i > 0$ ,  $q > 0$ ,  $\alpha > \beta_i + 1 \geq 0$  for all  $i = 1, \dots, N$ , and  $r > \max\{1, q, (\alpha - \beta_i)^{-1} : i = 1, \dots, N\}$ . Suppose  $f \in L_1[a, b]$  has an integrable left-sided fractional derivative  $\mathbf{D}_{a+}^{\alpha} f \in L_{\infty}[a, b]$  and  $\mathbf{D}_{a+}^{\alpha-j} f(a) = 0$  for  $j = 1, \dots, [\alpha] + 1$ . Then for any  $w_1, w_2 \in C[a, b]$  with  $w_1 \geq 0$  and  $w_2 > 0$ ,*

$$\int_a^x w_1(t) |\mathbf{D}_{a+}^{\alpha} f(t)|^q \prod_{i=1}^N |\mathbf{D}_{a+}^{\beta_i} f(t)|^{p_i} dt \leq A(x) \left( \int_a^x w_2(t) |\mathbf{D}_{a+}^{\alpha} f(t)|^r dt \right)^{\frac{p+q}{r}},$$

where

$$A(x) = \left( \frac{q}{p+q} \right)^{\frac{q}{r}} \left[ \int_a^x w_1(t)^{\frac{r}{r-q}} w_2(t)^{-\frac{q}{r-q}} \prod_{i=1}^N |P_i(t)|^{\frac{p_i(r-1)}{r-q}} dt \right]^{\frac{r-q}{r}},$$

$$P_i(t) = \int_a^t w_2(\tau)^{-\frac{1}{r-1}} K_i(t, \tau)^{\frac{r}{r-1}} d\tau, \quad a \leq t \leq x,$$

$$K_i(t, \tau) = \frac{(t - \tau)_+^{\alpha - \beta_i - 1}}{\Gamma(\alpha - \beta_i)}, \quad a \leq t, \tau \leq x,$$

$$(t - \tau)_+ = \max\{t - \tau, 0\}.$$

We will give two-weighted, one-weighted and non-weighted versions of this theorem involving the Canavati left-sided fractional derivatives. Also we will give versions of those inequalities which include decreasing or bounded weight functions.

The right-sided versions of all inequalities in this paper can be established and proven analogously.

## 2. TWO-WEIGHTED CASE

First theorem is the Canavati fractional derivatives analogy of Theorem 1.2, with relaxed conditions on the function (here the role of  $p_i$  and  $r$  from Theorem 1.2 have  $r_i p$  and  $p + q$  respectively).

**Theorem 2.1.** *Let  $N \in \mathbb{N}$ ,  $\alpha > \beta_i \geq 0$ ,  $m = \min\{[\beta_i] + 1 : i = 1, \dots, N\}$  and  $n = [\alpha] + 1$ . Let  $f \in C_{a+}^{\alpha}[a, b]$  be such that  $f^{(i)}(a) = 0$  for  $i = m - 1, \dots, n - 2$ . Let  $w_1$  and  $w_2$  be continuous weight functions on  $[a, x]$  with  $w_1 \geq 0$  and  $w_2 > 0$ . Let  $r_i \geq 0$ ,  $r = \sum_{i=1}^N r_i > 0$ . Let  $p > 0$ ,  $q \geq 0$ ,  $\sigma = \frac{1}{p+q} < 1$ ,  $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$*

<sup>1</sup> Riemann-Liouville left-sided fractional derivative is defined by  $\mathbf{D}_{a+}^{\alpha} f(x) = \frac{d^n}{dx^n} J_{a+}^{n-\alpha} f(x)$ .

and let  $\alpha > \beta_i + \sigma$  for  $i = 1, \dots, N$ . Let also  $D_{a+}^\alpha f \in L_{p+q}[a, b]$ . Then

$$\begin{aligned} & \int_a^x w_1(t) \left( \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i} \right)^p |D_{a+}^\alpha f(t)|^q dt \\ & \leq C_1 \left( \int_a^x w_2(t) |D_{a+}^\alpha f(t)|^{p+q} dt \right)^{\frac{rp+q}{p+q}} \\ & \quad \cdot \left[ \int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\alpha-\beta_i-1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p}, \end{aligned} \quad (2.1)$$

where

$$C_1 = \prod_{i=1}^N [\Gamma(\alpha - \beta_i)]^{-r_i p} \left( \frac{q}{rp + q} \right)^{\sigma q}. \quad (2.2)$$

