Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 2, 33–48
http://doi.org/10.15352/afa/06-2-4
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

CYCLICITY FOR UNBOUNDED MULTIPLICATION OPERATORS IN L^p- AND C_0-SPACES

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Communicated by G. Androulakis

Abstract. For every, possibly unbounded, multiplication operator in L^p-space, $p \in]0, \infty[$, on finite separable measure space we show that multicyclicity, multi-$*$-cyclicity, and multiplicity coincide. This result includes and generalizes Bram's much cited theorem from 1955 on bounded $*$-cyclic normal operators. It also includes as a core result cyclicity of the multiplication operator M_z by the complex variable z in $L^p(\mu)$ for every σ-finite Borel measure μ on $\mathbb{C}$. The concise proof is based in part on the result that the function $e^{-|z|^2}$ is a $*$-cyclic vector for M_z in $C_0(\mathbb{C})$ and further in $L^p(\mu)$. We characterize topologically those locally compact sets $X \subset \mathbb{C}$, for which M_z in $C_0(X)$ is cyclic.

1. Introduction

In 1955 Bram [4] proves his well-known and much cited theorem that a bounded $*$-cyclic normal operator is cyclic. It is also well-known that, as a consequence, a normal operator is cyclic if and only if it has multiplicity one or, equivalently, if it is simple. Only in 2009 Nagy [17] tackles the generalization of Bram's result to unbounded normal operators.

Due to the spectral theorem the question actually concerns multiplication operators in $L^2(\mu)$ for finite Borel measures on $\mathbb{C}$ with possibly unbounded support. We extend the frame to general (unbounded) multiplication operators in L^p-spaces for $p \in]0, \infty[$ on finite separable measure spaces. We prove that multicyclicity, multi-$*$-cyclicity, and multiplicity coincide for those operators.

Date: Received: Jun 20, 2013; Revised: Feb. 19, 2014; Accepted: Apr. 21, 2014.
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2010 *Mathematics Subject Classification.* Primary 47A16; Secondary 41A10, 47B15.
Key words and phrases. Bram's theorem, unbounded normal operator, multiplicity, cyclic vector, uniform approximation.

This result includes cyclicity of the multiplication operator M_z by z in $L^p(\mu)$ for any finite Borel measure μ on $\mathbb{C}$, which in turn is a main step in the proof of the above result.

Let us rapidly recall the case of bounded M_z in $L^2(\mu)$. Here $\operatorname{supp}(\mu)$ is compact. Then the set $\Pi(z,\overline{z})$ of polynomials in z and $\bar{z}$ is dense in $L^2(\mu)$, since $\Pi(z,\overline{z})$ is dense in $C(\operatorname{supp}(\mu))$ by the theorem of Stone/Weierstraß. Therefore, if $\Pi(z,\bar{z})$ is contained in the closure of the polynomials $\Pi(z)$, the latter are dense in $L^2(\mu)$, i.e., $\overline{\Pi(z)} = L^2(\mu)$, which means that the constant $1_{\mathbb{C}}$ is a cyclic vector for M_z. Actually, still due to the boundedness of M_z, it suffices to show that the function $\bar{z}$ is element of the closure of $\Pi(z)$. Bram [4] solves this approximation problem decomposing $\mathbb{C}$ into the union of an increasing sequence of α-sets and a μ-null set. An α-set is a compact subset of $\mathbb{C}$ such that every continuous function on it can be approximated uniformly by polynomials in z. By Lavrentev's theorem (see, e.g. [5, 11]) the α-sets are just the compact subsets of $\mathbb{C}$ with empty interior and connected complement.

In the unbounded case this way has to be modified, mostly by two reasons. First, due to unboundedness, the support of μ is not compact and the polynomials in z and $\bar{z}$ are not bounded on $\operatorname{supp}(\mu)$. Secondly, $\Pi(z,\overline{z})$ need not be dense in $L^2(\mu)$ (see e.g. Hamburger's example in Simon [23, example 1.3]). In [17] Nagy generalizes Bram's decomposition of $\mathbb{C}$ for any (non-compact) $\operatorname{supp}(\mu)$. This is an important result. We have considerably simplified its proof. By this, [17] succeeds in showing that $\bar{z}$ is in the closure of $\Pi(z)$ in $L^p(\sigma)$ for some finite Borel measure σ equivalent to μ. We proceed similarly to [17], but show at once by an induction argument that the whole of $\Pi(z,\bar{z})$ lies in the closure of $\Pi(z)$.

The ensuing tacit assumption by Nagy that $z\,\overline{\Pi(z)} \subset L^p(\sigma)$ however, as we will explain below, definitely restricts the proof in [17] to the case of bounded M_z, thus missing the aim.

In the unbounded case, in the literature there seems even to exist no explicit proof for $*$-cyclicity of M_z in $L^2(\mu)$, and there are several futile attempts in the literature concerning the Hilbert space case. However, Agricola/Friedrich [2, sec. 3] show that the functions $p\,\mathrm{e}^{-|x|^2}$, p polynomial on $\mathbb{R}^d$, are dense in $C_0(\mathbb{R}^d)$ with respect to uniform convergence. In particular this means that the function $e^{-|z|^2}$ is a $*$-cyclic vector for M_z in $C_0(\mathbb{C})$. As a ready consequence, $e^{-|z|^2}$ is $*$-cyclic for M_z in $L^p(\mu)$. We like to remark that we present in (4.3) a short classical proof of the density result of [2] (which is central in [2]) and that we apply successfully the same method for the proof of other results on cyclicity. Another proof for (4.3) can be found in [24]. Moreover, we add in (4) a direct proof of $*$–cyclicity of M_z in $L^2(\mu)$. It generalizes a proof in [1] for the self-adjoint case $\mu(\mathbb{C}\setminus\mathbb{R}) = 0$. We are indebted to the referee for having drawn our attention to [1].

Another important ingredient is the Rohlin decomposition of a measurable function which we apply to unbounded functions on finite separable measure spaces.

We get started on the multiplication operator M_z in $C_0(\mathbb{C})$ and extend also to M_z in $C_0(X)$ for locally compact $X \subset \mathbb{C}$. We find that M_z is $*$-cyclic and describe topologically those X, for which M_z is cyclic.

Finally, it is worth mentioning that the results on $(*)$-cyclicity for the most part are obtained by polynomial approximation, thus contributing to this field. We shall give some examples.

2. Definitions and Notations

Two measurable spaces $(\Omega, \mathcal{A})$, $(\Omega', \mathcal{A}')$ are said to be *measurable space isomorphic*, if there is a bijection $\iota : \Omega \to \Omega'$ such that ι and ι^{-1} are measurable. A *standard measurable space* is a measurable space which is isomorphic with a Polish space provided with the Borel σ–algebra. A *Borel measure* is a measure on the σ-algebra of Borel sets of a topological space.

