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ON GLOBAL BOUNDS FOR GENERALIZED JENSEN'S INEQUALITY

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Abstract. We offer a global bound for an abstract Jensen's functional. Particularly, the results from Simić [Rocky Mount. J. Math., 41 (2011), no. 6, 2021–2031] are reobtained. Applications to integral inequalities and the theory of means are pointed out.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a convex function, and $x_i \in [a, b]$ for $i = 1, 2, \ldots, n$. Let $\underline{p} = \{p_i\}$, $\sum_{i=1}^{n} p_i = 1$, $p_i > 0$ $(i = \overline{1, n})$ be a sequence of positive weights. Put $\underline{x} = \{x_i\}$. Then the Jensen functional $J_f(\underline{p}, \underline{x})$ is defined by

$$J_f(\underline{p}, \underline{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

In a recent paper [7] the following global bounds have been proved:

Theorem 1.1. *Let $f, \underline{p}, \underline{x}$ be defined as above, and let $p, q \geq 0$, $p + q = 1$. Then*

$$0 \leq J_f(\underline{p}, \underline{x}) \leq \max_{p}[pf(a) + qf(b) - f(pa + qb)]. \tag{1.1}$$

The left side of (1.1) is the classical Jensen inequality. Both bounds of $J_f(\underline{p}, \underline{x})$ in (1.1) are global, as they depend only on f and the interval $[a, b]$.

As it is shown in [7], the upper bound in relation (1.1) refines many earlier results, and in fact it is the best possible bound. In what follows, we will show

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that, this result has been discovered essentially by the present author in 1991 [4], and in fact this is true in a general framework for positive linear functionals defined on the space of all continuous functions defined on $[a, b]$.

In paper [4], as a particular case of a more general result, the following is proved:

Theorem 1.2. *Let $f, \underline{p}, \underline{x}$ as above. Then one has the double inequality:*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i)$$
$$\le \left(\sum_{i=1}^{n} p_i x_i\right)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{1.2}$$

The right side of (1.2) follows from the fact that the graph of f is below the graph of line passing through the points $(a, f(a))$, $(b, f(b))$:

$$f(x) \le (x-a)\frac{f(b)}{b-a} + (b-x)\frac{f(a)}{b-a}.$$

By letting $x = x_i$, and multiplying both sides with p_i, after summation we get the right side of (1.2) (the left side is Jensen's inequality).

Now, remark that the right side of (1.2) can be written also as

$$f(a)\left[\frac{b - \sum_{i=1}^{n} p_i x_i}{b-a}\right] + f(b)\left[\frac{\sum_{i=1}^{n} p_i x_i - a}{b-a}\right].$$

Therefore, by denoting

$$\frac{b - \sum_{i=1}^{n} p_i x_i}{b-a} = p \quad \text{and} \quad \frac{\sum_{i=1}^{n} p_i x_i - a}{b-a} = q,$$

we get $p \ge 0$, $p + q = 1$ and $\sum_{i=1}^{n} p_i x_i = pa + qb$. Thus, from (1.2) we get

$$0 \le J_f(\underline{p}, \underline{x}) \le pf(a) + qf(b) - f(pa + qb)$$

and this immediately gives Theorem 1.1.

2. An extension

Let $C[a, b]$ denote the space of all continuous functions defined on $[a, b]$, and let $L : C[a, b] \to \mathbb{R}$ be a linear and positive functional defined on $C[a, b]$ i.e. satisfying $L(f_1 + f_2) = L(f_1) + L(f_2)$, $L(\lambda f) = \lambda L(f)$ $(\lambda \in \mathbb{R})$ and $L(f) \ge 0$ for $f \ge 0$. Define $e_k(x) = x^k$ for $x \in [a, b]$ and $k = 0, 1, 2, \ldots$.

The following result has been discovered independently by Lupaş [2] and Sándor [4]:

Theorem 2.1. *Let f be convex and L, e_k as above and suppose that $L(e_0) = 1$. Then we have the double inequality*

$$f(L(e_1)) \leq L(f) \leq L(e_1)\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(a)-af(b)}{b-a}. \tag{2.1}$$

We note that the proof of (2.1) is based on basic properties of convex functions (e.g. $f \in C[a,b]$). Particularly, the right side follows on similar lines as shown for the right side of (1.2).

Define now the generalized Jensen functional as follows:

$$J_f(L) = L(f) - f(L(e_1)).$$

Then the following extension of Theorem 1.1 holds true:

Theorem 2.2. *Let f, L, p, q be as above. Then*

$$0 \leq J_f(L) \leq \max_p [pf(a) + qf(b) - f(pa+qb)] = T_f(a,b). \tag{2.2}$$

Proof. This is similar to the method shown in the case of Theorem 1.2. Remark that the right side of (2.1) can be rewritten as

$$f(a)p + f(b)q,$$

where

$$p = \frac{b - L(e_1)}{b-a} \quad \text{and} \quad q = \frac{L(e_1) - a}{b-a}.$$

As $e_1(x) = x$ and $a \leq x \leq b$, we get $a \leq L(e_1) \leq b$, the functional L being a positive one. Thus $p \geq 0$, $q \geq 0$ and $p + q = 1$. Moreover, $L(e_1) = pa + qb$; so relation (2.2) is an immediate consequence of (2.1).

