Annals of Functional Analysis
Ann. Funct. Anal. 4 (2013), no. 1, 1–10
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

THE BOCHNER INTEGRAL FOR MEASURABLE SECTIONS AND ITS PROPERTIES

INOMJON GANIEV* AND GHARIB S. MAHMUOD

Communicated by D. H. Leung

Abstract. In the present paper we introduce the notion Bochner integral for measurable sections and study some properties such integrals. Given necessary and successfully condition for integrability of a measurable section. Dominated convergence theorem and analogue of Hille's theorem are proved.

1. Introduction

Bochner integral is used in many mathematics field, such as probability theory, functional analysis, differential equations in vector spaces, theory of semigroup of linear operators and so on. The integral of Banach valued function was introduced by Bochner [1] and Pettis [2]. Integration of function with values in locally convex spaces considered by Phillips [3] and Rikkard [4]. The Bochner integral of Banach valued functions and its applications are given in many books and monographs, for example, in Hille and Fillips [5], Yosida [6] Bogachev [7], Vakhania et al [8], Schwabik [9]. The Bochner integral is used in Arendt et al [10] to solve different problems of analysis. For example in [11] it is used to study geometry of Banach spaces, in [12] it is used to study semigroup of linear operators in Banach spaces.

It is known that the theory of Banach bundles stemming from paper [13], where it was showed that such a theory has vast applications in analysis. For another applications of the measurable Banach bundles, we refer the reader [14, 16, 17]. In [14] and [15] Gutman introduced the notion measurable section and showed properties of measurable sections obtained by means of a measurability structure and proved that every Banach– Kantorovich space over ring measurable functions

Date: Received: 9 April 2012; Accepted: 27 June 2012.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46G10; Secondary 46G12, 46E40.
Key words and phrases. Measurable bundle; measurable section; integral Bochner.

is linearly isometric to the space of measurable sections of a measurable Banach bundle.

In present paper we generalize the notion Bochner integral for measurable sections and study properties of such integrals.

2. Preliminaries

Let $(\Omega, \Sigma, \lambda)$ be the space with finite measure, $L_0 = L_0(\Omega)$ algebra of classes measurable functions on $(\Omega, \Sigma, \lambda)$. $L_p(\Omega)$ be Banach space of measurable functions integrable with degree $p, p \geq 1$, with norm $\|f\|_p = (\int\limits_\Omega |f(\omega)|^p d\lambda)^{\frac{1}{p}}$.

An *ideal* space on $(\Omega, \Sigma, \lambda)$ is a linear subset E in L_0 such that

$$(x \in L_0, y \in E; |x| \leq |y|) \Rightarrow (x \in E)$$

i.e., with every function the set E contains its modulus and each function with smaller modulus. The basic examples are $L_0, L_p(\Omega), L^\infty$, Orlicz and Marsinkevicz spaces. Denote by E^+ the cone of positive elements or the positive cone of an ideal space E:

$$E^+ = \{x \in E : \quad x \geq 0\}.$$

A sequence b_n is said to be *order convergent* (or o-convergent) to b if there is a sequence a_n in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n. We write $b_n \overset{(o)}{\to} b$ or $b = (o) - \lim\limits_n b_n$ denote order convergence.

We will consider vector spaces F over field real numbers $\mathbb{R}$.

Definition 2.1. [18] A map $\|\cdot\| : F \longrightarrow E$ is called an $E-$ valued norm on F, if for any $x, y \in F, \lambda \in \mathbb{R}$ it satisfies the following conditions:
1) $\|x\| \geq 0; \|x\| = 0 \Longleftrightarrow x = 0$;
2) $\|\lambda x\| = |\lambda| \|x\|$;
3) $\|x + y\| \leq \|x\| + \|y\|$.

A pair $(F, \|\cdot\|)$ is called *lattice-normed space* (LNS) over E.

A LNS F is said to be *d-decomposable*, if for any $x \in F$ and for any decomposition $\|x\| = f + g$ to sum disjoint elements there exists such $y, z \in F$, that $x = y + z \quad \|x\| = f, \|z\| = g$.

