Ann. Funct. Anal. 4 (2013), no. 1, 40–52
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

VARIATIONS OF WEYL TYPE THEOREMS

M. H. M. RASHID[1]* AND T. PRASAD[2]

Communicated by J. Koliha

ABSTRACT. A Banach space operator T satisfies property(Bgw), a variant property(gw), if the complement in the approximate point spectrum $\sigma_a(T)$ of the semi-B-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$ coincides with set of isolated eigenvalues of T of finite multiplicity $E^0(T)$. We also introduce properties (Bb), and property (Bgb) in connection with Weyl type theorems, which are analogous, respectively, to generalized Browder's theorem and property(gb). We obtain relation among these new properties.

1. INTRODUCTION AND PRELIMINARIES

Let $B(X)$ denote the algebra of all bounded linear operator T acting on a Banach space X. For $T \in B(X)$, let T^*, $\ker(T)$, $R(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote respectively the *adjoint*, the *null space*, the *range*, the *spectrum*, the *point spectrum* and the *approximate point spectrum* of T. Let $\mathbb{C}$ denote the set of *complex numbers*. Let us denote by $\alpha(T)$ the dimension of the kernel and by $\beta(T)$ the codimension of the range. Recall that the operator $T \in B(X)$ is said to be *upper semi-Fredholm*, $T \in SF_+(X)$, if the range of $T \in B(X)$ is closed and $\alpha(T) < \infty$, while $T \in B(X)$ is said to be *lower semi-Fredholm*, $T \in SF_-(X)$, if $\beta(T) < \infty$. An operator $T \in B(X)$ is said to be *semi-Fredholm* if $T \in SF_+(X) \cup SF_-(X)$ and Fredholm if $T \in SF_+(X) \cap SF_-(X)$. If T is semi-Fredholm then the *index* of T is defined by $\mathrm{ind}\,(\mathrm{T}) = \alpha(\mathrm{T}) - \beta(\mathrm{T})$.

Let $a := a(T)$ be the *ascent* of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be the *descent* of an operator T ; i.e., the

Date: Received: 1 May 2012; Revised: 12 August 2012; Accepted: 8 September 2012.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47A11, 47A53.
Key words and phrases. Weyl's theorem, Property (w), Property (Bgw), Property (Bw), Property (Bgb), Property (Bb).

smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [21, Proposition 38.3]. Moreover, $0 < a(T - \lambda I) = d(T - \lambda I) < \infty$ precisely when λ is a pole of the resolvent of T , see Heuser [21, Proposition 50.2].

A bounded linear operator T acting on a Banach space X is *Weyl* if it is Fredholm of index zero and Browder if T is Fredholm of finite ascent and descent. The *Weyl spectrum* $\sigma_W(T)$ and *Browder spectrum* $\sigma_B(T)$ of T are defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$
$$\sigma_B(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

Let $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(\mathrm{T}) : 0 < \alpha(\mathrm{T} - \lambda) < \infty\}$ and let $\pi_0(T) := \sigma(T) \setminus \sigma_B(T)$ all *Riesz points* of T. According to Coburn [16], *Weyl's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_W(T) = E^0(T)$, and that *Browder's theorem* holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_W(T) = \pi^0(T)$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, iso A denotes the set of all isolated points of A and acc A denotes the set of all points of accumulation of A.

Let $SF_+^-(X) = \{T \in SF_+ : \operatorname{ind}(\mathrm{T}) \leq 0\}$. The *upper semi Weyl spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(X)\}$. According to Rakočević [23], an operator $T \in B(X)$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{T}) : 0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty\}.$$

It is known [23] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in\ B(X)$ and a non negative integer n define $T_{[n]}$ to be the restriction T to $R(T^n)$ viewed as a map from $R(T^n)$ to $R(T^n)$(in particular $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp., lower) semi-Fredholm operator, then T is called *upper* (resp., *lower*) *semi-B-Fredholm* operator. In this case index of T is defined as the index of semi-B-Fredholm operator $T_{[n]}$. A *semi-B-Fredholm operator* is an upper or lower semi-Fredholm operator [13]. Moreover, if $T_{[n]}$ is a Fredholm operator then T is called a *B-Fredholm* operator [7]. An operator T is called a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl operator } \}$ [9]. Let $E(T)$ be the set of all eigenvalues of T which are isolated in $\sigma(T)$. According to [10], an operator $T \in\ B(X)$ is said to satisfy *generalized Weyl's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$. In general, generalized Weyl's theorem implies Weyl's theorem but the converse is not true [14]. Following [9], we say that T satisfies *generalized Browders's theorem*, if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T.

