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A CHARACTERIZATION OF THE INNER PRODUCT SPACES INVOLVING TRIGONOMETRY

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ABSTRACT. In this paper we will give a new characterization of the inner product space which use the trigonometry. We conclude that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if there exists $\alpha \in \mathbb{R} \backslash \pi\mathbb{Q}$ so that

$$\|x \cos\alpha + y \sin\alpha\|^2 + \|y \cos\alpha - x \sin\alpha\|^2 = \|x\|^2 + \|y\|^2 ,$$

for any $x, y \in X$.

1. INTRODUCTION

The problem of finding some necessary and sufficient geometric conditions for a normed space to be an inner product space has been investigated by several mathematicians for a long time. Some characterizations of inner product spaces and their generalizations can be found in [1, 2, 4] and references therein.

In [3], Moslehian and Rassias gave a characterization of the inner product space using an Euler-Lagrange type identity. The result is presented in the next proposition.

Proposition 1.1. *Let be $(X, \|\cdot\|)$ a normed space. Then the norm is derived from an inner product space if and only if*

$$\|ax + by\|^2 + \|bx - ay\|^2 = \left(a^2 + b^2\right)\left(\|x\|^2 + \|y\|^2\right),$$

for any $x, y \in X$ and $a, b > 0$.

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Starting from this result, we will obtain another characterization for the inner product space involving the trigonometry.

2. Some Preliminary Results

The identity from previous proposition could be transformed. After we divide with $a^2 + b^2$, we obtain

$$\left\| \frac{a}{\sqrt{a^2+b^2}} x + \frac{b}{\sqrt{a^2+b^2}} y \right\|^2 + \left\| \frac{b}{\sqrt{a^2+b^2}} x - \frac{a}{\sqrt{a^2+b^2}} y \right\|^2 = \|x\|^2 + \|y\|^2 .$$

But, it easy to see that it exists a real number $\alpha \in \left(0, \frac{\pi}{2}\right)$ for which $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$. In this context, our identity becomes

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|x\sin\alpha - y\cos\alpha\|^2 = \|x\|^2 + \|y\|^2 .$$

In fact, the result is more general and it is exposed and proved in the next proposition.

Proposition 2.1. *Let be $(X, \|\cdot\|)$ a normed space. Then the norm is derived from an inner product space if and only if*

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2 ,$$

for any $x, y \in X$ and $\alpha \in \mathbb{R}$.

Proof. If X is an inner product spaces, we have

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2$$

$$= \langle x\cos\alpha + y\sin\alpha,\ x\cos\alpha + y\sin\alpha\rangle + \langle y\cos\alpha - x\sin\alpha,\ y\cos\alpha - x\sin\alpha\rangle$$

$$= \|x\cos\alpha\|^2 + \langle x\cos\alpha,\ y\sin\alpha\rangle + \langle y\sin\alpha,\ x\cos\alpha\rangle + \|y\sin\alpha\|^2 +$$

$$+ \|y\cos\alpha\|^2 - \langle x\sin\alpha,\ y\cos\alpha\rangle - \langle y\cos\alpha,\ x\sin\alpha\rangle + \|x\sin\alpha\|^2$$

$$= \left(\sin^2\alpha + \cos^2\alpha\right)\|x\|^2 + \left(\sin^2\alpha + \cos^2\alpha\right)\|y\|^2$$

$$+ \left(\langle x, y\rangle + \langle y, x\rangle - \langle x, y\rangle - \langle y, x\rangle\right)\sin\alpha\cos\alpha$$

$$= \|x\|^2 + \|y\|^2 .$$

For the second part of the proof, we choose $\alpha = \frac{\pi}{4}$ and identity

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|x\sin\alpha - y\cos\alpha\|^2 = \|x\|^2 + \|y\|^2$$

becomes

$$\frac{\|x+y\|^2}{2} + \frac{\|x-y\|^2}{2} = \|x\|^2 + \|y\|^2 ,$$

which conclude our proof . $\square$

But, our scope is to improve this result. For this we will remind two classical result from the mathematical analysis. First, we denote $C(O,1)$ the unit circle from $\mathbb{R}^2$, and $\pi\mathbb{Q} = \{\pi k | k \in \mathbb{Q}\}$.

Lemma 2.2. *For any $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$, the set $\{(\cos n\alpha, \sin n\alpha) \mid n \in \mathbb{N}\}$ is dense in $C(O,1)$.*

Lemma 2.3. *Let be $(X, \|\cdot\|)$ a normed space. The the function $f : X \to \mathbb{R}, \quad f(x) = \|x\|$ is continuous.*

Now we can present and prove the main result of our paper.

3. The Main Result

Using the lemmas reminded in previous paragraph, now we can improve the results from Proposition 2.1. in the next form:

Theorem 3.1. *Let be $(X, \|\cdot\|)$ a normed space. Then the norm is derived from an inner product space if and only if it exists $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ so that*

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

for any $x, y \in X$.

