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FUGLEDE-PUTNAM THEOREM FOR w-HYPONORMAL OR CLASS $\mathcal{Y}$ OPERATORS

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ABSTRACT. An asymmetric Fuglede-Putnam's Theorem for $w-$hyponormal operators and class $\mathcal{Y}$ operators is proved, as a consequence of this result, we obtain that the range of the generalized derivation induced by the above classes of operators is orthogonal to its kernel.

1. INTRODUCTION

For complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $B(\mathcal{H}), B(\mathcal{K})$ and $B(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators on $\mathcal{H}$, the set of all bounded linear operators on $\mathcal{K}$ and the set of all bounded linear transformations from $\mathcal{H}$ to $\mathcal{K}$ respectively. A bounded operator $A \in B(\mathcal{H})$ is called normal if $A^*A = AA^*$. An operator $A \in B(\mathcal{H})$ is said to be a class $\mathcal{Y}_\alpha$ for $\alpha \leq 1$ if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A-\lambda)^*(A-\lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigcup_{1\leq\alpha} \mathcal{Y}_\alpha$. (see [7])

Also A is called $p-$hyponormal [1, 8, 9, 20], if $(A^*A)^p \geq (AA^*)^p$ for some $0 < p \leq 1$, semi-hyponormal if $p = 1/2$, $\log-$hyponormal [18] if A is invertible operator and satisfies $\log(A^*A) \geq \log(AA^*)$, and $w-$hyponormal if $|\widetilde{A}| \geq |A| \geq |(\widetilde{A})^*|$, where $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ is the Aluthge transformation. It was shown in [2] and [3] that the class of $w-$hyponormal operators contains both the $p-$hyponormal and $\log-$hyponormal operators. We have the following inclusion

$$\{Normal\} \subset \{Hyponormal\} \subset \{p - Hyponormal\} \subset \{w - Hyponormal\}.$$

$$\{invertible - hyponormal\} \subset \{invertible - p - hyponrmal\}$$

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$$\subset \{\log - hyponormal\} \subset \{w - hyponormal\}.$$

If an operator A is $p-$hyponormal, then $\ker A \subset \ker A^*$, and if A is $\log-$hyponormal, then $\ker A = \ker A^*$. However, if A is $w-$hyponormal, the kernel condition $\ker A \subset \ker A^*$ does not necessarily hold. Nevertheless in ([2, 3]) $w-$hyponormal operators have many properties similar to those of $p-$hyponormal operators.

The familiar Fuglede-Putnam's theorem asserts that if $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are normal operators and $AX = XB$ for some operators $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$ ([12], [17]). Many authors have extended this theorem for several classes of operators, recently S. Mecheri, K. Tanahashi and A. Uchiyama [15] proved that Fuglede-Putnam's theorem holds for $p-$hyponormal or class $\mathcal{Y}$ operators, B. P. Duggal [10] and I. H. Jeon, K. Tanahashi and A. Uchiyama [14] proved that Fuglede-Putnam's theorem holds for $p-$hyponormal or $\log-$hyponormal. We say that the pair (A, B) satisfy Fuglede-Putnam's theorem if $AX = XB$ implies $A^*X = XB^*$.
Our aim is to extend the Fuglede-Putnam theorem [12], we prove that if either

(1) A is class $\mathcal{Y}$ and B^* is $w-$hyponormal such that $\ker B^* \subset \ker B$ or
(2) A is $w-$hyponormal such that $\ker A^* \subset \ker A$ and B^* is class $\mathcal{Y}$,

then the pair (A, B) satisfy Fuglede-Putnam's theorem. At the end of this paper we study the orthogonality of the range and the null space of the generalized derivation for some classes of operators.

Let $A, B \in L(\mathcal{H})$, we define the generalized derivation $\delta_{A,B}$ induced by A and B by

$$\delta_{A,B}(X) = AX - XB, \text{ for all } X \in B(\mathcal{H}).$$

Definition 1.1. [4] Given subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Banach space $\mathcal{V}$ with norm $\|\cdot\|$. $\mathcal{M}$ is said to be orthogonal to $\mathcal{N}$ if $\|m + n\| \geq \|n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$.

J.H. Anderson and C. Foias [4] proved that if A and B are normal, S is an operator such that $AS = SB$, then

$$\| \delta_{A,B}(X) - S \| \geq \| S \|, \text{ for all } X \in B(\mathcal{H}).$$

Where $\|\cdot\|$ is the usual operator norm. Hence the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. The orthogonality here is understood to be in the sense of definition [4].

2. Preliminaries

We will recall some known results which will be used in the sequel.

Definition 2.1. [1] Let $A \in B(\mathcal{H})$ and $A = U|A|$ be the polar decomposition of A, the Aluthge transformation of A is $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$.