Let $SBF_+^-(X)$ denote the class of all is *upper B-Fredholm* operators such that $\operatorname{ind}(\mathrm{T}) \leq 0$. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(X)\}.$$

Following [14], we say that *generalized a-Weyl's theorem* holds for $T \in B(X)$ if $\Delta_a^g(S) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\sigma_a(T) : \alpha(T - \lambda) >

$0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in B(X)$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$. It is proved in [4, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [14, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \pi_a(T)$ it is proved in [12, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Definition 1.1. ([8]) For any $T \in B(X)$ we define the sequences $(c_n(T))$ and $(c'_n(T))$ as follows:

(i) $c_n(T) = \dim\left(R(T^n)/R(T^{n+1})\right);$
(ii) $c'_n(T) = \dim\left(\ker(T^{n+1})/\ker(T^n)\right).$

Following [22], we say that $T \in B(X)$ possesses *property* (w) if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. The property (w) has been studied in [1, 2, 22]. In Theorem 2.8 of [2], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in B(X)$ possesses *property* (gw) if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [5]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [5] that an operator possessing property (gw) possesses property (w) but the converse is not true in general. According to [15], an operator $T \in B(X)$ is said to possess *property* (gb) if $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$, and is said to possess *property* (b) if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [15] that an operator possessing property (gb) possesses property (b) but the converse is not true in general. Recently in [24], property (gb) and perturbations were extensively studied by Rashid. According to [20], an operator $T \in B(X)$ is said to satisfy *property* (Bw) if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$.

In this paper we define and study three new *properties* (Bgw), (Bb) and (Bgb) (see Definitions 2.1 and 2.4) in connection with Weyl type theorems [14], which play roles analogous to Browder's theorem and generalized Browder's theorem, respectively. We prove in Theorem 2.3 that an operator possessing property (Bgw) possesses property (Bw) but the converse is not true in general as shown by Example 2.8. We show also in Theorem 2.7 that an operator possessing property (Bgw) possesses property (gb) and in Theorem 2.5 we show that an operator possessing property (Bgb) possesses property (b), but the converses of those theorems are not true in general. Conditions for the equivalence of properties (Bgw) and (gb), and properties (Bgw) and (Bw), are given in Theorem 2.7 and Theorem 2.17, respectively. We study conditions on Hilbert space operators T and S which ensure that $T \oplus S$ obeys property (Bgw).

In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems, extending a similar diagram given in [15].

2. PROPERTY(Bgw) AND WEYL TYPE THEOREMS

Now we define property (Bgw), a variant of generalized property (w), as follows.

Definition 2.1. A bounded linear operator $T \in B(X)$ is said to satisfy *property* (Bgw) if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T).$$

Definition 2.2. [19] Let $T \in B(X)$ and let $s \in \mathbb{N}$. Then T has *uniform descent* for $n \geq s$ if $R(T)+\ker(T^n) = R(T)+\ker(T^s)$ for all $n \geq s$. If in addition $R(T)+\ker(T^s)$ is closed then T said to have *topological uniform descent* for $n \geq s$.

Recall from [9] that an operator T is *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Drazin invertible$\}$. We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$.

Theorem 2.3. *If T satisfies property (Bgw), then it satisfies property (Bw).*

Proof. Suppose that T satisfies property (Bgw) and $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda I$ is B-Weyl and so $T - \lambda I$ is upper semi-B-Fredholm with index zero. Thus $\lambda \notin \sigma_{SBF_+^-}(T)$. Let $\lambda \notin \sigma_a(T)$. Since $T - \lambda I$ is an operator of topological uniform descent, then there exist $\epsilon > 0$ such that if $0 < |\lambda - \mu| < \epsilon$, then we have $c_n(T - \lambda I) = c_0(T - \mu I)$ and $c_n^{'}(T - \lambda I) = c_0^{'}(T - \mu I)$ for large enough n. Since $T - \lambda I$ is B-Weyl, $c_n(T - \lambda I) = c_n^{'}(T - \lambda I)$. We have $c_0^{'}(T - \lambda I) = 0$ because $\lambda \notin \sigma(T)$. Hence we have $c_0(T - \lambda I) = c_0^{'}(T - \lambda I) = 0$. Consequently $\lambda \notin \sigma_a(T)$, which is a contradiction. Hence $\lambda \in \sigma_a(T)$. Since T satisfies property (Bgw), $\lambda \in E^0(T)$. Conversely if $\lambda \in E^0(T)$. Then $\lambda \in E_a^0(T)$ which implies that $\lambda \notin \sigma_{SBF_+^-}(T)$. Hence $T - \lambda I$ is an operator of topological uniform descent, then there exist $\epsilon > 0$ such that $0 < |\lambda - \mu| < \epsilon$ implies that $c_n(T - \lambda I) = c_n(T - \mu I)$ and $c_n^{'}(T - \lambda I) = c_n^{'}(T - \mu I)$ for all large enough n. Since $\lambda \in \text{iso}\sigma(T)$, if ϵ is chosen small enough, then $c_n^{'}(T - \lambda I) = c_n^{'}(T - \mu I) = 0$. So $T - \lambda I$ is Drazin invertible. Therefore $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. □

Now we introduce property (Bb) and property (Bgb) a variant of generalized Browder's theorem and property (gb) respectively as follows:

Definition 2.4. A bounded linear operator $T \in B(X)$ is said to satisfy

(i) *property* (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi^0(T)$.
(ii) *property* (Bgb) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$.

Theorem 2.5. *If T satisfies property (Bgb), then T satisfies property (Bb).*

Proof. We get the desired result by similar argument in Theorem 2.3. □

An operator $T \in B(X)$ has the *single valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies $(T - \lambda)f(\lambda)$=0 for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$. For more information, see [1]. The following proposition [24] is important for the characterization of property (Bgw).