*Proof.* Let  $q \neq 0$ ,  $\delta_i = \alpha - \beta_i - 1$ ,  $i = 1, \dots, N$ . Using composition identity (1.2), the triangle inequality and Hölder's inequality for  $\frac{1}{1-\sigma}$  and  $\frac{1}{\sigma}$ , for  $t \in [a, x]$  follows

$$\begin{aligned} & |D_{a+}^{\beta_i} f(t)| \\ & \leq \frac{1}{\Gamma(\delta_i + 1)} \int_a^t [w_2(\tau)]^{-\sigma} [w_2(\tau)]^\sigma (t-\tau)^{\delta_i} |D_{a+}^\alpha f(\tau)| d\tau \\ & \leq \frac{1}{\Gamma(\delta_i + 1)} \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{1-\sigma} \left( \int_a^t w_2(\tau) |D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^\sigma. \end{aligned}$$

Now we have

$$\begin{aligned} & \int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^\alpha f(t)|^q dt \\ & \leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \int_a^x w_1(t) [w_2(t)]^{-\sigma q} [w_2(t)]^{\sigma q} |D_{a+}^\alpha f(t)|^q \\ & \quad \cdot \left( \int_a^t w_2(\tau) |D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{(1-\sigma)r_i p} dt. \end{aligned} \quad (2.3)$$

Applying Hölder's inequality for  $\frac{1}{\sigma p}$ ,  $\frac{1}{\sigma q}$  and simple integration, we get

$$\begin{aligned}
 & \int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^{\alpha} f(t)|^q dt \\
 & \leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \\
 & \quad \cdot \left[ \int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\
 & \quad \cdot \left[ \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} \left( \int_a^t w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\frac{pr}{q}} dt \right]^{\sigma q} \\
 & = \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \left( \frac{q}{rp + q} \right)^{\sigma q} \left( \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(rp+q)} \tag{2.4} \\
 & \quad \cdot \left[ \int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p},
 \end{aligned}$$

which gives us inequality (2.1).

If  $q = 0$  ( $\sigma = \frac{1}{p}$ ), then inequality (2.3) has the form

$$\begin{aligned}
 & \int_a^x w_1(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} dt \\
 & \leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \left( \int_a^x w_2(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^r \\
 & \quad \cdot \int_a^x w_1(t) \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{(1-\sigma)r_i p} dt,
 \end{aligned}$$

from which we get inequality (2.1) for  $q = 0$ . □

Next results complement Theorem 2.1. To obtain inequality (2.5) we need a monotonicity of  $w_1$  and  $w_2$ .

**Theorem 2.2.** *Suppose that the assumptions of Theorem 2.1 hold. Suppose also that  $w_1$  is an increasing and  $w_2$  is a decreasing functions. Then*

$$\begin{aligned} & \int_a^x w_1(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p |D_{a+}^{\alpha} f(t)|^q dt \\ & \leq C_2 w_1(x) [w_2(x)]^{-\sigma(rp+q)} (x-a)^{(\rho+\sigma)p} \left( \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{p+q} dt \right)^{\frac{rp+q}{p+q}}, \end{aligned} \quad (2.5)$$

where

$$C_2 = \frac{C_1 \sigma^{\sigma p} (1-\sigma)^{(1-\sigma)rp}}{(\rho+\sigma)^{\sigma p} \prod_{i=1}^N (\alpha - \beta_i - \sigma)^{r_i(1-\sigma)p}} \quad (2.6)$$

and  $C_1$  is defined by (2.2).

*Proof.* We start the proof with obtained inequality (2.1) from Theorem 2.1. By monotonicity of  $w_1$  and  $w_2$  follows

$$\begin{aligned} & \left[ \int_a^x [w_1(t)]^{\frac{1}{p\sigma}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\ & \leq w_1(x) [w_2(x)]^{-\sigma(rp+q)} \left[ \int_a^x \prod_{i=1}^N \left( \int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\ & = w_1(x) [w_2(x)]^{-\sigma(rp+q)} \frac{(1-\sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N (\delta_i + 1 - \sigma)^{(1-\sigma)r_i p}} (x-a)^{(\rho+\sigma)p} \frac{\sigma^{\sigma p}}{(\rho+\sigma)^{\sigma p}}. \end{aligned} \quad (2.7)$$

Inequality (2.5) now follows from (2.4) and (2.7).

