



## 2-LOCAL DERIVATIONS ON ALGEBRAS OF LOCALLY MEASURABLE OPERATORS

SHAVKAT ABDULLAEVICH AYUPOV<sup>1</sup>, KARIMBERGEN KUDAYBERGENOV<sup>2\*</sup> AND AMIR ALAUADINOV<sup>3</sup>

Communicated by Z. Lykova

ABSTRACT. The paper is devoted to 2-local derivations on the algebra  $LS(M)$  of all locally measurable operators affiliated with a type  $I_\infty$  von Neumann algebra  $M$ . We prove that every 2-local derivations on any  $*$ -subalgebra  $\mathcal{A}$  in  $LS(M)$ , such that  $M \subseteq \mathcal{A}$ , is a derivation.

### 1. INTRODUCTION

Given an algebra  $\mathcal{A}$ , a linear operator  $D : \mathcal{A} \rightarrow \mathcal{A}$  is called a *derivation*, if  $D(xy) = D(x)y + xD(y)$  for all  $x, y \in \mathcal{A}$  (the Leibniz rule). Each element  $a \in \mathcal{A}$  implements a derivation  $D_a$  on  $\mathcal{A}$  defined as  $D_a(x) = [a, x] = ax - xa$ ,  $x \in \mathcal{A}$ . Such derivations  $D_a$  are said to be *inner derivations*. If the element  $a$ , implementing the derivation  $D_a$ , belongs to a larger algebra  $\mathcal{B}$  containing  $\mathcal{A}$ , then  $D_a$  is called a *spatial derivation* on  $\mathcal{A}$ .

There exist various types of linear operators which are close to derivations [8, 9, 14]. In particular R. Kadison [8] has introduced and investigated so-called local derivations on von Neumann algebras and some polynomial algebras.

A linear operator  $\Delta$  on an algebra  $\mathcal{A}$  is called a *local derivation* if given any  $x \in \mathcal{A}$  there exists a derivation  $D$  (depending on  $x$ ) such that  $\Delta(x) = D(x)$ . The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations [8]. In particular Kadison [8] has proved that each continuous local derivation from a von Neumann algebra  $M$  into a dual  $M$ -bimodule is a derivation.

---

*Date:* Received: 26 September 2012; Accepted: 17 January 2013.

\* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 46L51; Secondary 47B47.

*Key words and phrases.* measurable operator, derivation, 2-local derivation.

In 1997, P. Semrl [14] introduced the concept of 2-local derivations and automorphisms. A map  $\Delta : \mathcal{A} \rightarrow \mathcal{A}$  (not linear in general) is called a *2-local derivation* if for every  $x, y \in \mathcal{A}$ , there exists a derivation  $D_{x,y} : \mathcal{A} \rightarrow \mathcal{A}$  such that  $\Delta(x) = D_{x,y}(x)$  and  $\Delta(y) = D_{x,y}(y)$ . Local and 2-local derivations have been studied on different operator algebras by many authors [2, 3, 4, 5, 7, 8, 9, 10, 11, 14].

In [14], P. Semrl described 2-local derivations on the algebra  $B(H)$  of all bounded linear operators on the infinite-dimensional separable Hilbert space  $H$ . A similar description for the finite-dimensional case appeared later in [9]. In the paper [11] 2-local derivations have been described on matrix algebras over finite-dimensional division rings. J. H. Zhang and H. X. Li [17] described 2-local derivations on symmetric digraph algebras and constructed a 2-local derivation on the algebra of all upper triangular complex  $2 \times 2$ -matrices which is not a derivation. In [3] first two authors considered 2-local derivations on the algebra  $B(H)$  of all linear bounded operators on an arbitrary (no separability is assumed) Hilbert space  $H$  and proved that every 2-local derivation on  $B(H)$  is a derivation.