A net $\{x_\alpha\}$ in F is called *(bo)-convergent* to $x \in F$, if the net $\{\|x_\alpha - x\|\}$ is (o)-convergent to zero in E.

A lattice normed space is called (bo)-*complete* if every (bo)-fundamental net is (bo)-convergent in it. A Banach–Kantorovich space (BKS) over E is a (bo)-complete d-decomposable lattice normed space over E. It is well known [18] that every Banach–Kantorovich space F over E admits an E-module structure such that $||\lambda x|| = |\lambda| ||x||$ for every $x \in F, \lambda \in E$.

Let X be a mapping, which maps every point $\omega \in \Omega$ to some Banach space $(X(\omega), \|\cdot\|_{X(\omega)})$. In what follows, we assume that $X(\omega) \neq \{0\}$ for all $\omega \in \Omega$. A function u is said to be a section of X, if it is defined almost everywhere in Ω and takes its value $u(\omega) \in X(\omega)$ for $\omega \in dom(u)$, where $\omega \in dom(u)$ is the domain of u.

Let L be some set of sections.

Definition 2.2. [14]. A pair (X, L) is said to be a *measurable bundle of Banach spaces* over Ω if

1. $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ and $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in dom(c_1) \cap dom(c_2) \to \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;
2. the function $||c|| : \omega \in dom(c) \to ||c(\omega)||_{X(\omega)}$ is measurable for all $c \in L$;
3. for every $\omega \in \Omega$ the set $\{c(\omega) : c \in L, \omega \in dom(c)\}$ is dense in $X(\omega)$;

A section s is said to be *step*, if there are $c_i \in L, A_i \in \Sigma, i = \overline{1, n}$ such that $s(\omega) = \sum\limits_{i=1}^{n} \chi_{A_i}(\omega) c_i(\omega)$ for almost all $\omega \in \Omega$.

A section u is called *measurable* if there is a sequence $\{s_n\}$ of step sections such that $s_n(\omega) \to u(\omega)$ almost everywhere on Ω.

The set of all measurable sections is denoted by $M(\Omega, X)$, and $L_0(\Omega, X)$ denotes the factorization of this set with respect to equality everywhere. We denote by $\hat{u}$ the class from $L_0(\Omega, X)$ containing a section $u \in M(\Omega, X)$, and by $||\hat{u}||$ the element of L_0 containing the function $||u(\omega)||_{X(\omega)}$.

It is known [14, 15] that $L_0(\Omega, X)$ is a BKS over L_0.

3. The Bochner integral for measurable sections and its properies

Let s be a step section and $m_i = \sup\limits_{\omega \in dom(c_i)} \|c_i(\omega)\|_{X(\omega)} < \infty$ for any $i = 1, 2, ..., n$ then $\int\limits_{\Omega} \|s(\omega)\|_{X(\omega)} d\lambda < \infty$. Actually, as $\|c_i(\omega)\|_{X(\omega)} < m_i$ we have

$$\|s(\omega)\|_{X(\omega)} = \sum_{i=1}^{n} \chi_{A_i}(\omega) \|c_i(\omega)\|_{X(\omega)} \leq \sum_{i=1}^{n} m_i \chi_{A_i}(\omega).$$

Therefore

$$\int\limits_{\Omega} \|s(\omega)\|_{X(\omega)} d\lambda \leq \sum_{i=1}^{n} m_i \lambda(A_i) < \infty.$$

We define the integral of step section by measure λ with equality

$$\int\limits_{\Omega} s(\omega) d\lambda = \sum_{i=1}^{n} c_i(\omega) \lambda(A_i).$$

From this definition it follows that

$$\left\| \int\limits_{\Omega} s(\omega) d\lambda \right\|_{X(\omega)} = \left\| \sum_{i=1}^{n} c_i(\omega) \lambda(A_i) \right\|_{X(\omega)} \leq \sum_{i=1}^{n} \|c_i(\omega)\|_{X(\omega)} \lambda(A_i) = \int\limits_{\Omega} \|s(\omega)\|_{X(\omega)} d\lambda.$$

Definition 3.1. The measurable section u is said to be *integrable by Bochner*, if there exists a sequence step sections s_n such that

$$\lim_{n\to\infty} \int\limits_{\Omega} \|s_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda = 0$$

In this case the integral $\int\limits_A u d\lambda$ for every $A \in \Sigma$ defined with equality

$$\int\limits_A u d\lambda = \lim_{n\to\infty} \int\limits_A s_n d\lambda \tag{3.1}$$

By analogy of Banach valued case, it can be proved, that definition is correct, i.e. (3.1) independent from choosing the sequence step sections.