For  $q = 0$ , we proceed the same as in the proof of Theorem 2.1.  $\square$

For the next theorem we suppose that weight functions are bounded.

**Theorem 2.3.** *Suppose that the assumptions of Theorem 2.1 hold. Suppose also  $w_1(t) \leq B$  and  $A \leq w_2(t)$  for  $t \in [a, x]$ . Then*

$$\begin{aligned} & \int_a^x w_1(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p |D_{a+}^{\alpha} f(t)|^q dt \\ & \leq C_2 B A^{-\sigma(rp+q)} (x-a)^{(\rho+\sigma)p} \left( \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{p+q} dt \right)^{\frac{rp+q}{p+q}}, \end{aligned} \quad (2.8)$$

where  $C_2$  is defined by (2.6).

*Proof.* The proof of (2.8) is the same as the one for (2.5), except one change: instead of inequalities  $w_1(t) \leq w_1(x)$ ,  $w_2(t) \geq w_2(x)$  we use  $w_1(t) \leq B$ ,  $w_2(t) \geq A$  respectively.  $\square$

With extra parameters  $s_1, s_2$  and  $s_3$  we can extract expressions containing just weight functions to get inequality (2.9).

**Theorem 2.4.** *Suppose that the assumptions of Theorem 2.1 hold. Suppose also that  $s_k > 1$  and  $\frac{1}{s_k} + \frac{1}{s'_k} = 1$  for  $k = 1, 2, 3$ . Then*

$$\begin{aligned} & \int_a^x w_1(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p |D_{a+}^{\alpha} f(t)|^q dt \\ & \leq C_3 P(x) Q(x) R(x) (x-a)^{\rho p + \frac{\sigma p}{s_2 s_3} - \frac{(1-\sigma) r p}{s'_1}} \left( \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{p+q} dt \right)^{\frac{r p + q}{p+q}}, \end{aligned} \tag{2.9}$$

where

$$C_3 = \frac{C_1 (1-\sigma)^{\frac{(1-\sigma) r p}{s_1}} (\sigma s_1)^{\frac{\sigma p}{s_2 s_3}}}{\prod_{i=1}^N [s_1(\alpha - \beta_i - 1) + 1 - \sigma]^{\frac{(1-\sigma) r_i p}{s_1}} \left[ \sum_{i=1}^N [s_1(\alpha - \beta_i - 1) + 1 - \sigma] r_i s_2 s_3 + \sigma s_1 \right]^{\frac{\sigma p}{s_2 s_3}}}$$

and

$$\begin{aligned} P(x) &= \left( \int_a^x [w_2(t)]^{-\frac{\sigma}{1-\sigma} s'_1} dt \right)^{\frac{(1-\sigma) r p}{s'_1}}, \\ Q(x) &= \left( \int_a^x [w_1(t)]^{\frac{s'_2}{\sigma p}} dt \right)^{\frac{\sigma p}{s'_2}}, \\ R(x) &= \left( \int_a^x [w_2(t)]^{-\frac{q}{p} s_2 s'_3} dt \right)^{\frac{\sigma p}{s_2 s'_3}}. \end{aligned} \tag{2.10}$$

*Proof.* We start the proof with obtained inequality (2.1) from Theorem 2.1. By Hölder's inequality we have

$$\begin{aligned} & \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \\ & \leq \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s'_1} d\tau \right)^{\frac{1}{s'_1}} \left( \int_a^t (t-\tau)^{\frac{\delta_i}{1-\sigma} s_1} d\tau \right)^{\frac{1}{s_1}} \\ & \leq \left( \int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s'_1} d\tau \right)^{\frac{1}{s'_1}} \left( \frac{1-\sigma}{\delta_i s_1 + 1 - \sigma} \right)^{\frac{1}{s_1}} (t-a)^{\frac{\delta_i}{1-\sigma} + \frac{1}{s_1}}. \end{aligned}$$

Now follows

$$\begin{aligned} & \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma) r_i}{\sigma}} \\ & \leq \left( \int_a^x [w_2(\tau)]^{-\frac{\sigma}{1-\sigma} s'_1} d\tau \right)^{\frac{(1-\sigma) r}{\sigma s'_1}} \frac{(1-\sigma)^{\frac{(1-\sigma) r}{\sigma s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma) r_i}{\sigma s_1}}} (t-a)^{\frac{\sum_{i=1}^N (\delta_i s_1 + 1 - \sigma) r_i}{\sigma s_1}}. \end{aligned}$$