Theorem 2.2. [13] An operator $A \in B(\mathcal{H})$ is $w-$hyponormal if and only if

$$(|A^*|^{\frac{1}{2}}|A||A^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |A^*|.$$

Lemma 2.3. [20] Let $A \in B(\mathcal{H})$ be $p-$hyponormal operator and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A, then the restriction of A to $\mathcal{M}$ is $p-$hyponormal.

Lemma 2.4. [21] Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A, then the restriction of A to $\mathcal{M}$ is class $\mathcal{Y}$.

Lemma 2.5. [21] Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A. If $A|_M$ is normal, then $\mathcal{M}$ reduces A.

Lemma 2.6. ([6], [16]) Let $A \in B(\mathcal{H})$ be $w-$hyponormal and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace for A, then the restriction of A to $\mathcal{M}$ is $w-$hyponormal.

Lemma 2.7. [19] Let $A \in B(\mathcal{H})$ be $w-$hyponormal operator, then its Aluthge transform

$$\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$$

is semi-hyponormal.

Theorem 2.8. [15] Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$. If either (1) A is $p-$hyponormal and B^* is a class $\mathcal{Y}$ or (2) A is a class $\mathcal{Y}$ operator and B^* is $p-$ hyponormal, then $AX = XB$ for some operator $X \in B(\mathcal{K},\mathcal{H})$ implies $A^*X = XB^*$. Moreover, $\overline{R(X)}$ reduces A, $\ker(X)^\perp$ reduces B, and $A\mid_{\overline{R(X)}}$, $B\mid_{(\ker X)^\perp}$ are unitarily equivalent normal operators.

Theorem 2.9. [18] Let $A \in B(\mathcal{H})$ and $B^* \in B(\mathcal{K})$. Then the following assertions are equivalent

(1) The pair (A, B) satisfy Fuglede-Putnam's theorem.
(2) If $AX = XB$ for some $X \in B(\mathcal{K},\mathcal{H})$, then $\overline{R(X)}$ reduces A, $\ker(X)^\perp$ reduces B, and $A\mid_{\overline{R(X)}}$, $B\mid_{(\ker X)^\perp}$ are normal operators.

Definition 2.10. We say that $A \in B(\mathcal{H})$ has the single valued extension property at λ (SVEP for short) if for every neighborhood U of λ, the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation $(A-\lambda)f(\lambda) = 0$, for all $\lambda \in U$ is the function $f \equiv 0$. We say that $A \in B(\mathcal{H})$ satisfies the SVEP property if A has the single valued extension property at every $\lambda \in \mathbb{C}$.

Remark 2.11.

(1) If well known that if $N \in B(\mathcal{H})$ is normal, then N has SVEP.
(2) If $A \in B(\mathcal{H})$ and $\sigma_p(A) = \emptyset$, then A has SVEP, where $\sigma_p(A)$ is the set of all eigenvalues of A.

3. Main results

Our goal is to investigate the orthogonality of $R(\delta_{A,B})$ (the range of $\delta_{A,B}$) and $\ker(\delta_{A,B})$ (the kernel of $\delta_{A,B}$) for some operators. We prove that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$ when either (1) A is a class $\mathcal{Y}$ and B^* is $w-$hyponormal such that $\ker B^* \subset \ker B$ or (2) A is $w-$hyponormal such that $\ker A \subset \ker A$ and B^* is a class $\mathcal{Y}$. Before proving these results, we need the following ones.

Lemma 3.1. If $A \in B(\mathcal{H})$ is semi-hyponormal, then A has SVEP.

Proof. Applying the properties of semi-hyponormal operators [22] and lemma 2.3, we can write A as $A = N \oplus A_0$ where N is normal and A_0 is a pure semi-hyponormal operator, i.e., $\sigma_p(A_0) = \emptyset$. $\square$

Theorem 3.2. Let $A \in B(\mathcal{H})$ be a class $\mathcal{Y}$ and $B^* \in B(\mathcal{K})$ be $w-$hyponormal such that $\ker B^* \subset \ker B$. If $AX = XB$ for some $X \in B(\mathcal{H}, \mathcal{K})$, then $A^*X = XB^*$.