A measure space $(\Omega, \mathcal{A}, \mu)$ is said to be *separable* if there is countable subset $\mathcal{D} \subset \mathcal{A}$ such that for all $A \in \mathcal{A}$, $\epsilon > 0$ there is $\Delta \in \mathcal{D}$ with $\mu(A \triangle \Delta) < \epsilon$. — For every measure space $(\Omega, \mathcal{A}, \mu)$ there is the equivalence relation $A \sim B \Leftrightarrow \mu(A \triangle B) = 0$ on $\mathcal{A}$. Let the set of equivalence classes $[A]$ be denoted by $\mathcal{A}/\mathcal{N}$ with $\mathcal{N}$ the ideal of null sets. The measure μ is constant on every equivalence class, and all set theoretical operations on $\mathcal{A}$ as well set inclusion are carried over to $\mathcal{A}/\mathcal{N}$ since they are compatible with the equivalence relation. So $(\mathcal{A}/\mathcal{N}, \mu)$ together with this structure is called the associated *measure algebra*. Moreover, let $M(\mu)$ denote the *function algebra* of all classes $[\varphi]$ of measurable functions $\varphi : \Omega \to \mathbb{C}$ modulo μ–a.e. vanishing functions. — Given a further measure space $(\Omega', \mathcal{A}', \mu')$, a bijection $T : \mathcal{A}/\mathcal{N} \to \mathcal{A}'/\mathcal{N}'$ preserving the measure algebra structure is called a *measure algebra isomorphism* if $\mu' \circ T = \mu$. — Every such isomorphism T induces a *function algebra isomorphism* $\tau : M(\mu) \to M(\mu')$ determined by $\tau([1_\Delta]) = [1_{\Delta'}]$ with $\Delta' \in T([\Delta])\ \forall\ \Delta \in \mathcal{A}$. It is multiplicative and preserves all p–metrics, $p \in]0, \infty]$. — A *measure space isomorphism* $\iota : (\Omega, \mathcal{A}, \mu) \to (\Omega', \mathcal{A}', \mu')$ is a measurable space isomorphism with $\mu' = \iota(\mu)$. It induces the measure algebra isomorphism $T([A]) := [\iota(A)]$ and the function algebra isomorphism $\tau([\varphi]) = [\varphi \circ \iota^{-1}]$. – In general we will omit the brackets $[\cdot]$.

Let $p \in]0, \infty[$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For a measurable function $\varphi : \Omega \to \mathbb{C}$ let M_φ denote the *multiplication operator* in $L^p(\mu)$ given by $M_\varphi f := \varphi f$ with domain $\mathcal{D}(M_\varphi) := \{f \in L^p(\mu) : \varphi f \in L^p(\mu)\}$. We will deal with separable L^p-spaces. Therefore it is no restriction to assume in the sequel that $(\Omega, \mathcal{A}, \mu)$ is finite and separable. It is well-known that M_φ is closed and, if $p = 2$, normal. Moreover, M_φ is bounded if and only if φ is μ-essentially bounded.

A set $Z \subset L^p(\mu)$ is called *cyclic* for M_φ if $p(\varphi)f \in L^p(\mu)$ for all polynomials $p \in \Pi(z)$, $f \in Z$ and if

$$\Pi(M_\varphi)Z := \{p(\varphi)f : p \in \Pi(z), f \in Z\}$$

is dense in $L^p(\mu)$. If there is no finite cyclic set the *multicyclicity* $\mathrm{mc}(M_\varphi)$ is set ∞. Otherwise it is defined as the smallest number of elements of a cyclic set. M_φ is called cyclic if $\mathrm{mc}(M_\varphi) = 1$. Similarly, a set $Z \subset L^p(\mu)$ is called $*$*-cyclic* for M_φ if $p(\varphi, \overline{\varphi})f \in L^p(\mu)$ for all polynomials $p \in \Pi(z, \bar{z})$, $f \in Z$ and if

$$\Pi(M_\varphi, M_{\overline{\varphi}})Z := \{p(\varphi, \overline{\varphi})f : p \in \Pi(z, \bar{z}), f \in Z\}$$

is dense in $L^p(\mu)$. The *multi-*$*$*-cyclicity* $\mathrm{mc}^*(M_\varphi)$ is defined analogously and M_φ is called $*$-cyclic if $\mathrm{mc}^*(M_\varphi) = 1$.

We define the multiplicity of M_φ on $L^p(\mu)$ by means of the *Rohlin decomposition* (π,ν) of (φ,μ). Let us briefly explain this decomposition. See also Seid [21, remark 3.4]. There is a measure algebra isomorphism T from $(\Omega,\mathcal{A},\mu)$ onto $([0,1]\times\mathbb{C},\mathcal{B},\nu)$ with $\mathcal{B}$ the Borel sets and ν a finite measure. The latter satisfies

$$\nu=\lambda\otimes\mu_c+\sum_{n\in\mathbb{N}}\delta_{1/n}\otimes\mu_n,$$

where λ denotes the Lebesgue measure on $[0,1]$, $\delta_{1/n}$ is the point measure at $1/n$, and μ_c, μ_n are Borel measures on $\mathbb{C}$ with $\mu_{n+1}\ll\mu_n$ for $n\in\mathbb{N}$. Moreover, the function algebra isomorphism τ induced by T satisfies $\tau(\varphi)=\pi$, where $\pi(t,z):=z$, $(t,z)\in[0,1]\times\mathbb{C}$. This implies that M_φ in $L^p(\mu)$ is isomorphic with M_π in $L^p(\nu)$ by $\tau\, M_\varphi\tau^{-1}f=M_\pi f\ \forall\ f\in L^p(\nu)$. By these properties the measures μ_n for $n\in\mathbb{N}\cup\{c\}$ are uniquely determined up to equivalence. Since the measures $\lambda\otimes\mu_c$, $\delta_1\otimes\mu_1$, $\delta_{1/2}\otimes\mu_2,\dots$ are mutually orthogonal, $L^p(\nu)$ and M_π are identified with the p-direct sums

$$L^p(\lambda\otimes\mu_c)\oplus\bigoplus_{n\in\mathbb{N}}L^p(\mu_n),\qquad M_\pi\oplus\bigoplus_{n\in\mathbb{N}}M_z.$$

Then M_z in $L^p(\mu_n)$, $n\in\mathbb{N}$, is cyclic, whereas M_π on a subspace $\{1_Sf:f\in L^p(\lambda\otimes\mu_c)\}$ with $S\in\mathcal{B}$ is cyclic only if $\lambda\otimes\mu_c(S)=0$. Hence, in view of $\mu_{n+1}\ll\mu_n$ for $n\in\mathbb{N}$, the *multiplicity* of M_φ is defined as $\mathrm{mp}(M_\varphi):=\sup\{n\in\mathbb{N}:\mu_n\neq0\}$ if $\mu_c=0$ and ∞ otherwise.

As it should, this definition of multiplicity is invariant under L^p-isomorphisms, which is due to the following known fact, see Seid [20]. Let $p\in]0,\infty[\setminus\{2\}$, let $(\Omega,\mathcal{A},\mu)$, $(\Omega',\mathcal{A}',\mu')$ be two finite measure spaces, and suppose that $\iota:L^p(\mu)\to L^p(\mu')$ is an isomorphism. Then ι equals the composition $\beta\circ\alpha$ of two isomorphisms $\alpha:L^p(\mu)\to L^p(|\iota(1_\Omega)|^p\mu')$ and $\beta:L^p(|\iota(1_\Omega)|^p\mu')\to L^p(\mu')$ of rather special types. Indeed, α comes from a measure algebra isomorphism. For β one simply has $\beta(g)=\iota(1_\Omega)g$.