Let  $\varepsilon = \sum_{i=1}^N (\delta_i s_1 + 1 - \sigma) r_i$ . Applying Hölder's inequalities we get

$$\begin{aligned}
& \left[ \int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} \prod_{i=1}^N \left( \int_a^t [w_2(\tau)]^{-\frac{\sigma}{1-\sigma}} (t-\tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} \\
& \leq P(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left[ \int_a^x [w_1(t)]^{\frac{1}{\sigma p}} [w_2(t)]^{-\frac{q}{p}} (t-a)^{\frac{\varepsilon}{\sigma s_1}} dt \right]^{\sigma p} \\
& \leq P(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left( \int_a^x [w_1(t)]^{\frac{s_2'}{\sigma p}} dt \right)^{\frac{\sigma p}{s_2'}} \\
& \quad \cdot \left( \int_a^x [w_2(t)]^{-\frac{qs_2}{p}} (t-a)^{\frac{\varepsilon}{\sigma s_1 s_2}} dt \right)^{\frac{\sigma p}{s_2}} \\
& \leq P(x) Q(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \\
& \quad \cdot \left( \int_a^x [w_2(t)]^{-\frac{q}{p} s_2 s_3'} dt \right)^{\frac{\sigma p}{s_2 s_3'}} \left( \int_a^x (t-a)^{\frac{\varepsilon}{\sigma s_1 s_2 s_3}} dt \right)^{\frac{\sigma p}{s_2 s_3}} \\
& = P(x) Q(x) R(x) \frac{(1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^N (\delta_i s_1 + 1 - \sigma)^{\frac{(1-\sigma)r_i p}{s_1}}} \left( \frac{\sigma s_1}{\varepsilon s_2 s_3 + \sigma s_1} \right)^{\frac{\sigma p}{s_2 s_3}} (x-a)^{\frac{\varepsilon p}{s_1} + \frac{\sigma p}{s_2 s_3}}.
\end{aligned} \tag{2.11}$$

Inequality (2.9) now follows from (2.4) and (2.11).

For  $q = 0$ , we proceed the same as in the proof of Theorem 2.1.  $\square$

If we choose a convenient parameter  $s_3$ , then we get next corollary.

**Corollary 2.5.** *Suppose that the assumptions of Theorem 2.4 hold. Suppose also that  $s_3 = \frac{\sigma p s_1'}{\sigma p s_1' - q(1-\sigma)s_2} > 1$ . Then*

$$\begin{aligned}
& \int_a^x w_1(t) \left( \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i} \right)^p |D_{a+}^{\alpha} f(t)|^q dt \\
& \leq \tilde{C}_3 \tilde{P}(x) Q(x) (x-a)^{\rho p + \frac{\sigma p}{s_2} - \frac{(1-\sigma)(rp+q)}{s_1'}} \left( \int_a^x w_2(t) |D_{a+}^{\alpha} f(t)|^{p+q} dt \right)^{\frac{rp+q}{p+q}},
\end{aligned}$$

where  $Q$  is defined by (2.10) and

$$\tilde{C}_3 = \frac{C_1 (1-\sigma)^{\frac{(1-\sigma)rp}{s_1}}}{\prod_{i=1}^N [(\alpha - \beta_i - 1)s_1 + 1 - \sigma]^{\frac{(1-\sigma)r_i}{s_1}}} \left[ \frac{\sigma p s_1' - q(1-\sigma)s_2}{(\rho s_2 + \sigma) p s_1' - (1-\sigma)(rp+q)s_2} \right]^{\frac{\sigma p}{s_2} - \frac{q(1-\sigma)}{s_1'}},$$



$$\tilde{P}(x) = \left( \int_a^x w_2(t)^{-\frac{\sigma}{1-\sigma}} s'_1 dt \right)^{\frac{(1-\sigma)(rp+q)}{s'_1}} .$$

### 3. ONE-WEIGHTED CASE

First result is a direct consequence of Theorem 2.1.