Proposition 2.6. *Let $T \in B(X)$ be have the SVEP. If $T - \lambda I$ has finite descent at every $\lambda \in E_a(T)$, then T satisfies property (gb).*

Theorem 2.7. *Let $T \in B(X)$. Then the following statements are equivalent:*

(i) *T satisfies property (Bgw),*
(ii) *T satisfies property (gb) and $\pi(T) = E^0(T)$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose T satisfies property (Bgw). To prove T satisfies property (gb), by Proposition 2.8 it is enough to show that T has SVEP. Let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since T satisfies property (Bgw), $\lambda \in E_0(T)$. Hence $\lambda \in \mathrm{iso}\,\sigma(\mathrm{T})$. Thus T has SVEP at λ. Now we have to prove $\pi(T) = E_0(T)$. Let $\lambda \in E_0(T)$. Since T satisfies property (Bgw), $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since T satisfies property (gb), $\lambda \in \pi(T)$. Conversely suppose $\lambda \in \pi(T)$. Since T satisfies property (gb), $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Hence $\lambda \in E^0(T)$ because T satisfies property (Bgw).
(ii) $\Longrightarrow$(i). If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then $\lambda \in \pi(T)$ by hypothesis and so $\lambda \in E^0(T)$. Conversely, if $\lambda \in E^0(T)$, then, $\lambda \in \pi(T)$ by hypothesis. Since T satisfies property (gb), $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. This completes the proof. □

The following example shows the converse of Theorem 2.3 is not true in general.

Example 2.8. Let $R \in (\ell^2(\mathbb{N})$ be the unilateral right shift and T the operator defined on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = \sigma_{BW}(T) = \mathbb{D}(0,1)$ the unit disc in $\mathbb{C}$, $iso\sigma(T) = \emptyset$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle in $\mathbb{C}$. This implies that $\sigma_a(T)$ has empty interior and T has SVEP. On the other hand, it easily seen that $\sigma_{SBF_+^-}(T) = C(0,1)$. Therefore, T does not possess property (Bgw), since $\Delta_a^g(T) = \{0\}$ and $E^0(T) = \emptyset$. On the other hand, property (Bw) holds for T since $\Delta^g(T) = \emptyset = E^0(T)$.

Theorem 2.9. *Let $T \in B(X)$ satisfy property (Bgw). Then generalized a-Browder's theorems holds for T and $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(\mathrm{T})$.*

Proof. By Theorem 3.1 of [18] it is sufficient to prove that T has SVEP at every $\lambda \in \sigma_{SBF_+^-}(T)$. Let us assume that $\lambda \in \sigma_{SBF_+^-}(T)$. If $\lambda \notin \sigma_a(T)$, then T has SVEP at λ. If $\lambda \in \sigma_a(T)$ then $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$ since T satisfy property (Bgw). Thus $\lambda \in \mathrm{iso}\,\sigma_a(\mathrm{T})$ which implies T has SVEP at λ. To prove $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(\mathrm{T})$. We observe that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$. Thus $\lambda \in \mathrm{iso}\,\sigma_a(\mathrm{T})$. Hence $\sigma_a(T) \subseteq \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(\mathrm{T})$. But $\sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(\mathrm{T}) \subseteq \sigma_\mathrm{a}(\mathrm{T})$ for every operator T. Therefore, $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup \mathrm{iso}\,\sigma_a(\mathrm{T})$. □

A characterization of property (Bgw) is given as follows:

Theorem 2.10. *Let $T \in B(X)$. Then the following assertions are equivalent:*
(i) *T satisfies property (Bgw),*
(ii) *generalized a-Browder's theorems holds for T and $\pi_a(T) = E^0(T)$.*

Proof. (i)$\Rightarrow$(ii). Assume that T satisfies property (Bgw). By Theorem 2.9 it sufficient to prove the equality $\pi_a(T) = E^0(T)$. If $\lambda \in E^0(T)$ then as T satisfies property (Bgw), it implies that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi_a(T)$, because generalized a-Browder's theorems holds for T. If $\lambda \in \pi_a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$, therefore the equality $\pi_a(T) = E^0(T)$.

(ii)⇒(i). If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then as generalized a-Browder's theorem holds for T, we have $\lambda \in \pi_a(T) = E^0(T)$. Conversely, if $\lambda \in E^0(T)$ then $\lambda \in \pi_a(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Thus $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$. □

Theorem 2.11. *Let $T \in B(X)$. If T or T^* has SVEP at points in $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, Then T satisfies property (Bgw) if and only if $E^0(T) = \pi_a(T)$.*

Proof. We conclude from Theorem 3.1 of [18] that if T or T^* has SVEP at points in $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then T satisfies generalized a-Browder's theorem. Hence, $\pi_a(T) = E^0(T)$ if and only if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$ and so, T satisfies property (Bgw) if and only if $\pi_a(T) = E^0(T)$. □

Theorem 2.12. *Let $T \in B(X)$. If T satisfies property (Bgw), then T satisfies property (w).*

Proof. Suppose that T satisfies property (Bgw), then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, then $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$. Conversely, if $\lambda \in E^0(T)$. Then $\lambda \in E^0(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Hence $T - \lambda I \in SBF_+(X)$. Since $\alpha(T - \lambda I) < \infty$, then it follows from Lemma 2.2 of [5] we have $T - \lambda I \in SF_+(X)$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Finally, $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. □