The present paper is devoted to study 2-local derivations on  $*$ -subalgebras of the algebra  $LS(M)$  of all locally measurable operators with respect to type  $I_\infty$  von Neumann algebra  $M$ . We prove that every 2-local derivations on any  $*$ -subalgebra  $\mathcal{A}$  in  $LS(M)$ , such that  $M \subseteq \mathcal{A}$ , is a derivation.

## 2. ALGEBRA OF LOCALLY MEASURABLE OPERATORS

Let  $B(H)$  be the  $*$ -algebra of all bounded linear operators on a Hilbert space  $H$ , and let  $\mathbf{1}$  be the identity operator on  $H$ . Consider a von Neumann algebra  $M \subset B(H)$ . Denote by  $P(M) = \{p \in M : p = p^2 = p^*\}$  the lattice of all projections in  $M$  and by  $P_{fin}(M)$  the set of all finite projections in  $P(M)$ .

A linear subspace  $\mathcal{D}$  in  $H$  is said to be *affiliated* with  $M$  (denoted as  $\mathcal{D}\eta M$ ), if  $u(\mathcal{D}) \subset \mathcal{D}$  for every unitary  $u$  from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra  $M$ .

A linear operator  $x : \mathcal{D}(x) \rightarrow H$ , where the domain  $\mathcal{D}(x)$  of  $x$  is a linear subspace of  $H$ , is said to be *affiliated* with  $M$  (denoted as  $x\eta M$ ) if  $\mathcal{D}(x)\eta M$  and  $u(x(\xi)) = x(u(\xi))$  for all  $\xi \in \mathcal{D}(x)$  and for every unitary  $u \in M'$ .

A linear subspace  $\mathcal{D}$  in  $H$  is said to be *strongly dense* in  $H$  with respect to the von Neumann algebra  $M$ , if

- $\mathcal{D}\eta M$ ;
- there exists a sequence of projections  $\{p_n\}_{n=1}^\infty$  in  $P(M)$  such that  $p_n \uparrow \mathbf{1}$ ,  $p_n(H) \subset \mathcal{D}$  and  $p_n^\perp = \mathbf{1} - p_n$  is finite in  $M$  for all  $n \in \mathbf{N}$ , where  $\mathbf{N}$  is the set of all natural numbers.

A closed linear operator  $x$  acting in the Hilbert space  $H$  is said to be *measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and  $\mathcal{D}(x)$  is strongly dense in  $H$ .

Denote by  $S(M)$  the set of all linear operators on  $H$ , measurable with respect to the von Neumann algebra  $M$ . If  $x \in S(M)$ ,  $\lambda \in \mathbf{C}$ , where  $\mathbf{C}$  is the field of complex numbers, then  $\lambda x \in S(M)$  and the operator  $x^*$ , adjoint to  $x$ , is also measurable

with respect to  $M$  (see [13]). Moreover, if  $x, y \in S(M)$ , then the operators  $x + y$  and  $xy$  are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators  $x$  and  $y$ , and are denoted by  $x \dot{+} y$  and  $x * y$ . It was shown in [13] that  $x \dot{+} y$  and  $x * y$  belong to  $S(M)$  and these algebraic operations make  $S(M)$  a  $*$ -algebra with the identity  $\mathbf{1}$  over the field  $\mathbf{C}$ . It is clear that,  $M$  is a  $*$ -subalgebra of  $S(M)$ . In what follows, the strong sum and the strong product of operators  $x$  and  $y$  will be denoted in the same way as the usual operations, by  $x + y$  and  $xy$ .

A closed linear operator  $x$  in  $H$  is said to be *locally measurable* with respect to the von Neumann algebra  $M$ , if  $x\eta M$  and there exists a sequence  $\{z_n\}_{n=1}^{\infty}$  of central projections in  $M$  such that  $z_n \uparrow \mathbf{1}$  and  $z_n x \in S(M)$  for all  $n \in \mathbf{N}$  (see [16]).