**Theorem 3.1.** *Let  $N \in \mathbb{N}$ ,  $\alpha > \beta_i \geq 0$ ,  $m = \min\{[\beta_i] + 1 : i = 1, \dots, N\}$  and  $n = [\alpha] + 1$ . Let  $f \in C_{a+}^\alpha[a, b]$  be such that  $f^{(i)}(a) = 0$  for  $i = m - 1, \dots, n - 2$ . Let  $w$  be continuous positive weight function on  $[a, x]$ . Let  $r_i \geq 0$ ,  $r = \sum_{i=1}^N r_i > 0$ . Let  $p > 0$ ,  $q \geq 0$ ,  $\sigma = \frac{1}{p+q} < 1$ ,  $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$  and let  $\alpha > \beta_i + \sigma$  for  $i = 1, \dots, N$ . Let also  $D_{a+}^\alpha f \in L_{p+q}[a, b]$ . Then*

$$\begin{aligned} & \int_a^x w(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p |D_{a+}^\alpha f(t)|^q dt \\ & \leq C_1 \left( \int_a^x w(t) |D_{a+}^\alpha f(t)|^{p+q} dt \right)^{\frac{rp+q}{p+q}} \\ & \quad \cdot \left[ \int_a^x w(t) \prod_{i=1}^N \left( \int_a^t [w(\tau)]^{-\frac{\sigma}{1-\sigma}} (t - \tau)^{\frac{\alpha - \beta_i - 1}{1-\sigma}} d\tau \right)^{\frac{(1-\sigma)r_i}{\sigma}} dt \right]^{\sigma p} , \end{aligned}$$

where  $C_1$  is defined by (2.2) .

If we have a decreasing weight function, then we need the assumption  $r \geq 1$ .

**Theorem 3.2.** *Suppose that the assumptions of Theorem 3.1 hold. Suppose also that  $r \geq 1$  and  $w$  is a decreasing function. Then*

$$\begin{aligned} & \int_a^x w(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p |D_{a+}^\alpha f(t)|^q dt \\ & \leq C_2 [w(x)]^{(1-r)\sigma p} (x - a)^{(\rho+\sigma)p} \left( \int_a^x w(t) |D_{a+}^\alpha f(t)|^{p+q} dt \right)^{\frac{pr+q}{p+q}} , \end{aligned} \quad (3.1)$$

where  $C_2$  is defined by (2.6) .

*Proof.* Let  $q \neq 0$ ,  $\delta_i = \alpha - \beta_i - 1$ ,  $i = 1, \dots, N$ . Since  $w$  is decreasing, then

$$1 \leq \left[ \frac{w(\tau)}{w(t)} \right]^\sigma , \quad \tau \leq t . \quad (3.2)$$

Using composition identity (1.2), the triangle inequality and Hölder's inequality for  $\frac{1}{1-\sigma}$  and  $\frac{1}{\sigma}$ , for  $t \in [a, x]$  follows

$$\begin{aligned} \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} &\leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \prod_{i=1}^N \left[ \int_a^t (t - \tau)^{\delta_i} |D_{a+}^{\alpha} f(\tau)| d\tau \right]^{r_i p} \\ &\leq \frac{[w(t)]^{-\sigma r p}}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \prod_{i=1}^N \left[ \int_a^t (t - \tau)^{\delta_i} [w(\tau)]^{\sigma} |D_{a+}^{\alpha} f(\tau)| d\tau \right]^{r_i p} \\ &\leq \frac{[w(t)]^{-\sigma r p} (1 - \sigma)^{(1-\sigma) r p}}{\prod_{i=1}^N [\Gamma(\delta_i + 1) (\delta_i + 1 - \sigma)^{1-\sigma}]^{r_i p}} \\ &\quad \cdot (t - a)^{\sum_{i=1}^N (\delta_i + 1 - \sigma) r_i p} \left( \int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_a^x w(t) \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^{\alpha} f(t)|^q dt \\ \leq \prod_{i=1}^N \left[ \left( \frac{1 - \sigma}{\delta_i + 1 - \sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\delta_i + 1)} \right]^{r_i p} \end{aligned} \quad (3.3)$$

$$\begin{aligned} \cdot \int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t - a)^{\sum_{i=1}^N (\delta_i + 1 - \sigma) r_i p} \\ \cdot \left( \int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p} dt. \end{aligned} \quad (3.4)$$