Theorem 3.2. *If u is a measurable section such that $u(\omega) = 0$ for almost all $\omega \in \Omega$ then u is integrable by Bochner and $\int\limits_\Omega u(\omega) d\lambda = 0$.*

Proof. The sequence of step sections from Definition 3.1 can be chosen as sections which are identically zero. □

Corollary 3.3. *If the section u is Bochner integrable and v is a section such that $u(\omega) = v(\omega)$ for almost all $\omega \in \Omega$ then v is Bochner integrable and $\int\limits_\Omega u(\omega) d\lambda = \int\limits_\Omega v(\omega) d\lambda$*

Proof. As $u = u - v + v$ and $u - v$ is Bochner integrable by Theorem 3.2, we get the statement immediately. □

Proposition 3.4. *A countably valued measurable section u of the form*

$$u(\omega) = \sum_{i=1}^{\infty} c_i(\omega)\chi_{A_i}(\omega), c_i \in L, A_i \in \Sigma, A_i \cap A_j = \emptyset$$

is Bochner integrable if

$$\sum_{i=1}^{\infty} \|c_i(\omega)\|_{X(\omega)} \lambda(A_i) < \infty.$$

Proof. For any $n \in \mathbb{N}$ define sections $s_n(\omega) = \sum\limits_{i=1}^{n} c_i(\omega)\chi_{A_i}(\omega)$. Then $\lim\limits_{n\to\infty} s_n(\omega) = u(\omega)$ a.e. on Ω. For a.e. on Ω and $k < n$ the following equality is valid equality

$$\|s_k(\omega) - s_n(\omega)\|_{X(\omega)} = \|\sum_{i=k+1}^{n} c_i(\omega)\chi_{A_i}(\omega)\|_{X(\omega)}.$$

As

$$\|\sum_{i=k+1}^{n} c_i(\omega)\chi_{A_i}(\omega)\|_{X(\omega)} = \sum_{i=k+1}^{n} \|c_i(\omega)\|_{X(\omega)}\chi_{A_i}(\omega)$$

we have

$$\int\limits_\Omega \|s_k(\omega) - s_n(\omega)\|_{X(\omega)} d\lambda = \sum_{i=k+1}^{n} \|c_i(\omega)\|_{X(\omega)}\lambda(A_i).$$

Since $\sum\limits_{i=1}^{\infty}\|c_i(\omega)\|_{X(\omega)}\lambda(A_i)<\infty$, we get that $\sum\limits_{i=1}^{\infty}c_i(\omega)\chi_{A_i}(\omega)$ is convergent in $X(\omega)$ to $u(\omega)$. Then by the definition of Bochner integral we have

$$\int\limits_{\Omega} u(\omega)d\lambda=\sum_{i=1}^{\infty}c_i(\omega)\lambda(A_i)$$

and

$$\int\limits_{\Omega}\|u(\omega)\|_{X(\omega)}d\lambda=\sum_{i=1}^{\infty}\|c_i(\omega)\|_{X(\omega)}\lambda(A_i).$$

□

Corollary 3.5. *A countably valued measurable section u for which $\|u(\omega)\|_{X(\omega)}\leq g(\omega)$ a.e. with $g\in L_1(\Omega)$ is Bochner integrable.*