Denote by  $LS(M)$  the set of all linear operators that are locally measurable with respect to  $M$ . It was proved in [16] that  $LS(M)$  is a  $*$ -algebra over the field  $\mathbf{C}$  with the identity  $\mathbf{1}$  under the strong addition and the strong multiplication. In such a case,  $S(M)$  is a  $*$ -subalgebra in  $LS(M)$ . In the case where  $M$  is a finite von Neumann algebra or a factor, the algebras  $S(M)$  and  $LS(M)$  coincide. This is not true in the general case. In [12] the class of von Neumann algebras  $M$  has been described for which the algebras  $LS(M)$  and  $S(M)$  coincide.

We say that a measure  $\mu$  on a measure space  $(\Omega, \Sigma, \mu)$  has the direct sum property if there is a family  $\{\Omega_i\}_{i \in J} \subset \Sigma$ ,  $0 < \mu(\Omega_i) < \infty$ ,  $i \in J$ , such that for any  $A \in \Sigma$ ,  $\mu(A) < \infty$ , there exist a countable subset  $J_0 \subset J$  and a set  $B$  with zero measure such that  $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$ .

It is well-known (see e.g. [13]) that for each commutative von Neumann algebra  $M$  there exists a measure space  $(\Omega, \Sigma, \mu)$  with  $\mu$  having the direct sum property such that  $M$  is  $*$ -isomorphic to the algebra  $L^\infty(\Omega, \Sigma, \mu)$  of all (equivalence classes of) complex essentially bounded measurable functions on  $(\Omega, \Sigma, \mu)$  and in this case  $LS(M) = S(M) \cong L^0(\Omega, \Sigma, \mu)$ , where  $L^0(\Omega, \Sigma, \mu)$  is the algebra of all (equivalence classes of) complex measurable functions on  $(\Omega, \Sigma, \mu)$ .

Further we consider the algebra  $S(Z(M))$  of operators which are measurable with respect to the center  $Z(M)$  of the von Neumann algebra  $M$ . Since  $Z(M)$  is an abelian von Neumann algebra, it is  $*$ -isomorphic to  $L^\infty(\Omega, \Sigma, \mu)$  for an appropriate measure space  $(\Omega, \Sigma, \mu)$ . Therefore the algebra  $S(Z(M))$  coincides with  $Z(LS(M))$  and can be identified with the algebra  $L^0(\Omega, \Sigma, \mu)$ .

Let  $M$  be a von Neumann algebra. Given an element  $x \in LS(M)$ , the smallest projection  $p$  in  $M$  with  $xp = x$  is called the *right support* of  $x$  and denoted by  $r(x)$ . The *left support*  $l(x)$  is the smallest projection  $p$  in  $M$  with  $px = x$ . For a  $*$ -subalgebra  $\mathcal{A} \subset LS(M)$  denote

$$\mathcal{F}(\mathcal{A}) = \{x \in \mathcal{A} : l(x) \in P_{fin}(M)\}.$$

From the definition of the algebra  $\mathcal{F}(\mathcal{A})$  we have that the following properties are equivalent:

- (1)  $x \in \mathcal{F}(\mathcal{A})$ ;
- (2)  $\exists p \in P_{fin}(M)$  such that  $px = x$ ;
- (3)  $\exists p \in P_{fin}(M)$  such that  $xp = x$ ;

(4)  $\exists p \in P_{fin}(M)$  such that  $pxp = x$ .

Note that  $\mathcal{F}(A)$  is an  $*$ -ideal in  $\mathcal{A}$ . Moreover the algebra  $\mathcal{F}(\mathcal{A})$  is semi-prime, i.e. if  $a \in \mathcal{F}(\mathcal{A})$  and  $a\mathcal{F}(\mathcal{A})a = \{0\}$  then  $a = 0$ . Indeed, let  $a \in \mathcal{F}(\mathcal{A})$  and  $a\mathcal{F}(\mathcal{A})a = \{0\}$ , i.e.  $axa = 0$  for all  $x \in \mathcal{F}(\mathcal{A})$ . In particular for  $x = a^*$  we have  $aa^*a = 0$  and hence  $a^*aa^*a = 0$ , i.e.  $|a|^4 = 0$ . Therefore  $a = 0$ .