By letting $L(f) = \sum_{i=1}^{n} p_i f(x_i)$, which is a linear and positive functional, we get $J_f(L) = J_f(\underline{p}, \underline{x})$, so Theorem 1.1 is reobtained.

Let now $k : [a,b] \to \mathbb{R}$ be a strictly positive, integrable function, and $g : [a,b] \to [a,b]$ such that $f[g(x)]$ is integrable on $[a,b]$. Define

$$L_g(f) = \frac{\int_a^b k(x) f[g(x)] dx}{\int_a^b k(x) dx}.$$

It is immediate that L_g is a positive linear functional, with $L_g(e_0) = 1$.

Since

$$L(e_1) = \frac{\int_a^b k(x) g(x) dx}{\int_a^b k(x) dx},$$

by denoting

$$J_f(k,g) = \frac{\int_a^b k(x)f[g(x)]dx}{\int_a^b k(x)dx} - f\left(\frac{\int_a^b k(x)g(x)dx}{\int_a^b k(x)dx}\right),$$

we can deduce from Theorem 2.2 a corollary. Moreover, as in the discrete case, the obtained bound is best possible:

Theorem 2.3. *Let f, k, g as above, and let $p, q \geq 0$, $p + q = 1$. Then*

$$0 \leq J_f(k,g) \leq T_f(a,b).$$

The upper bound in (2.3) is best possible.

Proof. Relation (2.3) is a particular case of (2.2) applied to L_g and $J_f(k,g)$ above.

In order to prove that the upper bound in (2.3) is best possible, let $p_0 \in [0,1]$ be the point at which the maximum $T_f(a,b)$ is attained (see [7]). Let $c \in [a,b]$ be defined as follows:

$$\int_a^c k(x)dx = p_0 \int_a^b k(x)dx. \tag{2.3}$$

If $p_0 = 0$ then put $c = a$; while for $p_0 = 1$, put $c = b$. When $p_0 \in (0,1)$ remark that the application

$$h(t) = \int_a^t k(x)dx - p_0 \int_a^b k(x)dx$$

has the property $h(a) < 0$ and $h(b) > 0$; so there exist $t_0 = c \in (a,b)$ such that $h(c) = 0$, i.e. (2.3) is proved.

Now, select $g(x)$ as follows:

$$g(x) = \begin{cases} a, & \text{if } \ a \leq x \leq c \\ b, & \text{if } \ c \leq x \leq b. \end{cases}$$

Then

$$\int_a^b k(x)g(x)dx \Big/ \int_a^b k(x)dx = a \int_a^c k(x)dx \Big/ \int_a^b k(x)dx$$
$$+ b \int_a^b k(x)dx \Big/ \int_a^b k(x)dx = ap_0 + bq_0,$$

where $q_0 = 1 - p_0$.

On the other hand,

$$\int_a^b k(x)f[g(x)]dx \Big/ \int_a^b k(x)dx = f(a) \int_a^c k(x)dx \Big/ \int_a^b k(x)dx$$
$$+ f(b) \int_c^b k(x)dx \Big/ \int_a^b k(x)dx = p_0 f(a) + q_0 f(b).$$

This means that

$$J_f(k,g) = p_0 f(a) + q_0 f(b) - f(ap_0 + bq_0) = T_f(a,b).$$

Therefore, the equality is attained at the right side of (2.3), which means that this bound is best possible.

3. Applications

a) The left side of (2.3) is the generalized form of the famous Jensen integral inequality

$$f\left(\frac{\int_a^b k(x)g(x)dx}{\int_a^b k(x)dx}\right) \le \frac{\int_a^b k(x)f[g(x)]dx}{\int_a^b k(x)dx}, \tag{3.1}$$

with many application in various fields of Mathematics.

For $f(x) = -\ln x$, this has a more familiar form.

Now, the right side of (2.1) applied to $L = L_g$ gives the inequality

$$\frac{\int_a^b k(x)f[g(x)]dx}{\int_a^b k(x)dx} \le \frac{b-u}{b-a}f(a) + \frac{u-a}{b-a}f(b), \tag{3.2}$$

where

$$u = L(e_1) = \frac{\int_a^b k(x)g(x)dx}{\int_a^b k(x)dx}.$$

Inequalities (3.1) and (3.2) offer an extension of the famous Hadamard inequalities (or Jensen–Hadamard, or Hermite–Hadamard inequalities) (see e.g. [1, 3, 4])