Applying Hölder's inequality for  $\frac{1}{\rho p}$  and  $\frac{1}{\sigma q}$  with  $\rho p = \sum_{i=1}^N (\delta_i + 1 - \sigma) r_i p$ , we obtain

$$\begin{aligned} \int_a^x [w(t)]^{1-\sigma r p} |D_{a+}^{\alpha} f(t)|^q (t - a)^{\rho p} \left( \int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma r p} dt \\ \leq \left( \int_a^x (t - a)^{\frac{\rho}{\sigma}} dt \right)^{\sigma p} \\ \cdot \left( \int_a^x [w(t)]^{\frac{1-\sigma r p - \sigma q}{\sigma q}} w(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} \left( \int_a^t w(\tau) |D_{a+}^{\alpha} f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\frac{r p}{q}} dt \right)^{\sigma q} \\ \leq (x - a)^{(\rho + \sigma) p} \left( \frac{\sigma}{\rho + \sigma} \right)^{\sigma p} [w(x)]^{\frac{p(1-r)}{p+q}} \\ \cdot \left( \frac{q}{r p + q} \right)^{\sigma q} \left( \int_a^x w(t) |D_{a+}^{\alpha} f(t)|^{\frac{1}{\sigma}} dt \right)^{\sigma(r p + q)}. \end{aligned} \quad (3.5)$$

The inequality (3.1) now follows from (3.3) and (3.5).

For  $q = 0$ , we proceed the same as in the proof of Theorem 2.1.  $\square$

If  $r = 1$  then we have Alzer's inequality [2, Theorem 1] for the Canavati left-sided fractional derivatives:

**Corollary 3.3.** *Suppose that the assumptions of Theorem 3.2 hold and let  $r = 1$ . Then*

$$\begin{aligned} & \int_a^x w(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\ & \leq \tilde{C}_2 (x-a)^{\sum_{i=1}^N r_i(\alpha-\beta_i)p} \int_a^x w(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt, \end{aligned}$$

where

$$\tilde{C}_2 = \sigma q^{\sigma q} \left[ \sum_{i=1}^N r_i(\alpha - \beta_i) \right]^{-\sigma p} \prod_{i=1}^N \left[ \left( \frac{1 - \sigma}{\alpha - \beta_i - \sigma} \right)^{1-\sigma} \frac{1}{\Gamma(\alpha - \beta_i)} \right]^{r_i p}.$$

For the next theorem we suppose that weight function is bounded.

**Theorem 3.4.** *Suppose that the assumptions of Theorem 3.1 hold. Suppose also that  $r \geq 1$  and  $A \leq w(t) \leq B$  for  $t \in [a, x]$ . Then*

$$\begin{aligned} & \int_a^x w(t) \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\ & \leq C_2 \left( \frac{B}{A^r} \right)^{\sigma p} (x-a)^{(\rho+\sigma)p} \left( \int_a^x w(t) \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \quad (3.6)$$

where  $C_2$  is given by (2.6).

*Proof.* The proof of (3.6) is the same as the one for (3.1), except two changes. Instead of inequality (3.2) we use  $1 \leq (w(\tau)/A)^\sigma$ . Moreover, in (3.4) we apply the inequality  $w(t) = [w(t)]^{\sigma p} [w(t)]^{\sigma q} \leq B^{\sigma p} [w(t)]^{\sigma q}$ . These two changes lead to the inequality (3.6).  $\square$

#### 4. NON-WEIGHTED CASE

The last result is a non-weighted case of previous theorems. Here we also give a case with a best possible solution.

**Proposition 4.1.** *Let  $N \in \mathbb{N}$ ,  $\alpha > \beta_i \geq 0$ ,  $m = \min\{[\beta_i] + 1 : i = 1, \dots, N\}$  and  $n = [\alpha] + 1$ . Let  $f \in C_{a+}^{\alpha}[a, b]$  be such that  $f^{(i)}(a) = 0$  for  $i = m-1, \dots, n-2$ . Let  $r_i \geq 0$ ,  $r = \sum_{i=1}^N r_i > 0$ . Let  $p > 0$ ,  $q \geq 0$ ,  $\sigma = \frac{1}{p+q} < 1$ ,  $\rho = \sum_{i=1}^N r_i(\alpha - \beta_i) - r\sigma$  and let  $\alpha > \beta_i + \sigma$  for  $i = 1, \dots, N$ . Let also  $D_{a+}^{\alpha} f \in L_{p+q}[a, b]$ . Then*

$$\begin{aligned} & \int_a^x \left( \prod_{i=1}^N \left| D_{a+}^{\beta_i} f(t) \right|^{r_i} \right)^p \left| D_{a+}^{\alpha} f(t) \right|^q dt \\ & \leq C_2 (x-a)^{(\rho+\sigma)p} \left( \int_a^x \left| D_{a+}^{\alpha} f(t) \right|^{p+q} dt \right)^{\frac{pr+q}{p+q}}, \end{aligned} \quad (4.1)$$

where  $C_2$  is defined by (2.6).