Recall the definition of the faithful normal semifinite extended center valued trace on the algebra  $M$  (see [15]).

Let  $M$  be an arbitrary von Neumann algebra with the center  $Z(M) \equiv L^\infty(\Omega, \Sigma, \mu)$  and let  $M_+$  be the set of all positive elements of  $M$ . By  $L_+$  we denote the set of all measurable functions  $f : (\Omega, \Sigma, \mu) \rightarrow [0, \infty]$  (modulo functions equal to zero  $\mu$ -almost everywhere). Then there exists a map  $\Phi : M_+ \rightarrow L_+$  with the following properties:

- (1)  $\Phi(x + y) = \Phi(x) + \Phi(y)$  for  $x, y \in M_+$ ;
- (2)  $\Phi(ax) = a\Phi(x)$  for  $a \in Z(M)_+, x \in M_+$ ;
- (3)  $\Phi(xx^*) = \Phi(x^*x)$ ;
- (4)  $\Phi(x^*x) = 0 \Rightarrow x = 0$ ;
- (5)  $\Phi\left(\sup_{i \in J} x_i\right) = \sup_{i \in J} \Phi(x_i)$  for any bounded increasing net  $\{x_i\}$  in  $M_+$ .

This map  $\Phi : M_+ \rightarrow L_+$ , is called the *extended center valued trace* on  $M$ .

The set  $\{x \in M : \Phi(x^*x) \in Z(M)\}$  is an ideal of  $M$ . If this ideal is  $\sigma$ -weakly dense in  $M$ , then  $\Phi$  is said to be *semifinite*.

It is well-known (see e.g. [15]) that a von Neumann algebra  $M$  is semifinite if and only if  $M$  admits a faithful, semifinite, normal extended center valued trace.

Let us remark that a projection  $p \in M$  is finite if and only if  $\Phi(p) \in S(Z(M))$ . Hence for any  $x \in \mathcal{F}(LS(M)) \cap M_+$  we have that  $\Phi(x) \in S(Z(M))$ .

Note that the algebra  $LS(M)$  has the following remarkable property: given any family  $\{z_i\}_{i \in I}$  of mutually orthogonal central projections in  $M$  with  $\bigvee_{i \in I} z_i = \mathbf{1}$  and a family of elements  $\{x_i\}_{i \in I}$  in  $LS(M)$  there exists a unique element  $x \in LS(M)$  such that  $z_i x = z_i x_i$  for all  $i \in I$ . This element is denoted by  $x = \sum_{i \in I} z_i x_i$  (see [12]). Conversely if  $M$  is a type I von Neumann algebra then for an arbitrary element  $x \in LS(M)$  there exists a sequence  $\{z_n\}$  of mutually orthogonal central projections with  $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$  such that  $z_n x \in M$  for all  $n \in \mathbb{N}$  (see [1]). For  $0 \leq x \in \mathcal{F}(LS(M))$  set

$$\Phi(x) = \sum_{n \in \mathbb{N}} z_n \Phi(z_n x). \quad (2.1)$$

Since the trace  $\Phi$  is  $Z(M)$ -homogeneous, the equality (2.1) gives a well-defined map from  $\mathcal{F}(LS(M))_+$  into  $S(Z(M))$ .

Since each element of  $\mathcal{F}(LS(M))$  is a finite linear combination of positive elements from  $\mathcal{F}(LS(M))_+$  we can naturally extend  $\Phi$  to a  $S(Z(M))$ -valued trace on  $\mathcal{F}(LS(M))$ .