As to the case $p=2$ note that for $\mu_c\neq0$ the normal operator M_π in $L^2(\lambda\otimes\mu_c)$ is Hilbert space isomorphic with the countably infinite orthogonal sum of copies of M_z in $L^2(\mu_c)$. Hence multiplicity $\mathrm{mp}(M_\varphi)$ coincides with the usual multiplicity for a normal operator. Finally note that if for the Rohlin decomposition $\mu_c=0$ occurs, then the Rohlin decomposition is just the spectral decomposition of the normal operator M_φ in $L^2(\mu)$.

3. Main Results

Theorem 3.1. *Let T be a multiplication operator in $L^p(\mu)$, $p\in]0,\infty[$, on a finite separable measure space. Then $\mathrm{mc}(T)=\mathrm{mc}^*(T)=\mathrm{mp}(T)$ holds.*

As already mentioned, using the spectral theorem, the classical theorem of Bram [4], by which any $*$-cyclic bounded normal operator is cyclic, is generalized by (3.1) to unbounded normal operators. By definition $\mathrm{mp}(T)=1$ holds if and only if T is isomorphic with M_z in $L^p(\mu)$ for some finite Borel measure on $\mathbb{C}$. Hence (3.1) includes also the result that M_z is cyclic. Recall that a normal operator T is said to be simple if its spectral measure is simple. So by (3.1), T is simple if and only if T is cyclic.

In case that $(\Omega, \mathcal{A})$ is a standard measurable space, multiplicity $\mathrm{mp}(M_\varphi)$ has the meaning one expects intuitively, i.e., it equals the maximal number in $\mathbb{N}\cup\{\infty\}$ of preimages of $z \in \mathbb{C}$ under some $\varphi' = \varphi$ μ-a.e. See (5.4) for some details.

If for a cyclic set Z for M_φ the subspace $\Pi(M_\varphi)Z$ is even a core of M_φ then Z is called *graph cyclic*. We have taken this expression from Szafraniec [26], which we consider appropriate in view of (4.8). In case that $1_{\mathbb{C}}$ is graph cyclic for M_z in $L^2(\mu)$ then the Borel measure μ on $\mathbb{C}$ is called *ultradeterminate* by Fuglede [10]. One has

Proposition 3.2. *Let $p \in]0, \infty[$. Let $(\Omega, \mathcal{A}, \mu)$ be a finite separable measure space and let $\varphi : \Omega \to \mathbb{C}$ be measurable. If Z is a cyclic set for M_φ in $L^p(\mu)$, then $Z e^{-|\varphi|}$ is graph cyclic for M_φ.*

In particular (3.2) shows that every cyclic normal operator is even graph cyclic.

For the proof of (3.1) we had first to establish that M_z in $L^p(\mu)$ is cyclic for every finite Borel measure μ on $\mathbb{C}$. More precisely we have

Theorem 3.3. *Let μ be a finite Borel measure on $\mathbb{C}$. Then there is a positive Borel measurable function ρ such that $\Pi(z)\rho$ is dense in $L^p(\mu')$ for all $p \in]0, \infty[$ if μ' is a finite Borel measure on $\mathbb{C}$ equivalent to μ. Moreover h is cyclic for M_z in $L^p(\mu')$ if h is Borel measurable and satisfies $0 < |h| \leq C\rho$ for some constant $C > 0$.*

An immediate consequence of (3.3) due to Nagy [17] concerns polynomial approximation. It generalizes the result in Conway [9, corollary V.14.22] for measures with compact support. *Let μ be a σ-finite Borel measure on $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be measurable. Then there is a sequence (p_n) of polynomials in z with $p_n \to f$ μ-a.e.* Indeed, without restriction μ is finite. Let $h := \inf\{\rho, \frac{1}{1+|f|}\}$. Then h is positive cyclic and fh is bounded. Therefore there is a sequence (p_n) satisfying $p_n h \to fh$ in $L^p(\mu)$, and the result follows for some subsequence of (p_n).

Cyclicity of M_z in $L^2(\mu)$ has already been tackled by Béla Nagy in [17] adapting in parts the original proof of Joseph Bram [4] for bounded normal operators. See also Conway [9, theorem V.14.21] for a proof of Bram's theorem. The first step (i) and important result achieved in [17] is the decomposition of the complex plane into a null set and a countable union of increasing α-sets. Secondly (ii) $\bar{z}$ is approximated by polynomials in $L^2(\sigma)$ for some finite Borel measure σ on $\mathbb{C}$ equivalent to the original measure μ. The third step (iii) in [17] deals with the proof for the denseness in $L^2(\sigma)$ of the polynomials $\Pi(z)$. However the result obtained by Hilbert space methods is valid only for bounded M_z. Indeed, [17] starts the third step with the (unfounded) assumption that any function in the closure of $\Pi(z)$ is still square-integrable if multiplied by z. In other words, $\overline{\Pi(z)} \subset \mathcal{D}(M_z)$ is assumed. Proceeding on this assumption [17] shows $\overline{\Pi(z)} = L^2(\sigma)$ by a reducing subspace argument. Consequently $\mathcal{D}(M_z)$ is the whole of $L^2(\sigma)$ implying that

M_z is bounded, whence Nagy [17] does not achieve its goal. In addition, in accomplishing the reducing space argument, [17] uses $*$-cyclicity of M_z relying on a reference, which proves to be erroneous.

Our first step (i) in proving cyclicity of M_z in $L^p(\mu)$ for $p \in]0, \infty[$ and every finite Borel measure μ on $\mathbb{C}$ is the same as in [17]. We present a short proof (4.4) of the decomposition valid for a large class of Polish spaces including e.g. separable Banach spaces with real dimension ≥ 2. In the second step (ii) we show by induction that even $\Pi(z, \bar{z})$ is contained in the closure of $\Pi(z)$, see (4.5). At this stage, in the third step (iii), we bring in $*$-cyclicity of M_z in $L^2(\mu)$ by (3.4) and thus avoid a reducing subspace argument, which anyway is not available in the case $p \neq 2$.

In the sequel we denote by $\Pi(f_1, \ldots, f_n)$ the set of complex polynomials in functions $f_1, \ldots, f_n$ on some set with $f_i^0 := 1$. Let $d \in \mathbb{N}$ and $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Moreover, let x_i also denote the i-th coordinate function on $\mathbb{R}^d$. Let $a > 0$.

Proposition 3.4. *Let $p \in]0, \infty[$ and let μ be a finite Borel measure on $\mathbb{R}^d$. Then $\Pi(x_1, \ldots, x_d)\, \mathrm{e}^{-a|x|^2}$ is dense in $L^p(\mu)$. In particular, $\Pi(z, \bar{z})\, \mathrm{e}^{-a|z|^2}$ is dense in $L^p(\mu)$ for any finite Borel measure μ on $\mathbb{C}$.*

Let h_n denote the n-th Hermite function in one real variable. Then (3.4) for $\mu := \mathrm{e}^{-a|x|^2} \lambda^d$ with λ^d the Lebesgue measure on $\mathbb{R}^d$ yields the completeness of the orthonormal system of Hermite functions $h_{n_1} \times \cdots \times h_{n_d}$, $n_1, \ldots, n_d \in \mathbb{N} \cup \{0\}$ in $L^2(\lambda^d)$.