The converse of Theorem 2.12 does not hold in general as shown by the following example:

Example 2.13. Let $T \in B(\ell^2(\mathbb{N}))$ be the unilateral right shift. It is known that $\sigma(T) = \mathbb{D}$, the closed unit disc in $\mathbb{C}$, $\sigma_a(T) = C(0,1)$, the unit circle of $\mathbb{C}$ and T has empty eigenvalues set. Moreover, $\sigma_{SF_+^-}(T) = C(0,1)$ and $\pi(T) = \emptyset$. Define S on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $S = 0 \oplus T$ then $S^{-1}(0) = \ell^2(\mathbb{N}) \oplus \{0\}, \sigma_{SF_+^-(S)} = \sigma_a(S) = \{0\} \cup C(0,1)$, $\sigma_{SBF_+^-}(S) = C(0,1), \pi_a(S) = \{0\}$ and $\pi(S) = \pi^0(S) = E^0(S) = \emptyset$. Hence $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(S)$ and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\} \neq E^0(S)$.

The following two examples show property (gw) and property (Bgw) are independent:

Example 2.14. Let $Q \in B(X)$ be any quasinilpotent operator acting on an infinite dimensional Banach space X such that $Q^n(X)$ is non-closed for all n. Let $T = 0 \oplus Q$ defined on the Banach space $X \oplus X$. Since $T^n(X \oplus X) = Q^n(X)$ is non-closed for all n, then T is not a semi-Fredholm operator, so $\sigma_{SBF_+^-}(T) = \{0\}$. Since $\sigma_a(T) = \{0\}$ and $E(T) = \{0\}$, then T does not satisfies property (gw). But T satisfies property (Bgw), since $E^0(T) = \emptyset$.

Example 2.15. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasinilpotent operator which is not nilpotent. We define T on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$, where I is the identity operator on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_a(T) = \{0,1\}$ and $E(T) = \{0\}$. It follows from Example 2 of [11] that $\sigma_{BW}(T) = \{0\}$. This implies that $\sigma_{SBF_+^-}(T) = \{0\}$. Hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{1\} = E(T)$ and

T satisfies property (gw). On the other hand, since $E^0(T) = \emptyset$. Then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{1\} \neq E^0(T)$ and so, T does not satisfy property (Bgw).

In the next theorem we give a characterization of operators satisfying property (Bgw).

Theorem 2.16. *Let $T \in B(X)$. Then T satisfies property (Bgw) if and only if*
(i) *T satisfies property (Bw);*
(ii) $\mathrm{ind}\,(\mathrm{T} - \lambda \mathrm{I}) = 0$ *for all* $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Proof. Suppose T satisfies property (Bgw) and let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{BW}(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T)$. If $\alpha(T - \lambda I) = 0$, as $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda I$ will be invertible. But this is impossible since $\lambda \in \sigma(T)$. Hence $0 < \alpha(T - \lambda I)$ and $\lambda \in \sigma_a(T)$. As T satisfies property (Bgw), then $\lambda \in E^0(T)$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) \subseteq E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Since T satisfies property (Bgw), then $\lambda \notin \sigma_{SBF_+^-}(T)$ and hence $\mathrm{ind}(T - \lambda I) \leq 0$. On the other hand, as $\lambda \in E^0(T)$, then λ is an isolated in $\sigma(T)$, and hence T^* has SVEP at λ. By Theorem 2.11 of [3], we have $\mathrm{ind}(T - \lambda I) \geq 0$. Hence $\mathrm{ind}(T - \lambda I) = 0$ and $\lambda \in \sigma_{BW}(T)$. So $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and $ind(T - \lambda I) = 0$ for all $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.
Conversely, assume that T satisfies property (Bw) and $\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then $T - \lambda I$ is a semi-B-Fredholm operator such that $\mathrm{ind}(T - \lambda I) = 0$. Hence $T - \lambda I$ is a B-Weyl operator. Since T satisfies property (Bw), then $\lambda \in E^0(T)$ and hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$, then $\lambda \notin \sigma_{BW}(T)$ and hence $T - \lambda I$ is a B-Weyl and since $\lambda \in \sigma(T)$, then $0 < \alpha(T - \lambda I) < \infty$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Consequently, T satisfies property (Bgw). □

Theorem 2.17. *Let $T \in B(X)$. Then T satisfies property (Bgb) if and only if*
(i) *T satisfies property (Bb); and*
(ii) $\mathrm{ind}\,(\mathrm{T} - \lambda \mathrm{I}) = 0$ *for all* $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Proof. Suppose T satisfies property (Bgw), then by Theorem 2.5, T satisfies property (Bb). If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, as T satisfies property (Bgb), then $\lambda \in \pi^0(T)$. Thus λ is isolated in $\sigma(T)$. So $\mathrm{ind}\,(T - \lambda I) = 0$. Conversely, assume that T satisfies property (Bb) and $\mathrm{ind}\,(T - \lambda I) = 0$ for all $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, then $T - \lambda I$ is an upper semi-B-Fredholm such that $\mathrm{ind}\,(T - \lambda I) = 0$. Hence $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Since T satisfies property (Bb), we have $\lambda \in \pi^0(T)$. On the other hand, if $\lambda \in \pi^0(T)$, then $T - \lambda I$ is Browder's operator and so $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Finally, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^0(T)$ and T satisfies property (Bgb). □

Theorem 2.18. *Let $T \in B(X)$. If T satisfies property (Bw), then T satisfies Weyl's theorem.*

Proof. Suppose T satisfies property (Bw), i.e., $\Delta^g(T) = E^0(T)$. Let $\lambda \in \Delta(T)$. Since $\sigma_{BW}(T) \subseteq \sigma_W(T)$, then $\lambda \in \Delta^g(T) = E^0(T)$. Hence, $\Delta(T) \subseteq E^0(T)$.