Now let  $\mu$  be an arbitrary faithful normal semifinite trace on  $Z(M)$ . Put  $\tau = \mu \circ \Phi$ . Then by [15, Lemma 2.16] we have that

$$\tau(xy) = \tau(yx)$$

for all  $x \in M$ ,  $y \in \mathcal{F}(LS(M)) \cap M$ . Therefore

$$\Phi(xy) = \Phi(yx)$$

for all  $x \in LS(M)$ ,  $y \in \mathcal{F}(LS(M))$ . Since the trace  $\Phi$  maps the set  $\mathcal{F}(LS(M))$  into  $S(Z(M))$  and  $\mathcal{F}(LS(M))$  is an ideal in  $LS(M)$  we have

$$\Phi(axy) = \Phi((ax)y) = \Phi((ya)x) = \Phi(xya),$$

i.e.

$$\Phi(axy) = \Phi(xya) \tag{2.2}$$

for all  $a, x \in LS(M)$ ,  $y \in \mathcal{F}(LS(M))$ .

### 3. MAIN RESULTS

Let  $D$  be a derivation on  $LS(M)$ . Then  $D$  maps the ideal  $\mathcal{F}(LS(M))$  into itself. Indeed, for any  $x \in \mathcal{F}(LS(M))$  there exists a finite projection  $p \in M$  such that  $x = xp$ . Then

$$D(x) = D(xp) = D(x)p + xD(p),$$

and therefore  $D(x) \in \mathcal{F}(LS(M))$ . Hence any 2-local derivation on  $LS(M)$  also maps  $\mathcal{F}(LS(M))$  into itself.

**Lemma 3.1.** *Let  $M$  be an arbitrary semifinite von Neumann algebra and let  $b \in LS(M)$  be an arbitrary element. If  $\Phi(xb) = 0$  for all  $x \in \mathcal{F}(LS(M))$  then  $b = 0$ .*

*Proof.* Let  $b \in LS(M)$ . For any finite projection  $e \in LS(M)$  we have  $eb^* \in \mathcal{F}(LS(M))$  and therefore by the assumption of the lemma it follows that  $\Phi(eb^*b) = 0$ . Thus

$$0 = \Phi(eb^*b) = \Phi(e^2b^*b) = \Phi(eb^*be) = \Phi((be)^*(be)),$$

i.e.

$$\Phi((be)^*(be)) = 0.$$

Since the trace  $\Phi$  is faithful, we obtain  $(be)^*(be) = 0$ , i.e.  $be = 0$ .

Now take a family of finite projections  $\{e_\alpha\}_{\alpha \in J}$  in  $M$  such that  $e_\alpha \uparrow \mathbf{1}$ . Then

$$0 = be_\alpha b^* \uparrow bb^*,$$

i.e.  $bb^* = 0$ . Thus  $b = 0$ . The proof is complete.  $\square$

**Lemma 3.2.** *Let  $M$  be an arbitrary von Neumann algebra of type  $I_\infty$  and let  $\Delta : LS(M) \rightarrow LS(M)$  be a 2-local derivation. Then*

- (1)  $\Delta$  is  $S(Z(M))$ -homogenous, i.e.  $\Delta(cx) = c\Delta(x)$  for all  $c \in S(Z(M))$ ,  $x \in LS(M)$ ;
- (2)  $\Delta(x^2) = \Delta(x)x + x\Delta(x)$  for all  $x \in LS(M)$ .

*Proof.* (1) For each  $x \in LS(M)$ , and for  $c \in S(Z(M))$  there exists a derivation  $D_{x,cx}$  such that  $\Delta(x) = D_{x,cx}(x)$  and  $\Delta(cx) = D_{x,cx}(cx)$ . Since  $M$  is a type  $I_\infty$  then by [1, Theorem 2.7] every derivation on  $LS(M)$  is inner, in particular,  $S(Z(M))$ -linear. Therefore

$$\Delta(cx) = D_{x,cx}(cx) = cD_{x,cx}(x) = c\Delta(x).$$

Hence,  $\Delta$  is  $S(Z(M))$ -homogenous.