As already mentioned, (3.4) is a corollary to the $*$-cyclicity of M_z in $C_0(\mathbb{C})$. The question we pose now is about $(*)$-cyclicity of M_z on $C_0(X)$ for $X \subset \mathbb{C}$.

Theorem 3.5. *Let $X \subset \mathbb{C}$ be a locally compact subspace. Then M_z in $C_0(X)$ is $*$-cyclic by $e^{-a|z|^2}\eta$ with any positive $\eta \in C_0(X)$, and M_z is cyclic if and only if every compact K contained in X is an α-set.*

In view of (3.5) we remark that a locally compact subspace of $\mathbb{C}$ is σ-compact and locally closed. Hence, if $X \subset \mathbb{C}$ is locally compact and every compact $K \subset X$ has empty interior then X is nowhere dense. If $X \subset \mathbb{C}$ has empty interior and $\mathbb{C} \setminus K$ is connected for every compact $K \subset X$, then $\mathbb{C} \setminus X$ is dense and has no bounded components, and vice versa.

If X is compact then M_z in $C_0(X)$ is cyclic if and only if $1_{\mathbb{C}}$ is cyclic for M_z. This is due to $||ph - fh||_{\infty,X} \geq C||p - f||_{\infty,X}$ with $C := \inf_{z \in X} |h(z)| > 0$ for $f \in C(X)$, $p \in \Pi(z)$, and h a cyclic vector for M_z. Hence in case of compact X one recovers Lavrentev's theorem on α-sets from (3.5).

As an example, (3.5) implies that M_z is cyclic by the function $\mathrm{e}^{-a|z|^2}$ in $C_0(X)$, where X is the spiral $\{\mathrm{e}^{(1+i)t} : t \in \mathbb{R}\}$.

In this context we mention the result by Lavrentev/Keldych [25] that for a closed subset X of $\mathbb{C}$ every continuous function on X can be approximated uniformly by entire functions if and only if $\mathbb{C} \setminus X$ is dense, has no bounded components, and is locally connected at infinity.

4. Proofs

If necessary, in order to avoid ambiguities, we write $\overline{M}^\mu$ for the closure in $L^p(\mu)$ of the subset M. Similarly $||f||_{p\mu}$ denotes the norm of $f \in L^p(\mu)$. We start with two preliminary elementary results.

Lemma 4.1. *Let $p \in]0,\infty[$. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $M \cup \{h\}$ be a set of measurable functions on Ω with $h \neq 0$ μ-a.e. Then Mh is dense in $L^p(\mu)$ if and only if M is dense in $L^p(|h|^p\mu)$.*

Proof. Set $\nu := |h|^p\mu$. — Suppose $\overline{Mh}^\mu = L^p(\mu)$. Let $g \in L^p(\nu)$, $\epsilon > 0$. Then $gh \in L^p(\mu)$ and there is an $f \in M$ such that $\epsilon > ||fh - gh||_{p\mu} = ||f - g||_{p\nu}$. This proves $\overline{M}^\nu = L^p(\nu)$. — Now suppose $\overline{M}^\nu = L^p(\nu)$ and let $f \in L^p(\mu)$, $\epsilon > 0$. Then $f/h \in L^p(\nu)$ and there is $g \in M$ with $\epsilon > ||f/h - g||_{p\nu} = ||f - gh||_{p\mu}$. This proves $\overline{Mh}^\mu = L^p(\mu)$. □

Lemma 4.2. *Let $p \in]0,\infty[$. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $M \cup \{h\}$ be a set of measurable functions on Ω with h bounded and $h \neq 0$ μ-a.e. If M is dense in $L^p(\mu)$, then so is Mh.*

Proof. Let $C > 0$ be a constant with $|h| < C$. Set $A_n := \{|\frac{1}{h}| \leq n\}$. For $\Delta \in \mathcal{A}$ with $\mu(\Delta) < \infty$ and $\epsilon > 0$ there exists $f \in M$ satisfying $||1_{\Delta\cap A_n}\frac{1}{h} - f||_p < \epsilon/C$. Then $||1_{\Delta\cap A_n} - fh||_p < \epsilon$ holds, which implies $1_{\Delta\cap A_n} \in \overline{Mh}$ for all $n \in \mathbb{N}$. Therefore $1_\Delta \in \overline{Mh}$ for all $\Delta \in \mathcal{A}$ with $\mu(\Delta) < \infty$. The result follows. □

As to the proof of (4.3) note that $\Pi(x_1,\ldots,x_d)\mathrm{e}^{-a|x|^2}$ is not a subalgebra of $C_0(\mathbb{R}^d)$, whence the Stone-Weierstraß theorem cannot be applied directly. In [2] a combination of the theorems of Hahn/Banach, Riesz, and Bochner is used to overcome this problem.

Proposition 4.3. $\Pi(x_1,\ldots,x_d)e^{-a|x|^2}$ *is dense in $C_0(\mathbb{R}^d)$. In particular, the set $\Pi(z,\bar{z})e^{-a|z|^2}$ is dense in $C_0(\mathbb{C})$.*

Proof. For convenience let $a = 2$. The subalgebra $\Pi(x_1,\ldots,x_d,\mathrm{e}^{-|x|^2})\mathrm{e}^{-2|x|^2}$ of $C_0(\mathbb{R}^d)$ satisfies the assumptions of the Stone/Weierstraß Theorem. Thus it is dense in $C_0(\mathbb{R}^d)$. Therefore it remains to show

$$\Pi(x_1,\ldots,x_d,\mathrm{e}^{-|x|^2})\mathrm{e}^{-2|x|^2} \subset \overline{\Pi(x_1,\ldots,x_d)\exp(-2|x|^2)},$$

which follows from

$$\Pi(x_1,\ldots,x_d)\mathrm{e}^{-n|x|^2}\mathrm{e}^{-2|x|^2} \subset \overline{\Pi(x_1,\ldots,x_d)\exp(-2|x|^2)}$$

for $n = 0, 1, 2\ldots$ by forming the linear span at the left hand side. Now this is shown by induction on n. For the step $n \to n+1$ let T_k denote the k-th Taylor polynomial of e^z and let $p \in \Pi(x_1,\ldots,x_d)$. Then

$$||p\,\mathrm{e}^{-(n+1)|x|^2}\,\mathrm{e}^{-2|x|^2} - pT_k(-|x|^2)\,\mathrm{e}^{-n|x|^2}\,\mathrm{e}^{-2|x|^2}\,||_\infty \leq$$

$$C\;||\mathrm{e}^{-|x|^2}\big(\,\mathrm{e}^{-|x|^2} - T_k(-|x|^2)\big)||_\infty$$

with $C := ||p\,\mathrm{e}^{-(n+1)|x|^2}\,||_\infty < \infty$. Estimating the remainder function according to Lagrange one gets $\mathrm{e}^{-t}|\mathrm{e}^{-t} - T_k(-t)| = \mathrm{e}^{-t}|R_{k+1}(-t,0)| = \mathrm{e}^{-t}\frac{\mathrm{e}^{\tau}}{(k+1)!}t^{k+1} \le \frac{\mathrm{e}^{-t}}{(k+1)!}t^{k+1}$ with maximum at $t = k+1$, and Sterling's formula yields

$$\frac{\mathrm{e}^{-(k+1)}}{(k+1)!}(k+1)^{k+1} \le \frac{1}{\sqrt{2\pi(k+1)}} \to 0$$

for $k \to \infty$. □

In other words, (4.3) means that $h_{n_1} \times \cdots \times h_{n_d}$, $n_1, \ldots, n_d \in \mathbb{N} \cup \{0\}$ is total in $C_0(\mathbb{R}^d)$. — In particular, for every continuous function f on $\mathbb{C}$ vanishing at infinity there is a sequence (p_n) of polynomials in z and $\bar{z}$ such that $p_n e^{-|z|^2} \to f$ uniformly on $\mathbb{C}$.