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{3.3}$$

Applying (3.1) and (3.2) for $g(x) = x$, we get from (3.1) and (3.2):

$$f(v) \le \frac{\int_a^b k(x)f(x)dx}{\int_a^b k(x)dx} \le \frac{(b-v)f(a)+(v-a)f(b)}{b-a}, \tag{3.4}$$

where

$$v = \frac{\int_a^b xk(x)dx}{\int_a^b k(x)dx}.$$

When $k(x) \equiv 1$, inequality (3.4) reduces to (3.3).

b) Let $a, b > 0$ and $G = G(a,b) = \sqrt{ab}$; $L = L(a,b) = \dfrac{b-a}{\ln b - \ln a}$ $(a \neq b)$, $L(a,a) = a$, $I = I(a,b) = \dfrac{1}{e}(b^b/a^a)^{1/(b-a)}$ $(a \neq b)$, $I(a,a) = a$ be the well-known geometric, logarithmic and identric means.

In our paper [5] the following generalized means have been introduced (assume $a \neq b$):

$$\ln I_k(a,b) = \int_a^b k(x) \ln x dx \Big/ \int_a^b k(x)dx,$$

$$A_k(a,b) = \int_a^b xk(x)dx \Big/ \int_a^b k(x)dx,$$

$$L_k(a,b) = \int_a^b k(x)dx \Big/ \int_a^b k(x)/xdx,$$

$$G_k^2(a,b) = \int_a^b k(x)dx \Big/ \int_a^b k(x)/x^2dx.$$

Clearly, $I_1 \equiv I$, $A_1 \equiv A$, $L_1 \equiv L$, $G_1 \equiv G$.

Applying inequality (2.3) for $f(x) = -\ln x$, and using the fact that in this case $T_f(a,b) = \ln \dfrac{L \cdot I}{G^2}$ (see [7]), we get the inequalities

$$0 \leq \ln \left(\frac{\int_a^b k(x)g(x)dx}{\int_a^b k(x)dx} \right) - \frac{\int_a^b k(x)\ln g(x)dx}{\int_a^b k(x)dx} \leq \ln \frac{L \cdot I}{G^2}.$$

For $g(x) = x$, with the above notations, we get

$$1 \leq \frac{A_k}{I_k} \leq \frac{L \cdot I}{G^2}. \tag{3.5}$$

Applying the right side of inequality (3.4) for the same function

$$f(x) = -\ln x$$

we get

$$\frac{A_k}{L} \leq 1 + \ln \left(\frac{I \cdot I_k}{G^2} \right), \tag{3.6}$$

where we have used the remark that

$$\ln(e \cdot I) = \frac{b \ln b - a \ln a}{b-a} \quad \text{and} \quad \ln G^2 - \ln(e \cdot I) = \frac{b \ln a - a \ln b}{b-a}.$$

Note that the more complicated inequality (3.6) is a slightly stronger than the right side of (3.5), as by the classical inequality $\ln x \leq x - 1$ $(x > 0)$ one has

$$\ln \left(\frac{I \cdot I_k}{G^2} \right) + 1 \leq \frac{I \cdot I_k}{G^2},$$

so

$$\frac{A_k}{L} \leq 1 + \ln \left(\frac{I \cdot I_k}{G^2} \right) \leq \frac{I \cdot I_k}{G^2}.$$

These inequalities seem to be new even in the case $k(x) \equiv 1$. For $k(x) = e^x$ one obtains the exponential mean $A_{e^x} = E$, where

$$E(a,b) = \frac{be^b - ae^a - 1}{b-a}.$$

The mean I_{e^x} has been called as the "identric exponential mean" in [6], where other inequalities for these means have been obtained.

c) Applying inequality (2.3) for $g(x) = \ln x$, $f(x) = e^x$, we get

$$0 \le A_k - I_k \le \frac{e^b - e^a}{b-a} \ln\left(\frac{e^b - e^a}{b-a}\right) + \frac{be^a - ae^b}{b-a} - \frac{e^b - e^a}{b-a},$$

where the right hand side is $T_f(a,b)$ for $f(x) = e^x$. This may be rewritten also as

$$0 \le A_k(a,b) - I_k(a,b) \le 2[A(x,y) - L(x,y)] - L(x,y) \ln \frac{I(x,y)}{L(x,y)}, \tag{3.7}$$

where $e^a = x$, $e^b = y$.

As in [5] it is proved that $\ln \frac{I}{L} \ge \frac{L-G}{L}$, the right side of (3.7) implies

$$0 \le A_k(a,b) - I_k(a,b) \le 2A(x,y) + G(x,y) - 3L(x,y).$$

d) Finally, applying (3.4) for $f(x) = x \ln x$ and $k(x)$ replaced with $k(x)/x$, we can deduce

$$\ln L_k \le \ln I_k \le 1 + \ln I - \frac{G^2}{L \cdot L_k}, \tag{3.8}$$

where the identity $\frac{b \ln b - a \ln a}{b-a} = \ln I + 1$ has been used. We note that for $k(x) \equiv 1$, inequality (3.8) offers a new proof of the classical relations

$$G \le L \le I.$$

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