Conversely, if $\lambda \in E^0(T) = \Delta^g(T)$, then $T - \lambda I$ is a B-Weyl operator. Since $\alpha(T - \lambda I) < \infty$ we conclude that $T - \lambda I$ is a Weyl operator. So, $\lambda \in \Delta(T)$. Therefore, T satisfies Weyl's theorem. □

The converse of the preceding theorem does not hold in general. Indeed, if we consider the operator T defined in Example 2.15, then $\sigma_{BW}(T) = \{0\}$, $E^0(T) = \emptyset$ and $\sigma_W(T) = \{0, 1\}$. Then $\Delta(T) = \emptyset = E^0(T)$ and so T satisfies Weyl's theorem. However, since $\Delta^g(T) = \{1\} \neq E^0(T)$ then T does not satisfy property (Bw).

Theorem 2.19. *Let $T \in B(X)$. If T satisfies property (Bgb), then T satisfies generalized a-Browder's theorem.*

Proof. Suppose T satisfies property (Bgb), i.e., $\Delta_a^g(T) = \pi^0(T)$. Let $\lambda \in \Delta_a^g(T)$. Then as T satisfies property (Bgb) we have $\lambda \in \pi^0(T)$ and so, $\lambda \in \pi_a(T)$. Conversely, if $\lambda \in \pi_a(T)$. Then we conclude from Remark 2.7 and Theorem 2.8 of [14] that $\lambda \notin \sigma_{SBF_+^-}(T)$ and λ is isolated in $\sigma_a(T)$. Hence, $\lambda \in \Delta_a^g(T) = \pi^0(T)$. Therefore, T satisfies generalized a-Browder's theorem. □

Theorem 2.20. *Let $T \in B(X)$. If T satisfies property (Bgb), then T satisfies property (b).*

Proof. We get the desired result by a similar argument in Theorem 2.12. □

Theorem 2.21. *Let $T \in B(X)$. If T satisfies property (Bgw), then T satisfies property (Bgb).*

Proof. Suppose T satisfies property (Bgw). Then we conclude from Theorem 2.12 and Theorem 2.13 of [15] that T satisfies property (w) and $E^0(T) = \pi^0(T)$. As T satisfies property (Bgw), we have $\Delta_a^g(T) = E^0(T)$. So, $\Delta_a^g(T) = \pi^0(T)$. That is, T satisfies property (Bgb). □

Theorem 2.22. *Let $T \in B(X)$. If T satisfies property (Bw), then T satisfies property (Bb).*

Proof. Suppose T satisfies property (Bw). Then it follows from Theorem 2.18 that T satisfies Weyl's theorem. Hence, by Theorem 3.85 of [1] it follows that T satisfies Browder's theorem and $\pi^0(T) = E^0(T)$. As T satisfies property (Bw) we have $\Delta^g(T) = E^0(T)$. Therefore, $\Delta^g(T) = \pi^0(T)$. That is, T satisfies property (Bb). □

Definition 2.23. An operator $T \in B(X)$ is said to be *finitely isoloid* (resp., *finitely a-isoloid*) if $\mathrm{iso}\,\sigma(\mathrm{T}) \subseteq \mathrm{E}^0(\mathrm{T})$ (resp., $\mathrm{iso}\,\sigma_\mathrm{a}(\mathrm{T}) \subseteq \mathrm{E}^0(\mathrm{T})$). An operator $T \in B(X)$ is said to be *finitely polaroid* (resp., *finitely a-polaroid*) if $\mathrm{iso}\,\sigma(\mathrm{T}) \subseteq \pi^0(\mathrm{T})$ (resp., $\mathrm{iso}\,\sigma_\mathrm{a}(\mathrm{T}) \subseteq \pi^0(\mathrm{T})$).

Theorem 2.24. *Let $T \in B(X)$ be finitely a-isoloid operator and satisfies generalized a-Weyl's theorem. Then T satisfies property (Bgw).*

Proof. If T satisfies generalized a-Weyl's theorem then $\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = E_a(T)$. To show that T satisfies property (Bgw), we need to prove that $E_a(T) = E^0(T)$. Suppose that $\lambda \in E_a(T)$ then as T is finitely a-isoloid we have $\lambda \in E^0(T)$. Since the other inclusion is always verified. Therefore, T satisfies property (Bgw). □

Recall that an operator $T \in B(X)$ is said to be *a-polaroid* if $E_a(T) = \pi(T)$.