Proof of (3.4). Recall that $C_0(\mathbb{R}^d)$ is dense in $L^p(\mu)$ (see e.g. [8]) and note that $||\cdot||_p \le \mu(\mathbb{R}^d)^{1/p}||\cdot||_\infty$. Therefore the result follows from (4.3). □

In some textbooks completeness of the orthonormal system (h_n) in $L^2(\lambda^1)$, λ^1 Lebesgue measure on $\mathbb{R}$, usually is shown using analytic function theory (e.g. [12, exercise 21.64]). Then the d-dimensional Hermite functions $(h_{n_1} \times \cdots \times h_{n_d})$ form a orthonormal basis in the d-fold Hilbert space tensor product $\bigotimes^d L^2(\lambda^1) \simeq L^2(\lambda^d)$. Taken this for granted one gets an alternative

Proof of (3.4) for $p = 2$ and $d = 2$. For convenience let $a = 1$. Let $f \in L^2(\mu)$ be orthogonal to $\Pi(x_1, x_2)\mathrm{e}^{-|x|^2}$. One has to show $f = 0$.

For $i = 1, 2$ put $\chi_i(x, y) := 1_{]-\infty, x_i[}(y_i) = 1_{]y_i, \infty[}(x_i)$, where $x, y \in \mathbb{R}^2$. Then for every $p \in \Pi(x_i)$ the function $(x, y) \mapsto h(x, y) := \partial_i(p(x_i)\,\mathrm{e}^{-|x|^2})\chi_i(x, y)f(y)$ on $\mathbb{R}^2 \times \mathbb{R}^2$ is integrable with respect to $\lambda^2 \otimes \mu$, since $|h(x, y)| \le |q(x_i)|\,\mathrm{e}^{-|x|^2}\,|f(y)|$ with some polynomial q and where $f \in L^1(\mu)$ as μ is finite. Hence Fubini's theorem applies to $\iint h\,\mathrm{d}\,\lambda^2 \otimes \mu$ yielding

$$-\int p(y_i)\,\mathrm{e}^{-y_i^2}\,\,f(y)\,\mathrm{d}\,\mu(y) = \int \partial_i\left(p(x_i)\,\mathrm{e}^{-x_i^2}\right)F_i(x_i)\,\mathrm{d}\,x_i$$

for $F_i(x_i) := \int \chi_i(x, y)f(y)\,\mathrm{d}\,\mu(y)$. By assumption the left hand side is zero. The right hand side becomes $\int q(t)\,\mathrm{e}^{-\frac{1}{2}t^2}\,F_i(t)\,\mathrm{e}^{-\frac{1}{2}t^2}\,\mathrm{d}\,t$ with $q(t) := p'(t) - 2tp(t)$. Note that $t \mapsto F_i(t)\,\mathrm{e}^{-\frac{1}{2}t^2}$ is square-integrable as $|F_i(t)| \le \sqrt{\mu(\mathbb{R}^2)}\,||f||_2$. Moreover q is the Hermite polynomial H_{n+1} if $p = -H_n$, $n = 0, 1, \ldots$. This implies that the only L^2-functions orthogonal to all $q\,\mathrm{e}^{-\frac{1}{2}t^2}$ are the constant multiples of $\mathrm{e}^{-\frac{1}{2}t^2}$. Consequently $F_i = 0$, $i = 1, 2$.

Next consider $\chi(x, y) := \prod_{i=1}^2 1_{]-\infty, x_i[}(y_i) = \prod_{i=1}^2 1_{]y_i, \infty[}(x_i)$ and, for every $p \in \Pi(x_1, x_2)$, the function $(x, y) \mapsto h(x, y) := \partial_1\partial_2\,(p(x)\,\mathrm{e}^{-|x|^2})\chi(x, y)f(y)$ on $\mathbb{R}^2 \times \mathbb{R}^2$. Reasoning analogously one finds for $F(x) := \int \chi(x, y)f(y)\,\mathrm{d}\,\mu(y)$

$$\int p(y)\,\mathrm{e}^{-|y|^2}\,\,f(y)\,\mathrm{d}\,\mu(y) = \int \partial_1\partial_2\left(p(x)\,\mathrm{e}^{-|x|^2}\right)F(x)\,\mathrm{d}\,\lambda^2(x)$$

and further $F(x) = g_1(x_1) + g_2(x_2)$ with measurable $g_i : \mathbb{R} \to \mathbb{C}$.

Now $0 = F_i(x_i) = \lim_{x_j \to \infty} F(x) = g_i(x_i) + \lim_{x_j \to \infty} g_j(x_j)$ with $i \neq j$.

Therefore g_i are constant and consequently $F = 0$.

Finally by 2-dimensional Lebesgue–Stieltjes integration the result

$$\int |f(y)|^2 \,\mathrm{d}\,\mu(y) = \int \overline{f(y)} \,\mathrm{d}\, F(y) = 0$$

follows. □

So the foregoing proof for $p = 2$ and $d = 2$ yields a direct proof of the fact that $\mathrm{e}^{-a|z|^2}$, $a > 0$ is a $*$–cyclic vector for M_z in $L^2(\mu)$ for every finite Borel measure μ on $\mathbb{C}$. It generalizes the proof in [1, sec. 83] of the case $d = 1$ which concerns the self-adjoint case $\mu(\mathbb{C} \setminus \mathbb{R}) = 0$.

Lemma 4.4. *Let X be a Polish space where every pair of distinct points are joined by infinitely many non-intersecting paths. Let μ be a σ-finite Borel measure on X. Then there is an increasing sequence of compact sets F_n with empty interior and connected complement such that $\mu(X \setminus \bigcup_n F_n) = 0$.*

Proof. Without restriction let μ be finite. All subspaces of X are separable. Choose a countable dense set $\{a_1, a_2, \dots\}$ in the complement of the set of mass points. Since the latter is countable, it does not contain an inner point by Baire's theorem, whence $\{a_1, a_2, \dots\}$ is dense in X. Since μ is finite and since there are infinitely many non-intersecting paths joining a_n to a_{n+1}, for every $m \in \mathbb{N}$ there are connected measurable sets A_n with $a_n, a_{n+1} \in A_n$ such that $B_m := \bigcup_n A_n$ is dense connected with $\mu(B_m) < \frac{1}{2m}$. By Ulam's theorem (see, e.g., [7, theorem 2.67]) μ is tight and in particular outer regular. Therefore there is an open V_m with $B_m \subset V_m$ and $\mu(V_m) < \frac{1}{m}$ and there is an increasing sequence (C_n) of compact sets with $\mu(X \setminus \bigcup_n C_n) = 0$.