Inequality (4.1) is sharp if and only if  $\alpha = \beta_i + 1$ ,  $i = 1, \dots, N$  and  $q = 1$ . The equality in this case is attained for a function  $f$  such that  $D_{a+}^\alpha f(t) = 1$ ,  $t \in [a, x]$ .

*Proof.* Let  $q \neq 0$ ,  $\delta_i = \alpha - \beta_i - 1$ ,  $i = 1, \dots, N$ . As in the proof of Theorem 2.1, using composition identity (1.2), the triangle inequality and Hölder's inequality for  $\frac{1}{1-\sigma}$  and  $\frac{1}{\sigma}$ , for  $t \in [a, x]$  follows

$$\begin{aligned} & \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} \\ & \leq \frac{1}{\prod_{i=1}^N [\Gamma(\delta_i + 1)]^{r_i p}} \prod_{i=1}^N \left[ \left( \int_a^t (t - \tau)^{\frac{\delta_i}{1-\sigma}} d\tau \right)^{1-\sigma} \left( \int_a^t |D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^\sigma \right]^{r_i p} \quad (4.2) \\ & = \frac{(1 - \sigma)^{(1-\sigma)rp} (t - a)^{\sum_{i=1}^N (\delta_i + 1 - \sigma)r_i p}}{\prod_{i=1}^N [\Gamma(\delta_i + 1) (\delta_i + 1 - \sigma)^{1-\sigma}]^{r_i p}} \left( \int_a^t |D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma rp}. \end{aligned}$$

Now we have

$$\begin{aligned} & \int_a^x \prod_{i=1}^N |D_{a+}^{\beta_i} f(t)|^{r_i p} |D_{a+}^\alpha f(t)|^q dt \\ & \leq \frac{(1 - \sigma)^{(1-\sigma)rp}}{\prod_{i=1}^N [\Gamma(\delta_i + 1) (\delta_i + 1 - \sigma)^{1-\sigma}]^{r_i p}} \\ & \quad \cdot \int_a^x (t - a)^{\sum_{i=1}^N (\delta_i + 1 - \sigma)r_i p} \left( \int_a^t |D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} d\tau \right)^{\sigma rp} |D_{a+}^\alpha f(t)|^q dt. \end{aligned}$$

By Hölder's inequality for  $\frac{1}{\sigma p}$  and  $\frac{1}{\sigma q}$  and a simple integration inequality (4.1) follows.

For  $q = 0$ , we proceed as in the proof of Theorem 2.1.

Using the equality condition in Hölder's inequality we have equality in (4.2) if and only if  $|D_{a+}^\alpha f(\tau)|^{\frac{1}{\sigma}} = \lambda(t - \tau)^{\frac{\delta_i}{1-\sigma}}$ ,  $i = 1, \dots, N$ , which implies (since  $D_{a+}^\alpha f(\tau)$  depends only on  $\tau$ ) that  $\delta_i = 0$ , that is  $\alpha = \beta_i + 1$  for  $i = 1, \dots, N$ . Due to homogeneous property of inequality (4.1) we can take  $D_{a+}^\alpha f(\tau) = 1$ , which gives  $D_{a+}^{\beta_i} f(\tau) = D_{a+}^{\alpha-1}(\tau) = \tau - a$ ,  $i = 1, \dots, N$ . Substituting this in equality (4.1) for the left side we get

$$\int_a^x \prod_{i=1}^N (t - a)^{r_i p} dt = \int_a^x (t - a)^{rp} dt = \frac{(x - a)^{rp+1}}{rp + 1}.$$

For the right side, with  $\rho = r - r\sigma$ , follows

$$C_2 (x - a)^{(\rho+\sigma)p} \left( \int_a^x dt \right)^{\frac{pr+q}{p+q}} = \frac{\sigma^{\sigma p} q^{\sigma q}}{(r - r\sigma + \sigma)^{\sigma p} (rp + q)^{\sigma q}} (x - a)^{rp+1}.$$