Theorem 2.25. *Let $T \in B(X)$ be a-polaroid operator and satisfy property (Bgw). Then T satisfies generalized a-Weyl's theorem.*

Proof. T is a-polaroid and satisfy property (Bgw) if and only if $\sigma_a(T) \backslash \sigma_{SBF_+^-}(T) = E^0(T) \subseteq E_a(T) = \pi(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, because T satisfies property (gb) by Theorem 2.7. □

Theorem 2.26. *Let $T \in B(X)$ be a finitely a-polaroid operator. If T or T^* has SVEP, then T satisfies property (Bgw).*

Proof. If T or T^* has SVEP, then T satisfies generalized a-Browder's theorem. Suppose that $\lambda \in E^0(T)$. It implies that $\lambda \in \operatorname{iso}\sigma(\mathrm{T}) \subseteq \pi^0(\mathrm{T}) \subseteq \pi_{\mathrm{a}}(\mathrm{T})$, as T is finitely polaroid. Therefore, $E^0(T) \subseteq \pi^0(T)$. For the reverse inclusion, suppose $\lambda \in \pi_a(T)$, then $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{T}) \subseteq \pi^0(\mathrm{T}) \subseteq \mathrm{E}^0(\mathrm{T})$. Hence $\pi_a(T) \subseteq E^0(T)$. Using Theorem 2.11, we have that T satisfies property (Bgw). □

3. Property (Bgw) for Direct Sum

Let H and K be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on H and K respectively and at least one of them satisfies property (Bgw) then their direct sum $T \oplus S$ obeys property (Bgw). We also explore various conditions on T and S to ensure that $T \oplus S$ satisfies property (Bgw).

Theorem 3.1. *Suppose that property (Bgw) holds for $T \in B(H)$ and $S \in B(K)$. If T and S are isoloid and $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$, then property (Bgw) holds for $T \oplus S$.*

Proof. We know that $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pairs of operators. If T and S are isoloid, then

$$E^0(T \oplus S) = \left[E^0(T) \cap \rho_a(S)\right] \cup \left[\rho_a(T) \cap E^0(S)\right] \cup \left[E^0(T) \cap E^0(S)\right],$$

where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.
If property (Bgw) holds for T and S, then

$$\begin{aligned}[\sigma_a(T) \cup \sigma_a(S)] \setminus \left[\sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)\right] \\ = \left[E^0(T) \cap \rho_a(S)\right] \cup \left[\rho_a(T) \cap E^0(S)\right] \cup \left[E^0(T) \cap E^0(S)\right].\end{aligned}$$

Thus, $E^0(T \oplus S) = [\sigma_a(T) \cup \sigma_a(S)] \setminus \left[\sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)\right]$.
if $\sigma_{SBF_+^-}(T \oplus S) = \sigma_{SBF_+^-}(T) \cup \sigma_{SBF_+^-}(S)$, then

$$E^0(T \oplus S) = \sigma_a(T \oplus S) \setminus \sigma_{SBF_+^-}(T \oplus S).$$

Hence property (Bgw) holds for $T \oplus S$. □

Theorem 3.2. *Suppose that $T \in B(H)$ such that $\operatorname{iso}\sigma_{\mathrm{a}}(\mathrm{T}) = \emptyset$ and $S \in B(K)$ satisfies property (Bgw). If $\sigma_{SBF_+^-}(T \oplus S) = \sigma_a(T) \cup \sigma_{SBF_+^-}(S)$, then property (Bgw) holds for $T \oplus S$.*

Proof. We know that $\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_a(S)$ for any pairs of operators. Then

$$\begin{aligned}\sigma_a(T\oplus S)\setminus\sigma_{SBF_+^-}(T\oplus S)&=[\sigma_a(T)\cup\sigma_a(S)]\setminus\Big[\sigma_a(T)\cup\sigma_{SBF_+^-}(S)\Big]\\&=\sigma_a(S)\setminus\Big[\sigma_a(T)\cup\sigma_{SBF_+^-}(S)\Big]\\&=\Big[\sigma_a(S)\setminus\sigma_{SBF_+^-}(S)\Big]\setminus\sigma_a(T)\\&=E^0(S)\cap\rho_a(T)\end{aligned}$$

If $\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})=\emptyset$ it implies that $\sigma_a(T)=\mathrm{acc}\,\sigma_{\mathrm{a}}(\mathrm{T})$, where $\mathrm{acc}\,\sigma_{\mathrm{a}}(\mathrm{T})=\sigma_{\mathrm{a}}(\mathrm{T})\setminus\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})$ is the set of all accumulation points of $\sigma_a(T)$. Thus we have

$$\begin{aligned}\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T}\oplus\mathrm{S})&=[\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})\cup\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})]\setminus[(\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})\cap\mathrm{acc}\,\sigma_{\mathrm{a}}(\mathrm{S}))\cup(\mathrm{acc}\,\sigma_{\mathrm{a}}(\mathrm{T})\cap\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S}))]\\&=[\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})\setminus\mathrm{acc}\sigma_{\mathrm{a}}(\mathrm{S})]\cup[\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\setminus\mathrm{acc}\,\sigma_{\mathrm{a}}(\mathrm{T})]\\&=\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\setminus\sigma_{\mathrm{a}}(\mathrm{T})\\&=\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\cap\rho_{\mathrm{a}}(\mathrm{T}).\end{aligned}$$