Proof. Using the sequence $s_n(\omega)=\sum\limits_{i=1}^{n}c_i(\omega)\chi_{A_i}(\omega)$ we get

$$\int\limits_{\Omega}\|s_n(\omega)\|_{X(\omega)}d\lambda\leq\int\limits_{\Omega}g(\omega)d\lambda<\infty$$

for any $n\in\mathbb{N}$. Then by Proposition 3.4 u is Bochner integrable. □

Theorem 3.6. *A measurable section u is integrable by Bochner if and only if*

$$\int\limits_{\Omega}\|u(\omega)\|_{X(\omega)}d\lambda<\infty.$$

Proof. Let the measurable section u be integrable by Bochner and s_n be a sequence of step sections such that $\int\limits_{\Omega}u(\omega)d\lambda=\lim\limits_{n\to\infty}\int\limits_{\Omega}s_n(\omega)d\lambda$. Then

$$\int\limits_{\Omega}\|u(\omega)\|_{X(\omega)}d\lambda\leq\int\limits_{\Omega}\|s_n(\omega)-u(\omega)\|_{X(\omega)}d\lambda+\int\limits_{\Omega}\|s_n(\omega)\|_{X(\omega)}d\lambda<\infty.$$

On the contrary, let u be measurable section and

$$\int\limits_{\Omega}\|u(\omega)\|_{X(\omega)}d\lambda<\infty.$$

By [15, Proposition 4.1.8 (2)] there is a sequence of measurable sections g_n in form

$$\sum_{i=1}^{\infty}c_i^{(n)}(\omega)\chi_{A_i^{(n)}}(\omega),\, c_i^{(n)}\in L,\, A_i^{(n)}\in\Sigma,\, A_i^{(n)}\cap A_j^{(n)}=\emptyset$$

when $i\neq j$, such, that $\|g_n(\omega)-u(\omega)\|_{X(\omega)}<\frac{1}{n}$ for almost all $\omega\in\Omega$. Then $\|g_n(\omega)\|_{X(\omega)}\leq\|u(\omega)\|_{X(\omega)}+\frac{1}{n}$.

For any n we will choose $p_n\in\mathbb{N}$ such, that

$$\int\limits_{\bigcup\limits_{n=p_n+1}^{\infty}A_i^{(n)}}\|g_n(\omega)\|_{X(\omega)}d\lambda<\frac{\lambda(\Omega)}{n}.$$

Put

$$s_n(\omega) = \sum_{i=1}^{p_n} c_i^{(n)}(\omega)\chi_{A_i^{(n)}}(\omega).$$

Then s_n is a step section and

$$\int\limits_\Omega \|s_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda \le \int\limits_\Omega \|g_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda +$$

$$+ \int\limits_\Omega \|s_n(\omega) - g_n(\omega)\|_{X(\omega)} d\lambda \le \frac{\lambda(\Omega)}{n} + \frac{\lambda(\Omega)}{n} = \frac{2\lambda(\Omega)}{n}$$

So the section u is integrable by Bochner. □

Corollary 3.7. *A measurable section u for which $\|u(\omega)\|_{X(\omega)} \le g(\omega)$ a.e. with $g \in L_1(\Omega)$ is Bochner integrable.*

The following simple properties of integral Bochner are hold:

Theorem 3.8. *If a section u is integrable by Bochner, then*

(1) $\left\|\int\limits_A u(\omega)d\lambda\right\|_{X(\omega)} \le \int\limits_A \|u(\omega)\|_{X(\omega)} d\lambda$ *for all $A \in \Sigma$;*

(2) $\lim\limits_{\lambda(A)\to 0} \int\limits_A u(\omega)d\lambda = 0;$

(3) If $c \in L, f \in L_1(\Omega)$ and $\sup\limits_{\omega \in dom(c)} \|c(\omega)\|_{X(\omega)} < \infty$ then cf is integrable by Bochner and

$$\int\limits_\Omega c(\omega)f(\omega)d\lambda = c(\omega)\int\limits_\Omega f(\omega)d\lambda.$$

Proof. (1). $\left\|\int\limits_A u(\omega)d\lambda\right\|_{X(\omega)} = \lim\limits_{n\to\infty}\left\|\int\limits_\Omega s_n(\omega)d\lambda\right\|_{X(\omega)} \le \lim\limits_{n\to\infty}\int\limits_\Omega \|s_n(\omega)\|_{X(\omega)} d\lambda =$
$= \int\limits_\Omega \|u(\omega)\|_{X(\omega)} d\lambda.$