Proof. **Case 1.** If B^* is injective. Assume that $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$.
Since $\overline{R(X)}$ is invariant by A and $(\ker X)^\perp$ is invariant by B^*, we consider the following decompositions:

$$\mathcal{H} = \overline{R(X)} \oplus (R(X))^\perp, \quad \mathcal{K} = (\ker X)^\perp \oplus (\ker X),$$

then it yields

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & B_3 \end{pmatrix}$$

and

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} : \quad (\ker X)^\perp \oplus (\ker X) \longrightarrow \overline{R(X)} \oplus (R(X))^\perp.$$

From $AX = XB$ we get

$$A_1X_1 = X_1B_1 \tag{3.1}$$

Let $B_1^* = U^*|B_1^*|$ be the polar decomposition of B_1^*. Multiply the both members of (3.1) by $|B_1^*|^{1/2}$, we obtain

$$A_1X_1|B_1^*|^{1/2} = X_1B_1|B_1^*|^{1/2},$$

hence

$$A_1X_1|B_1^*|^{1/2} = X_1|B_1^*|^{1/2}(\widetilde{B_1^*})^*$$

Since A_1 is class $\mathcal{Y}$ by Lemma 2.4 and B_1^* is $w-$hyponormal by Lemma 2.6, then $(\widetilde{B_1^*})^*$ is semi-hyponormal. Applying Theorem 2.8(2) we get the pair $(A_1, \widetilde{B_1^*})$ satisfies the Fuglede-Putnam's theorem. Therefore $A_1|_{R(X_1|B_1^*|^{1/2})}$ and $\widetilde{B_1^*}|_{(\ker(X_1|B_1^*|^{1/2})^\perp}$ are normal operators.
Since X_1 is injective with dense range and $|B_1^*|^{1/2}$ is injective, then

$$\overline{R(X_1|B_1^*|^{1/2})} = \overline{R(X_1)} = \overline{R(X)},$$

and

$$\ker(X_1|B_1^*|^{1/2}) = \{0\}.$$

It follows that $\widetilde{B_1^*}$ is normal and $(\ker X)^\perp$ reduces B^*. Therefore $\overline{R(X)}$ reduces A and $(\ker X)^\perp$ reduces B. Thus, $A_2 = B_2 = 0$. Since $A_1X_1 = X_1B_1$ are normal operators, then $A_1^*X_1 = X_1B_1^*$. Consequently $A^*X = XB^*$.

Case 2. If B^* is not injective, the condition $\ker B^* \subset \ker B$ implies that $\ker B^*$ reduces B^*, since $\ker A$ reduces A, the operators A and B can be written on the following decompositions

$$\mathcal{H} = (\ker A)^\perp \oplus \ker A, \ \ \mathcal{K} = (\ker B^*)^\perp \oplus \ker B^*,$$

as follows

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since A_1 is injective class $\mathcal{Y}$ operator and B_1^* is injective $w-$hyponormal operator. Let

$$X : (\ker B^*)^\perp \oplus \ker B^* \to (\ker A)^\perp \oplus \ker A,$$

and let $X = [X_{ij}]_{i,j=1}^2$ be the matrix representation, then $AX = XB$ implies that $A_1X_{11} = X_{11}B_1$ and $X_{12} = 0, X_{21} = 0$. From case 1, we deduce that $A_1^*X_{11} = X_{11}B_1^*$. Thus $A^*X = XB^*$. □

Theorem 3.3. Let $A \in B(\mathcal{H})$ be an injective $w-$hyponormal operator and $B^* \in B(\mathcal{K})$ be a class $\mathcal{Y}$. If $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.

Proof. Since B^* is of class $\mathcal{Y}$, there exist positive numbers α and k_α^2 such that

$$|BB^* - B^*B|^\alpha \leq k_\alpha^2 (B - \lambda)(B - \lambda)^* \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Hence for all $v \in |BB^* - B^*B|^{\alpha/2}\mathcal{K}$ there exists a bounded function $f : \mathbb{C} \longrightarrow \mathcal{K}$ such that

$$(B - \lambda)f(\lambda) = v \quad \text{for all} \quad \lambda \in \mathbb{C}$$

by [10]. Let $A = U|A|$ be the polar decomposition of A and defines its Aluthge transform by $\widetilde{A} = |A|^{1/2}U|A|^{1/2}$. Then $\widetilde{A}$ is semi-hyponormal by [2] and so

$$\begin{aligned} (\widetilde{A} - \lambda)|A|^{1/2}Xf(\lambda) &= |A|^{1/2}(A - \lambda)Xf(\lambda) \\ &= |A|^{1/2}X(B - \lambda)f(\lambda) \\ &= |A|^{1/2}Xv, \quad \text{for all} \quad \lambda \in \mathbb{C}. \end{aligned}$$