(2) For each  $x \in LS(M)$ , there exists a derivation  $D_{x,x^2}$  such that  $\Delta(x) = D_{x,x^2}(x)$  and  $\Delta(x^2) = D_{x,x^2}(x^2)$ . Then

$$\Delta(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \Delta(x)x + x\Delta(x)$$

for all  $x \in LS(M)$ . The proof is complete.  $\square$

**Lemma 3.3.** *Let  $M$  be an arbitrary von Neumann algebra of type  $I_\infty$ . If  $\Delta : LS(M) \rightarrow LS(M)$  is a 2-local derivation such that  $\Delta|_{\mathcal{F}(LS(M))} \equiv 0$ , then  $\Delta \equiv 0$ .*

*Proof.* Let  $\Delta : LS(M) \rightarrow LS(M)$  be a 2-local derivation such that  $\Delta|_{\mathcal{F}(LS(M))} \equiv 0$ . For arbitrary  $x \in LS(M)$  and  $y \in \mathcal{F}(LS(M))$  there exists a derivation  $D_{x,y}$  on  $LS(M)$  such that  $\Delta(x) = D_{x,y}(x)$  and  $\Delta(y) = D_{x,y}(y)$ . By [1, Theorem 2.7] there exists element  $a \in LS(M)$  such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Since  $y \in \mathcal{F}(LS(M))$  we have  $\Delta(y) = 0$ , and therefore  $[a, xy] = \Delta(x)y$ . By the equality (2.2) we obtain that

$$0 = \Phi(axy - xy a) = \Phi([a, xy]) = \Phi(\Delta(x)y),$$

i.e.  $\Phi(\Delta(x)y) = 0$  for all  $y \in \mathcal{F}(LS(M))$ . By Lemma 3.1 we have that  $\Delta(x) = 0$ . The proof is complete.  $\square$

**Lemma 3.4.** *Let  $M$  be an arbitrary von Neumann algebra of type  $I_\infty$  and let  $\Delta : LS(M) \rightarrow LS(M)$  be a 2-local derivation. Then the restriction  $\Delta|_{\mathcal{F}(LS(M))}$  of the operator  $\Delta$  on  $\mathcal{F}(LS(M))$  is additive.*

*Proof.* Let  $\Delta : LS(M) \rightarrow LS(M)$  be a 2-local derivation. For each  $x, y \in \mathcal{F}(LS(M))$  there exists a derivation  $D_{x,y}$  on  $LS(M)$  such that  $\Delta(x) = D_{x,y}(x)$  and  $\Delta(y) = D_{x,y}(y)$ . By [1, Theorem 2.7] there exists an element  $a \in LS(M)$  such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \Delta(x)y + x\Delta(y),$$

i.e.

$$[a, xy] = \Delta(x)y + x\Delta(y).$$

Similarly as in Lemma 3.3 we have

$$0 = \Phi(axy - xy a) = \Phi([a, xy]) = \Phi(\Delta(x)y + x\Delta(y)),$$

i.e.  $\Phi(\Delta(x)y) = -\Phi(x\Delta(y))$ . For arbitrary  $u, v, w \in \mathcal{F}(LS(M))$ , set  $x = u + v$ ,  $y = w$ . Then from above we obtain

$$\begin{aligned} \Phi(\Delta(u+v)w) &= -\Phi((u+v)\Delta(w)) = \\ &= -\Phi(u\Delta(w)) - \Phi(v\Delta(w)) = \Phi(\Delta(u)w) + \Phi(\Delta(v)w) = \Phi((\Delta(u) + \Delta(v))w), \end{aligned}$$

and so

$$\Phi((\Delta(u+v) - \Delta(u) - \Delta(v))w) = 0$$

for all  $u, v, w \in \mathcal{F}(LS(M))$ . Denote  $b = \Delta(u + v) - \Delta(u) - \Delta(v)$  and put  $w = b^*$ . Then  $\Phi(bb^*) = 0$ . Since the trace  $\Phi$  is faithful it follows that  $bb^* = 0$ , i.e.  $b = 0$ . Therefore

$$\Delta(u + v) = \Delta(u) + \Delta(v),$$

i.e.  $\Delta$  is an additive map on  $\mathcal{F}(LS(M))$ . The proof is complete.  $\square$

The following theorem is the main result of this paper.