Now set $U_n := \bigcup_{m \geq n} V_{2^m}$ and $F_n := C_n \setminus U_n$. Clearly, (F_n) is increasing, $\mu(\complement F_n) \to 0$, and F_n is compact. Its interior is empty since $B_{2^n} \subset U_n \subset \complement F_n$ is dense. $\complement F_n$ is connected. Indeed, let U, V be open sets covering $\complement F_n$ with $U \neq \emptyset$ and $U \cap V \cap \complement F_n = \emptyset$. Since B_{2^n} is dense, $U \cap V = \emptyset$ follows, and since B_{2^n} is connected, $V = \emptyset$ follows. □

Note that separable Banach spaces with real dimension ≥ 2 satisfy the assumptions on X in (4.4). For $X = \mathbb{C}$ the proof can be further shortened taking in place of all B_m a single dense null set B consisting of countably many straight lines through one common point, and of course, Ulam's theorem is not needed. The first (more cumbrous) proof for $X = \mathbb{C}$ and any finite Borel measure μ is given in Nagy [17]. If $\operatorname{supp}(\mu)$ is compact there is the original proof by Bram [4], a similar one in Conway [9], and a simpler one in Shields [22].

Lemma 4.5. *Let μ be a finite Borel measure on $\mathbb{C}$. Then there is a positive Borel measurable function ρ such that $\Pi(z) \subset L^p(\nu)$ and $\Pi(z, \bar{z}) \subset \overline{\Pi(z)}^{\nu}$ for all $p \in]0, \infty[$ and all $\nu := |h|^p \mu'$ with μ' a finite Borel measure on $\mathbb{C}$ equivalent to μ and h Borel measurable satisfying $0 < |h| \leq \rho$.*

Proof. Set $k : \mathbb{C} \to \mathbb{C}$, $k(z) := \bar{z}$. Let (F_n) be an increasing sequence of α-sets of $\mathbb{C}$ satisfying $\mu(N) = 0$ for $N := \mathbb{C} \setminus \bigcup_n F_n$, see (4.4). For every $n \in \mathbb{N}$ there is $q_n \in \Pi(z)$ satisfying $||1_{F_n}(k - q_n)||_\infty < \mathrm{e}^{-\delta_n}$ with $\delta_n := n||1_{F_n} k||_\infty$. Set $M_n := \max\{1, ||q_1 \mathrm{e}^{-|z|}||_\infty, \dots, ||q_n \mathrm{e}^{-|z|}||_\infty\}$ and let ρ be the positive function on

$\mathbb{C}$ given by $\rho|N := 1$ and $\rho|(F_n \setminus F_{n-1}) := \mathrm{e}^{-2|z|}/M_n$ for $n \in \mathbb{N}$ with $F_0 := \emptyset$.

Since $q\rho$ for $q \in \Pi(z)$ is bounded, $\Pi(z) \subset L^p(\nu)$ holds. For the proof of $\Pi(z,\bar{z}) \subset \overline{\Pi(z)}^\nu$ obviously it suffices to show $\Pi(z)\bar{z}^m \subset \overline{\Pi(z)}^\nu$ for $m = 0,1,2,\dots$ This occurs by induction on $m \in \mathbb{N}\cup\{0\}$. Let $j \in \mathbb{N}\cup\{0\}$ and write $\bar{z}^{m+1} = \bar{z}^m k$. Then $||z^j\bar{z}^m k - z^j\bar{z}^m q_n||_{p\nu} \le \nu(\mathbb{C})^{1/p}\delta_n^{(j+m)}\,\mathrm{e}^{-\delta_n} + ||1_{\complement F_n}(z^j\bar{z}^m k - z^j\bar{z}^m q_n)||_{p\nu}$. The first summand vanishes for $n \to \infty$, the latter is less or equal to $||1_{\complement F_n} z^j\bar{z}^m k||_{p\nu} + ||1_{\complement F_n} z^j\bar{z}^m q_n\rho||_{p\mu'}$, up to the constant factor $\sqrt[p]{2}/2$ in the case $p < 1$. Now $|1_{\complement F_n} z^j\bar{z}^m k| \le |z^{(j+m+1)}|$ and $|1_{\complement F_n} z^j\bar{z}^m q_n\rho| \le |z^{(j+m)}|\mathrm{e}^{-|z|}$, whence both summands vanish for $n \to \infty$ by dominated convergence. Since $z^j q_n\bar{z}^m \in \overline{\Pi(z)}^\nu$ by assumption, we infer $z^j\bar{z}^m k \in \overline{\Pi(z,\bar{z})}^\nu$ for every j, thus concluding the proof. □

Proof of (3.3). By (4.5) there exists a Borel measurable function ρ with $0 < \rho \le \mathrm{e}^{-|z|^2}$ such that $\Pi(z) \subset L^p(\nu)$ and $\Pi(z,\bar{z}) \subset \overline{\Pi(z)}^\nu$ for all $p \in]0,\infty[$ if $\nu := \rho^p\mu'$. By (3.4) and (4.2) we have $\overline{\Pi(z,\bar{z})\rho}^{\mu'} = L^p(\mu')$, whence $\overline{\Pi(z,\bar{z})}^\nu = L^p(\nu)$ by (4.1). Therefore $\overline{\Pi(z)}^\nu = L^p(\nu)$ and hence $\overline{\Pi(z)\rho}^{\mu'} = L^p(\mu')$ by (4.1). The last assertion follows from (4.2) for $M = \Pi(z)$. □

The decomposition (π,ν) of (φ,μ) can be derived from Rohlin's disintegration theorem (see Rohlin [19]), and can be found in Seid [21, remark 3.4]. There φ is a bounded Borel function on the finite measure space $([0,1],\mathcal{B},\mu)$ with $\mathcal{B}$ the Borel sets. By the following two lemmata we generalize this result in that φ is a measurable not necessarily bounded function on a finite separable measure space $(\Omega,\mathcal{A},\mu)$ and the multiplication operator M_φ is isomorphic with M_π by means of a measure algebra isomorphism.

Lemma 4.6. *Let $(\Omega,\mathcal{A},\mu)$ be a finite separable measure space. Then there is a measurable function $a : \Omega \to [0,1]$ such that $[B] \mapsto [a^{-1}(B)]$, $B \in \mathcal{B}$ is a measure algebra isomorphism from $(\mathcal{B}, a(\mu))$ onto $(\mathcal{A},\mu)$ and that for every measurable $\varphi : \Omega \to \mathbb{C}$ there is an $a(\mu)$-almost unique measurable $\psi : [0,1] \to \mathbb{C}$ with $\varphi = \psi\circ a$ μ-a.e.*