Proof. The only if part of the proof is true for any $\alpha \in \mathbb{R}$ how we seen in Proposition 2.1., so particullary for an $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$. For the if part, we consider that there exists $\alpha \in \mathbb{R}\backslash\pi\mathbb{Q}$ with

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

for any $x, y \in X$. We replace x with $x\cos\alpha + y\sin\alpha$ and y with $y\cos\alpha - x\sin\alpha$. We obtain

$$\|x\cos\alpha + y\sin\alpha\|^2 + \|x\sin\alpha - y\cos\alpha\|^2 =$$

$$= \|(x\cos\alpha + y\sin\alpha)\cos\alpha + (y\cos\alpha - x\sin\alpha)\sin\alpha\|^2 +$$

$$+ \|(y\cos\alpha - x\sin\alpha)\cos\alpha - (x\cos\alpha + y\sin\alpha)\sin\alpha\|^2$$

$$= \left\|x\left(\cos^2\alpha - \sin^2\alpha\right) + y\cdot 2\sin\alpha\cos\alpha\right\|^2 + \left\|y\left(\cos^2\alpha - \sin^2\alpha\right) - x\cdot 2\sin\alpha\cos\alpha\right\|^2$$

$$= \|x\cos 2\alpha + y\sin 2\alpha\|^2 + \|y\cos 2\alpha - x\sin 2\alpha\|^2.$$

So, we obtain the identity

$$\|x\cos 2\alpha + y\sin 2\alpha\|^2 + \|y\cos\alpha - x\sin\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

for all $x, y \in X$. Further, we will use the mathematical induction and we suppose that the identity

$$\|x\cos k\alpha + y\sin k\alpha\|^2 + \|y\cos k\alpha - x\sin k\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

is true for some $k \in \mathbb{N}$ and we prove it for $k+1$. In the initial identity, we replace x with $x\cos k\alpha + y\sin k\alpha$ and y with $y\cos k\alpha - x\sin k\alpha$ and we have

$$\|x\cos k\alpha + y\sin k\alpha\|^2 + \|y\cos k\alpha - x\sin k\alpha\|^2 =$$

$$= \|(x\cos k\alpha + y\sin k\alpha)\cos\alpha + (y\cos k\alpha - x\sin k\alpha)\sin\alpha\|^2 +$$

$$+ \|(y\cos k\alpha - x\sin k\alpha)\cos\alpha - (x\cos k\alpha + y\sin k\alpha)\sin\alpha\|^2$$

$$= \|x(\cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha) + y(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha)\|^2 +$$

$$+ \|y(\cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha) - x(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha)\|^2$$

$$= \|x\cos(k+1)\alpha + y\sin(k+1)\alpha\|^2 + \|y\cos(k+1)\alpha - x\sin(k+1)\alpha\|^2.$$

So, we have that the identity

$$\|x\cos n\alpha + y\sin n\alpha\|^2 + \|y\cos n\alpha - x\sin n\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

is true for all $n \in \mathbb{N}^*$.

Now, we apply Lemma 2.2. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers for which

$$\lim_{n\to\infty} a_n = \infty$$

and

$$\lim_{n\to\infty} (\cos a_n\alpha,\ \sin a_n\alpha) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right).$$

Then, we obtain

$$\|x\cos a_n\alpha + y\sin a_n\alpha\|^2 + \|y\cos a_n\alpha - x\sin a_n\alpha\|^2 = \|x\|^2 + \|y\|^2.$$

If we make $n \to \infty$ and use Lemma 2.3., we have

$$\frac{\|x+y\|^2}{2} + \frac{\|x-y\|^2}{2} = \|x\|^2 + \|y\|^2,$$

for all $x, y \in X$ and our proof is ready. $\square$

Final remark: Obviously, the set $\mathbb{R}\backslash\pi\mathbb{Q}$ can be enlarged. For example, if $\alpha \in \left\{\frac{2k+1}{4n}\pi : k \in \mathbb{Z}, n \in \mathbb{N}\right\}$, then $(\sin n\alpha, \cos n\alpha) = \left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and from the identity

$$\|x\cos n\alpha + y\sin n\alpha\|^2 + \|y\cos n\alpha - x\sin n\alpha\|^2 = \|x\|^2 + \|y\|^2,$$

we get the parallelogram identity. The question is: what is the biggest set A, such that

$$\mathbb{R}\backslash\pi\mathbb{Q} \subset A \subset \mathbb{R}$$

and for which the theorem holds true.

References

1. C. Alsina, J. Sikorska and M.S. Tomás, *Norm derivatives and characterizations of inner product spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.
2. D. Amir, *Characterizations of inner product spaces*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 1986.
3. M.S. Moslehian and J.M. Rassias, *A characterization of inner product spaces concerning an Euler-Lagrange identity*, Commun. Math. Anal. **8** (2010), no. 2, 16–21.
4. K. Nikodem and Z. Páles, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (2011), no. 1, 83–87.

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