We assert $|A|^{1/2}Xv = 0$. Because if $|A|^{1/2}Xv \neq 0$, there exists an analytic function $\psi : \mathbb{C} \to \mathcal{H}$ such that $(\widetilde{A} - \lambda)\psi(\lambda) = |A|^{1/2}Xv$ by lemma 3.1. Since

$$\psi(\lambda) = (\widetilde{A} - \lambda)|A|^{1/2}Xv \to 0 \quad \text{as} \quad \lambda \to \infty,$$

we have $\psi(\lambda) = 0$ and hence $|A|^{1/2}Xv = 0$. This is a contradiction.
Then

$$|A|^{1/2}X|BB^* - B^*B|^{\alpha/2}\mathcal{K} = \{0\}.$$

Since $\ker A = \ker |A| = \{0\}$, we have

$$X(BB^* - B^*B) = 0.$$

Since $\overline{R(X)}$ is invariant under A and $(\ker X)^\perp$ is invariant under B^*, we can write

$$\begin{aligned} A &= \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{R(X)} \oplus (R(X))^\perp, \\ B &= \begin{pmatrix} B_1 & 0 \\ B_3 & B_2 \end{pmatrix} \quad \text{on} \quad \mathcal{K} = (\ker X)^\perp \oplus (\ker X), \\ X &= \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \quad (\ker X)^\perp \oplus (\ker X) \to \overline{R(X)} \oplus (R(X))^\perp. \end{aligned}$$

Then

$$\begin{array}{rcl} 0 & = & X(BB^* - B^*B) \\ & = & \begin{pmatrix} X_1(B_1B_1^* - B_1^*B_1 - B_3^*B_3) & X_1(B_1B_3^* - B_3^*B_2) \\ 0 & 0 \end{pmatrix} \end{array}$$

and

$$X_1(B_1B_1^* - B_1^*B_1 - B_3^*B_3) = 0$$

Since X_1 is injective and has dense range,

$$B_1B_1^* - B_1^*B_1 - B_3^*B_3 = 0$$

and

$$B_1B_1^* = B_1^*B_1 + B_3^*B_3 \geq B_1^*B_1.$$

This implies B_1^* is hyponormal. Since $AX = XB$ we have

$$A_1X_1 = X_1B_1$$

where A_1 is $w-$hyponormal by [6]. Hence A_1 and B_1 are normal and

$$A_1^*X_1 = X_1B_1^*$$

by [11]. Then $A_3 = 0$ by [6] and $B_3 = 0$ by Lemma 2.5. Hence

$$A^*X = \begin{pmatrix} A_1^*X_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1B_1^* & 0 \\ 0 & 0 \end{pmatrix} = XB^*.$$

□

Theorem 3.4. Let $A \in B(\mathcal{H})$ be $w-$hyponormal operator such that $\ker A \subseteq \ker A^*$ and $B^* \in B(\mathcal{K})$ be a class $\mathcal{Y}$. If $AX = XB$ for some $X \in B(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$.

Proof. Decompose A into normal part A_1 and pure part A_2 as

$$A = A_1 \oplus A_2 \ \text{ on } \ \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

and let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} : \mathcal{K} \longrightarrow \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Since $\ker A_2 \subseteq \ker A_2^*$ and A_2 is pure, A_2 is injective. $AX = XB$ implies

$$\begin{pmatrix} A_1X_1 \\ A_2X_2 \end{pmatrix} = \begin{pmatrix} X_1B \\ X_2B \end{pmatrix}.$$

Hence

$$A^*X = \begin{pmatrix} A_1^*X_1 \\ A_2^*X_2 \end{pmatrix} = \begin{pmatrix} X_1B^* \\ X_2B^* \end{pmatrix} = XB^*$$

by applying theorem 3.3. □

Theorem 3.5. Let $A, B \in B(\mathcal{H})$. If one of the following assertions

(1) A is a class $\mathcal{Y}$ and B^* is w-hyponormal such that $\ker B^* \subset \ker B$.
(2) A is $w-$hyponormal such that $\ker A \subset \ker A^*$ and B^* is a class $\mathcal{Y}$.

holds, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

Proof. The pair (A, B) verify the Fuglede-Putman's theorem by Theorem 2.9 and Theorem 3.4 respectively. Let $C \in B(\mathcal{H})$ be such that $AC = CB$. According to the following decompositions of $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_1 = \overline{R(C)} \oplus \overline{R(C)}^{\perp},\ \mathcal{H} = \mathcal{H}_2 = (\ker C)^{\perp} \oplus \ker C,$$

We can write A, B, C and X

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},\ C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix},\ X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

Where A_1 and B_1 are normal operators and X is an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. Since $AC = CB$, then $A_1C_1 = C_1B_1$. Hence

$$AX - XB - C = \begin{pmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{pmatrix}.$$

Since $C_1 \in \ker(\delta_{A_1,B_1})$ and $A_1,\ B_1$ are normal, it follows by [4]

$$\| AX - XB - C \| \geq \| A_1X_1 - X_1B_1 - C_1 \| \geq \| C_1 \| = \| C \|,\ \forall X \in L(\mathcal{H}).$$

This implies that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

□

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