Proof. (i) Let (A_n) be a sequence in $\mathcal{A}$ such that for all $\Delta \in \mathcal{A}$, $\epsilon > 0$ there is $n \in \mathbb{N}$ with $\mu(\Delta \triangle A_n) < \epsilon$. Let $\Delta \in \mathcal{A}$. Choose a subsequence (n_k) such that $\mu(\Delta \triangle A_{n_k}) < 2^{-2k}$. Set $\tilde{\Delta} := \bigcup_l \bigcap_{k\ge l} A_{n_k}$. Then $\mu(\Delta \triangle \tilde{\Delta}) = 0$ holds. Indeed, for $\Delta_l := \bigcap_{k\ge l} A_{n_k}$ one has $\Delta \triangle \Delta_l = \bigcup_{k\ge l}(\Delta \setminus A_{n_k}) \cup (\Delta_l \setminus \Delta) \subset \bigcup_{k\ge l}(\Delta \triangle A_{n_k})$, whence $\mu(\Delta \triangle \Delta_l) \le \sum_{k\ge l} 2^{-2k} < 2^{-l}$. Therefore $\mu(\Delta \setminus \Delta_l) < 2^{-l}$ $\forall\, l$ and hence $\mu(\Delta \setminus \tilde{\Delta}) = 0$. Since $\tilde{\Delta}\setminus\Delta = \bigcup_{l\ge m}(\Delta_l \setminus \Delta)$ $\forall\, m$ one has also $\mu(\tilde{\Delta}\setminus\Delta) < \sum_{l\ge m} 2^{-l}$ $\forall\, m$ and hence $\mu(\tilde{\Delta}\setminus\Delta) = 0$.

The set $A_0 := \bigcup_l \bigcap_{n\ge l} A_n$ is a μ–null set. Indeed, obviously $\tilde{\Delta} \supset A_0$ for all $\Delta \in \mathcal{A}$. In particular $\tilde{\Delta} \supset A_0$ for $\Delta := \Omega \setminus A_0$, whence $0 = \mu(\Delta \triangle \tilde{\Delta}) \ge \mu(A_0)$. Since $\mu(A_0) = 0$ one may replace the original sequence (A_n) by $(A_n \setminus A_0)$ achieving $A_0 = \emptyset$.

The function $a : \Omega \to [0,1[$, $a := \sum_{n=1}^\infty 2^{-n} 1_{A_n}$ is measurable. As $A_0 = \emptyset$ every $a(\omega) \in [0,1[$ is represented as a binary number without the period $\{1\}$. For $x \in [0,1[$ let $d_n(x) \in \{0,1\}$ denote the n'th figure of its dual representation. Then $d_n \circ a = 1_{A_n}$ holds. Moreover, d_n is measurable, since $d_n^{-1}(\{1\}) = \bigcup\{[x, x+2^{-n}[:$

$x = \eta_1 2^{-1} + \cdots + \eta_{n-1} 2^{-n+1} + 2^{-n}$ with $\eta_i = 0, 1\} \in \mathcal{B}$.

For every subsequence (n_k) consider the sets $A := \bigcup_l \bigcap_{k \geq l} A_{n_k}$ and $B := \bigcup_l \bigcap_{k \geq l} d_{n_k}^{-1}(\{1\})$. Then $a^{-1}(B) = \bigcup_l \bigcap_{k \geq l} a^{-1}(d_{n_k}^{-1}(\{1\})) = \bigcup_l \bigcap_{k \geq l} A_{n_k} = A$. Therefore for every $\Delta \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $[\Delta] = [\tilde{\Delta}] = [a^{-1}(B)]$. This means that $T : \mathcal{B}/\mathcal{N}_{a(\mu)} \to \mathcal{A}/\mathcal{N}_\mu$, $T([B]) := [a^{-1}(B)]$ is surjective. But note first that, by $\mu(a^{-1}(B)) = a(\mu)(B)$, T is well-defined injective with $\mu \circ T = a(\mu)$. Obviously T preserves the measure algebra structure.

(ii) The linear map $L^\infty_{a(\mu)} \to L^\infty_\mu$, $\psi \mapsto \psi \circ a$ is isometric as $a(\mu)(\{|\psi| \geq c\}) = \mu(a^{-1}(\{|\psi| \geq c\})) = \mu(\{|\psi \circ a| \geq c\})$. Its range contains the total set $\{1_\Delta : \Delta \in \mathcal{A}\}$, since $1_\Delta = 1_{\tilde{\Delta}} = 1_{a^{-1}(B)} = 1_{B_\Delta} \circ a$. Hence it is surjective, too. This means that for every bounded measurable $\varphi : \Omega \to \mathbb{C}$ there is an $a(\mu)$-almost unique measurable $\psi : [0,1] \to \mathbb{C}$ with $\varphi = \psi \circ a$ μ-a.e.

This result is easily generalized to unbounded measurable $\varphi : \Omega \to \mathbb{C}$. Let $\Delta_n := \{n - 1 \leq |\varphi| < n\} \in \mathcal{A}$ and let $\psi_n : [0,1[\to \mathbb{C}$ satisfy $\varphi 1_{\Delta_n} = \psi_n \circ a$ μ–a.e. Then $\varphi = \sum_n \varphi 1_{\Delta_n} = \sum_n (\psi_n \circ a) = (\sum_n \psi_n) \circ a$ holds on the complement of some μ–null set Γ. Let $M := \{x \in [0,1[: \sum_n \psi_n(x)$ not convergent$\}$. Then M is measurable and $a^{-1}(M) \subset \Gamma$. Therefore $\varphi = \psi \circ a$ μ–a.e. with $\psi := \sum_n \psi_n 1_{[0,1[\setminus M}$. Moreover, ψ is $a(\mu)$-almost unique as every ψ_n is so. □

Let $(\Omega, \mathcal{A})$ be an uncountable standard measurable space. Then, by the well-known Isomorphism Theorem [18, theorem I. 2.12], there is a measurable space isomorphism a onto $([0,1], \mathcal{B})$. Hence $a : (\Omega, \mathcal{A}, \mu) \to ([0,1], \mathcal{B}, a(\mu))$ is even a measure space isomorphisms.

Lemma 4.7. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite separable measure space and let $\varphi : \Omega \to \mathbb{C}$ be measurable. Let $k : \mathbb{C} \to \mathbb{D}$, $k(z) := \frac{z}{1+|z|}$, and let κ denote its inverse. Let $\gamma : [0,1] \times \mathbb{D} \to [0,1] \times \mathbb{C}$, $\gamma(t,z) := (t, \kappa(z))$. Then $k \circ \varphi$ is bounded and, if (π, ν) is a Rohlin decomposition of $(k \circ \varphi, \mu)$, then $(\pi, \gamma(\nu))$ is a Rohlin decomposition of (φ, μ).*

Proof. Let $\tau : M(\mu) \to M(\nu)$ be the function algebra isomorphism accomplishing the decomposition (π, ν) of $(k \circ \varphi, \mu)$. Then $\tau(k \circ \varphi) = \pi$ and hence $\tau(\varphi) = \kappa \circ \pi$. Note that $\mathrm{supp}(\pi(\nu)) \subset \mathbb{D}$ as $\pi(\nu) = (k \circ \varphi)(\mu)$, whence $\nu([0,1] \times \complement\mathbb{D}) = 0$.

Let $\nu' := \gamma(\nu) = \lambda \otimes \mu'_c + \sum_{n \neq c} \delta_{1/n} \otimes \mu'_n$ with $\mu'_n := \kappa(\mu_n)$ $\forall$ n. Then γ is a measure space isomorphism from $([0,1] \times \mathbb{C}, \mathcal{B}, \nu)$ onto $([0,1] \times \mathbb{C}, \mathcal{B}, \nu')$. Let $\tau_\gamma : M(\nu) \to M(\nu')$, $f \mapsto f \circ \gamma^{-1}$ be the induced function algebra isomorphism.