We know that $\sigma_p(T\oplus S)=\sigma_p(T)\cup\sigma_p(S)$ and $\alpha(T\oplus S)=\alpha(T)+\alpha(S)$ for any pairs of operators T and S, so that

$$\sigma_{PF}(T\oplus S)=\{\lambda\in\sigma_{PF}(T)\cup\sigma_{PF}(S):\alpha(T-\lambda I)+\alpha(S-\lambda I)<\infty\}.$$

Therefore,

$$\begin{aligned}E^0(T\oplus S)&=\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T}\oplus\mathrm{S})\cap\sigma_{\mathrm{PF}}(\mathrm{T}\oplus\mathrm{S})\\&=\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\cap\rho_{\mathrm{a}}(\mathrm{T})\cap\sigma_{\mathrm{PF}}(\mathrm{S})\\&=E^0(S)\cap\rho(T).\end{aligned}$$

Thus $\sigma_a(T\oplus S)\setminus\sigma_{SBF_+^-}(T\oplus S)=E^0(T\oplus S)$. Hence $T\oplus S$ satisfies property (Bgw). □

Corollary 3.3. *Suppose that $T\in B(H)$ is such that $\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})=\emptyset$ and $S\in B(K)$ satisfies property (Bgw) with $\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\cap\sigma_{\mathrm{p}}(\mathrm{S})=\emptyset$, and $\Delta_a^g(T\oplus S)=\emptyset$, then $T\oplus S$ satisfies property (Bgw).*

Proof. Since S satisfies property (Bgw), therefore given condition $\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{S})\cap\sigma_{\mathrm{p}}(\mathrm{S})=\emptyset$ implies that $\sigma_a(S)=\sigma_{SBF_+^-}(S)$. Now $\Delta_a^g(T\oplus S)=\emptyset$ gives that $\sigma_{SBF_+^-}(T\oplus S)=\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_{SBF_+^-}(S)$. Thus from Theorem 3.2, we have that $T\oplus S$ satisfies property (Bgw). □

Corollary 3.4. *Suppose that $T\in B(H)$ is such that $\mathrm{iso}\,\sigma_{\mathrm{a}}(\mathrm{T})\cup\Delta_{\mathrm{a}}^{\mathrm{g}}(\mathrm{T})=\emptyset$ and $S\in B(K)$ satisfies property (Bgw). If $\sigma_{SBF_+^-}(T\oplus S)=\sigma_{SBF_+^-}(T)\cup\sigma_{SBF_+^-}(S)$, then $T\oplus S$ satisfies property (Bgw).*

Theorem 3.5. *Let $T\in B(H)$ be an isoloid operator that satisfies property (Bgw). If $S\in B(K)$ is a normal operator satisfies property (Bgw). Then property (Bgw) holds for $T\oplus S$.*

Proof. If S is normal, then both S and S^* have SVEP, and $\mathrm{ind}\,(\mathrm{S}-\lambda\mathrm{I})=0$ for every λ such that $S-\lambda I$ is a B-Fredholm. Observe that $\lambda\notin\sigma_{SBF_+^-}(T\oplus S)$ if and only if $S-\lambda I \in SBF_{+}(K)$ and $T-\lambda I \in SBF_{+}(H)$ and $\operatorname{ind}(\mathrm{T}-\lambda \mathrm{I})+\operatorname{ind}(\mathrm{S}-\lambda \mathrm{I})=\operatorname{ind}(\mathrm{T}-\lambda \mathrm{I}) \leq 0$. if and only if $\lambda \notin \Delta_a^g(T) \cap \Delta_a^g(S)$. Hence $\sigma_{SBF_{+}^{-}}(T \oplus S)=\sigma_{SBF_{+}^{-}}(T) \cup \sigma_{SBF_{+}^{-}}(S)$. It is well known that the isolated points of the approximate point spectrum of a normal operator are simple poles of the resolvent of the operator implies that S is isoloid. So the result follows now from Theorem 3.1. □

4. Conclusion

In this last part, we give a summary of the known Weyl type theorems as in [14], including the properties introduced in [5, 15, 22], and in this paper. We use the abbreviations $gaW, aW, gW, W, (gw), (w), (Bw)$ and (Bgw) to signify that an operator $T \in B(X)$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (Bw) and property (Bgw). Similarly, the abbreviations $gaB, aB, gB, B, (gb), (b), (Bb)$ and (Bgb) have analogous meaning with respect to Browder's theorem or the new properties introduced in this paper.
The following table summarizes the meaning of various theorems and properties.

gaW	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=\pi_a(T)$
gW	$\sigma(T) \setminus \sigma_{BW}(T)=E(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T)=\pi(T)$
aW	$\sigma_a(T) \setminus \sigma_{SF_{+}^{-}}(T)=E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{SF_{+}^{-}}(T)=\pi_a^0(T)$
W	$\sigma(T) \setminus \sigma_W(T)=E^0(T)$	B	$\sigma(T) \setminus \sigma_W(T)=\pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=\pi(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_{+}^{-}}(T)=E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{SF_{+}^{-}}(T)=\pi^0(T)$
(Bw)	$\sigma(T) \setminus \sigma_{BW}(T)=E^0(T)$	(Bb)	$\sigma(T) \setminus \sigma_{BW}(T)=\pi^0(T)$
(Bgw)	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=E^0(T)$	(Bgb)	$\sigma_a(T) \setminus \sigma_{SBF_{+}^{-}}(T)=\pi^0(T)$