(2). As $\lim\limits_{\lambda(A)\to 0}\int\limits_A \|u(\omega)\|_{X(\omega)} d\lambda = 0$ from (1) follows, that

$$\left\|\int\limits_A u(\omega)d\lambda\right\|_{X(\omega)} \le \lim_{\lambda(A)\to 0}\int\limits_A \|u(\omega)\|_{X(\omega)} d\lambda = 0$$

i.e. $\lim\limits_{\lambda(A)\to 0}\int\limits_A u(\omega)d\lambda = 0.$

(3). Let f be a simple function from $L_1(\Omega)$ i.e. $f(\omega) = \sum\limits_{i=1}^n \lambda_i \chi_{A_i}(\omega)$, where $\lambda_i \in \mathbb{R}, A_i \in \Sigma, i = \overline{1,n}, A_i \cap A_j = \varnothing$ when $i \ne j$. Then $c(\omega)f(\omega) = \sum\limits_{i=1}^n c(\omega)\lambda_i\chi_{A_i}(\omega)$ is step section and by definition $\int\limits_\Omega c(\omega)f(\omega)d\lambda = \sum\limits_{i=1}^n c(\omega)\lambda_i\lambda(A_i) = c(\omega)\sum\limits_{i=1}^n \lambda_i\lambda(A_i) = c(\omega)\int\limits_\Omega f(\omega)d\lambda.$

Now let $f \in L_1(\Omega)$. Then there exists a sequence f_n of simple functions such, that

$$\int_\Omega f(\omega)d\lambda = \lim_{n\to\infty} \int_\Omega f_n(\omega)d\lambda.$$

Hence
$\int_\Omega c(\omega)f(\omega)d\lambda = \lim_{n\to\infty} \int_\Omega c(\omega)f_n(\omega)d\lambda = c(\omega)\lim_{n\to\infty} \int_\Omega f_n(\omega)d\lambda = c(\omega)\int_\Omega f(\omega)d\lambda.$ □

Theorem 3.9. *(Dominated convergence) Let u_n be a sequence of sections, each of which is Bochner integrable and there exist a section u and an integrable function g such*

1) $\lim_{n\to\infty} u_n(\omega) = u(\omega)$ for almost all $\omega \in \Omega$;

2)$\|u_n(\omega)\|_{X(\omega)} \le |g(\omega)|$ for almost all $\omega \in \Omega$.

Then u is Bochner integrable and $\lim_{n\to\infty} \int_\Omega \|u_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda = 0$. In particular we have that

$$\lim_{n\to\infty} \int_\Omega u_n(\omega)d\lambda = \int_\Omega u(\omega)d\lambda.$$

Proof. Since $\|u_n(\omega)\|_{X(\omega)} \to \|u(\omega)\|_{X(\omega)}$ almost everywhere on Ω, we get that $\|u(\omega)\|_{X(\omega)} \le |g(\omega)|$. Therefore, $\|u_n(\omega) - u(\omega)\|_{X(\omega)} \le 2|g(\omega)|$ for almost all $\omega \in \Omega$ and the result follows from the scalar dominated convergence theorem. □

Let L be the set of sections from Definition 2.2.

Theorem 3.10. *(Hille) Let $T_\omega : X(\omega) \to X(\omega)$, $\omega \in \Omega$ is a family of bounded linear operators, such that $T_\omega(c(\omega)) \in L$ for any $c \in L$ and $\|T_\omega\| \le 1$. If section u is Bochner integrable, then the section $T_\omega(u(\omega))$ is Bochner integrable and*

$$\int_\Omega T_\omega(u(\omega))d\lambda = T_\omega\left(\int_\Omega u(\omega)d\lambda\right)$$

Proof. Let s be simple section. Then

$$T_\omega(s(\omega)) = \begin{cases} T_\omega(c_1(\omega)), & \text{if } \omega \in A_1; \\ T_\omega(c_2(\omega)), & \text{if } \omega \in A_2; \\ \dots\dots\dots & \dots\dots\dots \\ T_\omega(c_n(\omega)), & \text{if } \omega \in A_n. \end{cases}$$

Since $T_\omega(c_i(\omega)) \in L$, we have that $T_\omega(s(\omega))$ is simple section. Therefore, $T_\omega(s(\omega))$ is Bochner integrable and

$$\int_\Omega T_\omega s(\omega)d\lambda = \sum_{i=1}^n T_\omega(c_i(\omega))\lambda(A_i) = T_\omega\left(\sum_{i=1}^n c_i(\omega)\lambda(A_i)\right) = T_\omega\left(\int_\Omega s(\omega)d\lambda\right)$$

for almost all $\omega \in \Omega$.