**Theorem 3.5.** *Let  $M$  be an arbitrary von Neumann algebra of type  $I_\infty$  and let  $\mathcal{A}$  be a  $*$ -subalgebra of  $LS(M)$  such that  $M \subseteq \mathcal{A}$ . Then every 2-local derivation  $\Delta : \mathcal{A} \rightarrow \mathcal{A}$  is a derivation.*

*Proof.* First we consider the case  $\mathcal{A} = LS(M)$ . By Lemma 3.4 the restriction  $\Delta|_{\mathcal{F}(LS(M))}$  of the operator  $\Delta$  on  $\mathcal{F}(LS(M))$  is additive. Further by Lemma 3.2  $\Delta$  is a homogeneous, and therefore  $\Delta|_{\mathcal{F}(LS(M))}$  is a linear. Again by Lemma 3.4 we have  $\Delta(x^2) = \Delta(x)x + x\Delta(x)$  for all  $x \in LS(M)$ . So the map  $\Delta|_{\mathcal{F}(LS(M))}$  is a linear Jordan derivation on  $\mathcal{F}(LS(M))$  in the sense of [6]. In [6, Theorem 1] it is proved that any Jordan derivation on a semi-prime algebra is a derivation. Since  $\mathcal{F}(LS(M))$  is semiprime, therefore the linear operator  $\Delta|_{\mathcal{F}(LS(M))}$  is a derivation on  $\mathcal{F}(LS(M))$ .

Since by Lemma 3.2  $\Delta$  is  $S(Z(M))$ -homogeneous then by [4, Corollary 3] the derivation  $\Delta|_{\mathcal{F}(LS(M))} : \mathcal{F}(LS(M)) \rightarrow \mathcal{F}(LS(M))$  is spatial, i.e.

$$\Delta(x) = ax - xa, \quad x \in \mathcal{F}(LS(M)) \quad (3.1)$$

for an appropriate  $a \in LS(M)$ .

Let us show that  $\Delta(x) = ax - xa$  for all  $x \in LS(M)$ . Consider the 2-local derivation  $\Delta_0 = \Delta - D_a$ . Then from the equality (3.1) we obtain that  $\Delta_0|_{\mathcal{F}(LS(M))} \equiv 0$ . Now by Lemma 3.3 it follows that  $\Delta_0 \equiv 0$ . This means that  $\Delta = D_a$ .

Now let  $\mathcal{A}$  be an arbitrary  $*$ -subalgebra of  $LS(M)$  such that  $M \subseteq \mathcal{A}$ . Since  $M$  is a type I von Neumann algebra for any element  $x \in LS(M)$  there exists a sequence  $\{z_n\}$  of mutually orthogonal central projections with  $\bigvee_{n \in \mathbb{N}} z_n = \mathbf{1}$  such that  $z_n x \in M$  for all  $n \in \mathbb{N}$ . Set

$$\tilde{\Delta}(x) = \sum_{n \in \mathbb{N}} z_n \Delta(z_n x). \quad (3.2)$$

Since the map  $\Delta$  is  $Z(M)$ -homogeneous, the equality (3.2) gives a well-defined 2-local derivation on  $LS(M)$ . From above we have that  $\tilde{\Delta}$  is a derivation. Therefore  $\Delta$  is a derivation. The proof is complete.  $\square$

**Corollary 3.6.** *Let  $M$  be an arbitrary von Neumann algebra of type  $I_\infty$ . Then every 2-local derivation  $\Delta : LS(M) \rightarrow LS(M)$  is a derivation.*

**Acknowledgement.** The second named author would like to acknowledge the hospitality of the Chern Institute of Mathematics, Nankai University (China). The authors are indebted to referees for useful remarks.