In the following diagram, which extends the similar diagram presented in [15], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw), property (gb), property (Bw), property (Bgw), property (Bb) and property (Bgb). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$
\begin{tikzcd}
gb \arrow[r, "{[15]}"] & gB \arrow[r, leftrightarrow, "{[4]}"] & B & aB \arrow[l, "{[17]}"'] \arrow[r, "{[17]}"] & B & W \arrow[l, "{[6]}"'] \\
gw \arrow[u, "{[15]}"'] \arrow[r, "{[5]}"] \arrow[d, "{[5]}"] & gW \arrow[u, "{[14]}"'] \arrow[d, "{[14]}"] & gaW \arrow[l, "{[14]}"'] \arrow[r, "{[14]}"] \arrow[d, "{[14]}"] & gaB \arrow[d, leftrightarrow, "{[4]}"] & Bgw \arrow[l, "2.10"'] \arrow[r, "2.21"] \arrow[d, "2.7"] & Bgb \arrow[d, "2.19"] \\
w \arrow[r, "{[15]}"] & W & aW \arrow[l, "{[23]}"] \arrow[r, "{[14]}"'] & aB & gb \arrow[l, "{[15]}"] \arrow[r, "{[15]}"'] \arrow[d, "{[15]}"] & gaB \\
Bgw \arrow[u, "2.12"'] \arrow[r, "2.3"'] & Bw \arrow[u, "2.18"'] \arrow[r, "2.22"'] & Bb & Bgb \arrow[l, "2.17"] \arrow[r, "2.20"'] & b \arrow[r, "{[15]}"'] & aB \arrow[u, leftrightarrow, "{[4]}"']
\end{tikzcd}
$$

Acknowledgement. The authors thank the referee and the editor for useful comments and suggestions that improved the quality of this paper.

References

[1] P. Aiena, *Fredhlom and Local Spectral Theory with Application to Multipliers*, Kluwer Acad. Publishers, Dordrecht, 2004.

[2] P. Aiena and P. Peña, *Variations of Weyl's theorem*, J. Math. Anal. Appl. **324** (2006), 566–579.

[3] P. Aiena, *Quasi-Fredholm operators and localized SVEP*, Acta Sci. Math. (Szeged) **73** (2007), 285–300.

[4] M. Amouch and H. Zguitti, *On the equivalence of Browder's and generalized Browder's theorem*, Glasg. Math. J. **48** (2006), 179-185.

[5] M. Amouch and M. Berkani, *On property* (gw), Mediterr. J. Math. **5** (2008), 371–378.

[6] B.A. Barnes, *Riesz points and Weyl's theorem*, Integral Equations and Operator Theory **34** (1999), 566–603.

[7] M. Berkani, *On a class of quasi-Fredholm operators*, Integral Equations and Operator Theory. **34** (1999), no.2, 244–249.

[8] M. Berkani, *Restriction of an operator to the range of its power*, Studia Math. **140** (2000), 163–175.

[9] M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272** (2002), 596–603.

[10] M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc. **130** (2002),1717-1723.

[11] M. Berkani and A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Aust. Math. Soc. **76** (2004), 1–12.

[12] M. Berkani, *On the equivalence of Weyl's theorem and generalized Weyl's theorem*, Acta Math. Sin. **272** (2007) 103–110.

[13] M. Berkani and M. Sarih, *On semi B-Fredholm operators*, Glasg. Math. J. **43** (2001), 457–465.

[14] M. Berkani and J.J Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003) 359–376.

[15] M. Berkani and H. Zariouh, *Extended Weyl type theorems*, Math. Bohem. **134** (2009), 369–378.

[16] L.A. Coburn, *Weyl's theorem for non-normal operators*, Michigan. Math. J. **13** (1966), 285–288.

[17] S.V. Djordjević and Y.M. Han, *Browder's theorems and spectral continuity*, Glasg. Math. J. **43** (2001), 457–465.

[18] B.P. Duggal, *SVEP and generalized Weyl's theorem*, Mediterr. J. Math. **4** (2007), 309–320.

[19] S. Grabiner , *Uniform ascent and descent of bounded operators*, J. Math. Soc. Japan **34** (1982), 317–337.

[20] A. Gupta and N. Kashyap, *Property* (Bw) *and Weyl Type theorems*, Bull. Math. Anal. Appl. **3** (2011), 1–7.

[21] H. Heuser, *Functional Analysis*, Dekker, New York, 1982.

[22] V. Rakočević, *On a class of operators*, Mat. Vesnik **37** (1985), 423-426.

[23] V. Rakočević, *Operators obeying a-Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **10** (1986), 915–919.

[24] M.H.M. Rashid, *Property* (gb) *and perturbations*, J. Math. Anal. Appl. **383** (2011), 82–94.

[1] Department of Mathematics, Faculty of Science P.O. Box(7), Mu'tah university, Al-Karak, Jordan.

E-mail address: malik_okasha@yahoo.com

[2] Department of Science and Humanities, Ahalia School of Engineering and Technology, Palakkad -678557, Kerala, India.
E-mail address: prasadvalapil@gmail.com