Hence,

$$\frac{1}{rp+1} = \frac{q^{\frac{q}{p+q}}}{[r(p+q) - r + 1]^{\frac{p}{p+q}} (rp+q)^{\frac{q}{p+q}}}$$

which is equivalent with

$$[r(p+q) - r + 1]^p [rp+q]^q = q^q (rp+1)^{p+q}. \tag{4.3}$$

For  $q = 1$  equality (4.3) obviously holds. For  $q = 0$  equality (4.3) implies  $r = 0$ , which gives trivial identity in (4.1). By simple rearrangements equation (4.3) becomes

$$\left[1 + r \frac{q-1}{rp+1}\right]^p \left[1 + r \frac{p}{q} \frac{1-q}{rp+1}\right]^q = 1. \tag{4.4}$$

For  $p = q$  the left-hand side of equation (4.4) is equal to  $\left[1 - \left(r \frac{1-p}{rp+1}\right)^2\right]^p$ , which is strictly less than 1, except in trivial cases. For  $0 < p < q$ ,  $q \neq 1$ ,  $r > 0$ , using the Bernoulli inequality, we have

$$\left[1 + r \frac{q-1}{rp+1}\right]^{\frac{p}{q}} \left[1 + r \frac{p}{q} \frac{1-q}{rp+1}\right] < \left[1 + r \frac{p}{q} \frac{q-1}{rp+1}\right] \left[1 + r \frac{p}{q} \frac{1-q}{rp+1}\right],$$

which is obviously strictly less than 1. For  $0 < q < p$ ,  $q \neq 1$ ,  $r > 0$ , using the Bernoulli inequality, we have

$$\left[1 + r \frac{q-1}{rp+1}\right] \left[1 + r \frac{p}{q} \frac{1-q}{rp+1}\right]^{\frac{p}{q}} < \left[1 + r \frac{q-1}{rp+1}\right] \left[1 + r \frac{1-q}{rp+1}\right],$$

which is again obviously strictly less than 1. It follows that (4.3) holds if and only if  $q = 1$ . □

*Remark 4.2.* Let  $N = 1$ ,  $\alpha = 1$ ,  $\beta_1 = 0$ ,  $r_1 = r = 1$ ,  $p = q = 1$ ,  $a = 0$  and  $x = h$ . Then inequality (4.1) becomes Beesack's inequality [4]

$$\int_0^h |f(t) f'(t)| dt \leq \frac{h}{2} \int_0^h [f'(t)]^2 dt. \tag{4.5}$$

He proved that inequality (4.5) is valid for any function  $f$  absolutely continuous on  $[0, h]$  satisfying single boundary condition  $f(0) = 0$ .

In order to get classical Opial's inequality (1.1) we need right-sided version of inequality (4.1) for  $N = 1$ ,  $r_1 = r = 1$  and  $p = q = 1$ :

$$\int_x^b |D_{b-}^{\beta_1} f(t)| |D_{b-}^{\alpha} f(t)| dt \leq C_2 (b-x) \int_x^b |D_{b-}^{\alpha} f(t)|^2 dt \tag{4.6}$$

satisfying  $f(b) = 0$ . Observe the inequality

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} |D_{a+}^{\beta_1} f(t)| |D_{a+}^{\alpha} f(t)| dt + \int_{\frac{a+b}{2}}^b |D_{b-}^{\beta_1} f(t)| |D_{b-}^{\alpha} f(t)| dt \\ & \leq C_2 \left(\frac{b-a}{2}\right) \left( \int_a^{\frac{a+b}{2}} |D_{a+}^{\alpha} f(t)|^2 dt + \int_{\frac{a+b}{2}}^b |D_{b-}^{\alpha} f(t)|^2 dt \right). \end{aligned} \tag{4.7}$$

If we put  $\alpha = 1$ ,  $\beta_1 = 0$ ,  $a = 0$  and  $b = h$ , then inequality (4.7) becomes Opial's inequality (1.1) (having boundary conditions  $f(0) = f(h) = 0$ ).

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<sup>1</sup> FACULTY OF CIVIL ENGINEERING, ARCHITECTURE AND GEODESY, UNIVERSITY OF SPLIT, SPLIT, CROATIA.

*E-mail address:* [maja.andric@gradst.hr](mailto:maja.andric@gradst.hr)

<sup>2</sup> FACULTY OF TEXTILE TECHNOLOGY, UNIVERSITY OF ZAGREB, ZAGREB, CROATIA.

*E-mail address:* [pecaric@element.hr](mailto:pecaric@element.hr)

<sup>3</sup> FACULTY OF FOOD TECHNOLOGY AND BIOTECHNOLOGY, UNIVERSITY OF ZAGREB, ZAGREB, CROATIA.

*E-mail address:* [iperic@pbf.hr](mailto:iperic@pbf.hr)