## REFERENCES

1. S. Albeverio, Sh.A. Ayupov and K.K. Kudaybergenov, *Structure of derivations on various algebras of measurable operators for type I von Neumann algebras*, J. Funct. Anal. **256** (2009), 2917–2943.
2. S. Albeverio, Sh. A. Ayupov, K.K. Kudaybergenov and B.O. Nurjanov, *Local derivations on algebras of measurable operators*, Commun. Contemp. Math. **13** (2011), 643–657.
3. Sh.A. Ayupov and K.K. Kudaybergenov, *2-local derivations and automorphisms on  $B(H)$* , J. Math. Anal. Appl. **395** (2012), 15–18.
4. Sh.A. Ayupov, K.K. Kudaybergenov and B.O. Nurjanov, *Local derivations on algebras of measurable operators*, Uzbek. Mat. Zh. **13** (2010), 643–657.
5. Sh.A. Ayupov and K.K. Kudaybergenov, B. O. Nurjanov, A. K. Alauatdinov, *Local and 2-local derivations on noncommutative Arens algebras*, Math. Slovaca (to appear), preprint available at <http://arxiv.org/abs/1110.1557>.
6. M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), 1003–1006.
7. M. Brešar and P. Šemrl, *Mapping which preserve idempotents, local automorphisms, and local derivations*, Canad. J. Math. **45** (1999), 483–496.
8. R.V. Kadison, *Local derivations*, J. Algebra. **130** (1990), 494–509.
9. S.O. Kim and J.S. Kim, *Local automorphisms and derivations on  $M_n$* , Proc. Amer. Math. Soc. **132** (2004), 1389–1392.
10. D.R. Larson and A.R. Sourour, *Local derivations and local automorphisms of  $B(X)$* , Proc. Sympos. Pure Math. **51** (1990), 187–194.
11. Y. Lin and T. Wong, *A note on 2-local maps*, Proc. Edinb. Math. Soc. **49** (2006), 701–708.
12. M.A. Muratov and V.I. Chilin, *\*-Algebras of unbounded operators affiliated with a von Neumann algebra*, J. Math. Sci. (N. Y.) **140** (2007), 445–451.
13. I.A. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. **57** (1953), 401–457.
14. P. Šemrl, *Local automorphisms and derivations on  $B(H)$* , Proc. Amer. Math. Soc. **125** (1997), 2677–2680.
15. M. Takesaki, *Theory of operator algebras I*, Springer, Heidelberg, 1979.
16. F.J. Yeadon, *Convergence of measurable operators*, Proc. Camb. Phil. Soc. **74** (1973), 257–268.
17. J.H. Zhang and H.X. Li, *2-local derivations on digraph algebras*, Acta Math. Sin. (Chin. Ser.) **49** (2006), 1401–1406.

<sup>1</sup> INSTITUTE OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN, 100125 TASHKENT, UZBEKISTAN AND THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP), TRIESTE, ITALY

*E-mail address:* [sh\\_ayupov@mail.ru](mailto:sh_ayupov@mail.ru)

<sup>2</sup> DEPARTMENT OF MATHEMATICS, KARAKALPAK STATE UNIVERSITY, CH. ABDIROV 1, 230113, NUKUS, UZBEKISTAN

*E-mail address:* [karim2006@mail.ru](mailto:karim2006@mail.ru)

<sup>3</sup> INSTITUTE OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN, 100125 TASHKENT, UZBEKISTAN

*E-mail address:* [amir\\_t85@mail.ru](mailto:amir_t85@mail.ru)