Now let section u be Bochner integrable. Then there exists a sequence simple sections s_n such that $\|s_n(\omega) - u(\omega)\|_{X(\omega)} \to 0$ for almost all $\omega \in \Omega$ and $\lim_{n\to\infty} \int_\Omega \|s_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda = 0.$

Since $\|T_\omega\| \leq 1$ we have that

$$\|T_\omega(s_n(\omega)) - T_\omega(u(\omega))\|_{X(\omega)} \leq \|s_n(\omega) - u(\omega)\|_{X(\omega)}$$

and

$$\|T_\omega(s_n(\omega)) - T_\omega(u(\omega))\|_{X(\omega)} \to 0$$

for almost all $\omega \in \Omega$. From equality

$$\int\limits_\Omega \|T_\omega(s_n(\omega)) - T_\omega(u(\omega))\|_{X(\omega)} d\lambda \leq \int\limits_\Omega \|s_n(\omega) - u(\omega)\|_{X(\omega)} d\lambda$$

we obtain that

$$\lim_{n\to\infty} \int\limits_\Omega \|T_\omega(s_n(\omega)) - T_\omega(u(\omega))\|_{X(\omega)} d\lambda = 0.$$

As $T_\omega(s_n(\omega))$ is sequence of simple sections, $T_\omega(u(\omega))$ is Bochner integrable. Then

$$\int\limits_\Omega T_\omega(u(\omega)) d\lambda = \lim_{n\to\infty} \int\limits_\Omega T_\omega(s_n(\omega)) d\lambda = \lim_{n\to\infty} T_\omega \left(\int\limits_\Omega s_n(\omega) d\lambda \right) =$$

$$T_\omega \left(\lim_{n\to\infty} \int\limits_\Omega s_n(\omega) d\lambda \right) = T_\omega \left(\int\limits_\Omega u(\omega) d\lambda \right).$$

□

Let $p \geq 1$. We define by $L_p(\Omega, X)$ all classes measurable sections for which $\int\limits_\Omega \|u(\omega)\|^p_{X(\omega)} d\lambda < \infty$, i.e.

$$L_p(\Omega, X) = \{u \in L_0(\Omega, X) : \|u\|^p \in L_1(\Omega)\}.$$

Then $L_p(\Omega, X)$ is a Banach-Kantorovich space over $L_p(\Omega)$ (see [14]) and according to [18, Theorem 7.13 (2)] it is Banach space with respect to the mixed norm

$$\|u\|_p = \left\| \|u\| \right\| = \left(\int\limits_\Omega \|u(\omega)\|^p_{X(\omega)} d\lambda \right)^{\frac{1}{p}}.$$

Let $\varphi_1, \varphi_2, ..., \varphi_n \in L_p$ and $c_1, c_2, ..., c_n \in L$, $m_i = \sup\limits_{\omega \in dom(c_i)} \|c_i(\omega)\|_{X(\omega)} < \infty$ for any $i = 1, 2, ..., n$. Then we can define a section $u : \Omega \to X(\omega)$ in $L_p(\Omega, X)$ by setting $u(\omega) = \sum\limits_{i=1}^n \varphi_i(\omega) c_i(\omega)$ for almost all $\omega \in \Omega$. We will denote by $L_p(\Omega) \bigotimes L = \{u : u(\omega) = \sum\limits_{i=1}^n \varphi_i(\omega) c_i(\omega)\}$ subspace of $L_p(\Omega, X)$.

Theorem 3.11. *The subspace $L_p(\Omega) \bigotimes L$ is dense in $L_p(\Omega, X)$.*

Proof. Let $u \in L_p(\Omega, X)$ and $s_n(\omega)$ be the sequence step sections from $L_p \bigotimes L$ such that $s_n(\omega) \to u(\omega)$ a.e. on Ω. Then $\|s_n(\omega)\|_{X(\omega)} \to \|u(\omega)\|_{X(\omega)}$ almost everywhere on Ω. Let $A_n = \{\omega : \|s_n(\omega)\|_{X(\omega)} < 2\|u(\omega)\|_{X(\omega)}\}$. If we set $g_n(\omega) = s_n(\omega)\chi_{A_n}(\omega)$ we have that $g_n(\omega) \to u(\omega)$ a.e. on Ω and

$$\sup_n \|g_n(\omega) - u(\omega)\|_{X(\omega)} \leq \sup_n \|g_n(\omega)\|_{X(\omega)} + \|u(\omega)\|_{X(\omega)} \leq 3\|u(\omega)\|_{X(\omega)}.$$

By dominated convergence we have that

$$\int_\Omega \|g_n(\omega) - u(\omega)\|^p_{X(\omega)} d\lambda \to 0$$

and of course $g_n \in L_p(\Omega) \bigotimes L$. □

Acknowledgement. This work has been supported by IIUM-RMC through projects EDW B 11-185-0663 and EDW B 11-199-0677.

References

1. S. Bochner, *Integration von Functionen, deren Werte die Elemete eines Vectorraumes sind*, Fund. Math. **20** (1933), 262–276.
2. B.J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), no. 2, 277–304.
3. R.S. Phillips. *Integration in a convex linear topological space*, Trans. Amer. Math. Soc. **47** (1940), 114–145.
4. C.E. Rickartd *Integration in a convex linear topological space*, Trans. Amer. Math. Soc. **52** (1942), 498–521.
5. E. Hille and R.S. Phillips, *Functional Analysis and Semi-Groups*, Providence, 1957.
6. K. Yosida *Functional Analysis*, Reprint of the sixth (1980) edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
7. V.I. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs, 62, American Mathematical Society, Providence, RI, 1998.
8. N.N. Vakhania, V.I. Tarieladze and S.A. Chobanyan, *Probability Distributions on Banach Spaces*, Mathematics and its Applications, 14, D. Reidel Publishing Co., Dordrecht, 1987.
9. S. Schwabik, *Topics in Banach Space Integration*, Series in Real Analysis, World Scientific Publishing Company, 2005.
10. W. Arendt and C.J.K. Batty, M. Hieber and F. Neubrander, *Vectorvalued Laplace Transforms and Cauchy problems*, Monographs in Mathematics, 96, Birkhauser Verlag, Basel, 2001.
11. J. Diestel and J.J. Uhl, Jr.,*Vector Measures*, American Mathematical Society, Providence, R.I., 1977.
12. N. Dunford and J.T. Schwartz, *Linear Operators, Part I: General Theory*, Wiley Classics Library, John Wiley and Sons Inc., New York, 1988.
13. J. Neumann, *On rings of operators III*, Ann. Math. **41** (1940), 94–161.
14. A.E. Gutman, *Banach bundles in the theory of lattice-normed spaces. II. Measurable Banach bundles*, Siberian Adv. Math. **3** (1993), no. 4, 8–40.
15. A.E. Gutman, *Banach fiberings in the theory of lattice-normed spaces* (Russian), Order-compatible linear operators (Russian), 63–211, 293, Trudy Inst. Mat., 29, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 1995.
16. I.G. Ganiev, *Measurable bundles of lattices and their applications* (Russian), Investigations in functional analysis and its applications (Russian), 9–49, "Nauka", Moscow, 2006.
17. A.G. Kusraev, *Measurable bundles of Banach lattices*, Positivity **12** (2010), 785–799.
18. A.G. Kusraev, *Dominated Operators*, Mathematics and and its Applications, Kluwer Academic Publishers, Dordrecht 2000.

Department of Science in Engineering, Faculty of Engineering, International Islamic University Malaysia, P.O. Box 10, 50728 Kuala Lumpur, Malaysia.
E-mail address: inam@iium.edu.my
E-mail address: gharib@iium.edu.my