Then the function algebra isomorphism $\tau_\gamma \circ \tau : M(\mu) \to M(\nu')$ satisfies $(\tau_\gamma \circ \tau)(\varphi)(t,z) = \tau(\varphi)(\gamma^{-1}(t,z)) = \tau(\varphi)(t, k(z)) = (\kappa \circ \pi)(t, k(z)) = \kappa(k(z)) = z$. Therefore $(\tau_\gamma \circ \tau)(\varphi) = \pi$ thus accomplishing the proof. □

Proof of (3.1). By definition of multicyclicity, multi-$*$-cyclicity, and multiplicity it suffices to show $\mathrm{mc}(M_\pi) = \mathrm{mc}^*(M_\pi) = \mathrm{mp}(M_\pi)$ for a Rohlin decomposition (π, ν). Plainly $\mathrm{mc}^*(T) \leq \mathrm{mc}(T)$.

Consider first the case $\mu_c \neq 0$. Then $\mathrm{mp}(M_\pi) = \infty$ holds by definition. In order to show $\mathrm{mc}^*(M_\pi) = \infty$, it suffices to treat the case $\nu = \lambda \otimes \mu_c$, since in general $L^p(\nu)$ is the direct sum of the subspaces $L^p(\lambda \otimes \mu_c)$ and $\bigoplus_n L^p(\delta_{1/n} \otimes \mu_n)$, which are invariant under M_π and $M_{\overline{\pi}}$. Let us assume that there is a finite

$*$-cyclic set $\{f_1, \dots, f_d\}$ for M_π with $d \in \mathbb{N}$. We consider $\chi_n 1_\mathbb{C} \in L^p(\nu)$ with $\chi_n := 1_{]1/(n+1),1/n]}$. Then for $n \in \mathbb{N}$ and $\delta = 1, \dots, d$ there is a sequence $(p_k^{n\delta})_k$ in $\Pi(z, \bar{z})$ such that $q_k^{n\delta}(z) := p_k^{n\delta}(z, \bar{z})$ satisfy $\sum_{\delta=1}^d q_k^{n\delta} f_\delta \to \chi_n 1_\mathbb{C}$ for $k \to \infty$ in $L^p(\nu)$. Set $f_{\delta z} := f_\delta(\cdot, z)$. By Tonelli's theorem there is a subsequence $(k_l)_l$, without restriction $(k)_k$ itself, with $\sum_{\delta=1}^d q_k^{n\delta}(z) f_{\delta z} \to \chi_n$ for $k \to \infty$ in $L^p(\lambda)$ for μ_c-almost all $z \in \mathbb{C}$. We consider this convergence for $n = 1, \dots, d$. Since $\chi_1, \dots, \chi_d$ are linear independent, it follows that $f_{1z}, \dots, f_{d\,z}$ are so for μ_c-almost all $z \in \mathbb{C}$. Consequently $q_k^{n\delta}(z)$ for $k \to \infty$ converge to the coordinates $\alpha_{\delta n}(z)$ of χ_n with respect to $(f_{\delta z})_\delta$. Hence one gets $f_{\delta z} = \sum_{n=1}^d \beta_{n\delta}(z) \chi_n$, $\delta = 1, \dots, d$ with coordinates $\beta_{n\delta}(z)$. — Repeating these considerations for $\chi_{d+1}, \dots, \chi_{2d}$ in place of $\chi_1, \dots, \chi_d$ we obtain $f_{\delta z} = \sum_{n=1}^d \beta'_{n\delta}(z) \chi_{d+n}$, $\delta = 1, \dots, d$. Since $\chi_1, \dots, \chi_{2d}$ are linear independent, this implies for μ_c-almost all $z \in \mathbb{C}$ that $f_\delta(t, z) = 0$ for λ-almost all $t \in [0, 1]$. Hence $\int |f_\delta|^p \mathrm{d}\, \lambda \otimes \mu_c = 0$ by Tonelli's theorem. This means $f_\delta = 0$ for $\delta = 1, \dots, d$, which is not possible.

Now let $\mu_c = 0$ and set $\chi_n := 1_{\{1/n\}}$, $n \in \mathbb{N}$. We consider first the case $\operatorname{mp}(M_\pi) = \infty$. Then $\mu_n \neq 0\ \forall\, n \in \mathbb{N}$. Assuming the existence of a $*$-cyclic set of d elements, as in the previous case $f_{\delta z} = \sum_{n=1}^d \beta_{n\delta}(z) \chi_n = \sum_{n=1}^d \beta'_{n\delta}(z) \chi_{d+n}$, $\delta = 1, \dots, n$ follows. This implies for μ_c-almost all $z \in \mathbb{C}$ that $f_\delta(t, z) = 0$ for all $1/t \in \mathbb{N}$. This means $f_\delta = 0$ for all $\delta = 1, \dots, d$, which is not possible. — We turn to the last case $N := \operatorname{mp}(M_\pi) \in \mathbb{N}$. Then $\mu_n \neq 0$ for $n = 1, \dots, N$ und $\mu_n = 0$ else. Since M_π is cyclic in $L^p(\delta_{1/n} \otimes \mu_n)$ according to (3.3), $\operatorname{mc}(M_\pi) \leq N$ follows. Let us assume now that there is a $*$-cyclic set $\{f_1, \dots, f_d\}$ for M_π with $d < N$. By considerations as in the case $\mu_c \neq 0$ we get $f_{\delta z} = \sum_{n=1}^d \beta_{n\delta}(z) \chi_n = \sum_{n=2}^d \beta'_{n\delta}(z) \chi_n + \beta'_{m\delta} \chi_m$ with $m \notin \{1, \dots, d\}$. Since all χ's are linear independent this implies $\beta_{1\delta}(z) = 0$ for μ_c-almost all $z \in \mathbb{C}$. Analogously $\beta_{n\delta}(z) = 0$ for μ_c-almost all $z \in \mathbb{C}$ for every $n \in \{1, , \dots, d\}$. Therefore $f_{\delta z} = 0$ for μ_c-almost all $z \in \mathbb{C}$. This means $f_\delta = 0$ for every δ, which is not possible. □

The next lemma is not new but it puts together the equivalences for convenience.

Lemma 4.8. *Let $p \in]0, \infty[$. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $Z \cup \{\varphi\}$ be a set of measurable functions. Then* (a) – (e) *are equivalent.*

(a) Z is graph cyclic for M_φ in $L^p(\mu)$
(b) $\Pi(M_\varphi)Z$ is a core for M_φ in $L^p(\mu)$
(c) $\{(f, M_\varphi f) : f \in \Pi(M_\varphi)Z\}$ is dense in $\{(f, M_\varphi f) : f \in \mathcal{D}(M_\varphi)\}$
(d) $\Pi(M_\varphi)Z \sqrt[p]{1 + |\varphi|^p}$ is dense in $L^p(\mu)$
(e) $\Pi(M_\varphi)Z$ is dense in $L^p\big((1 + |\varphi|^p)\, \mu\big)$

Proof. The equivalences of (a) and (b) and (c) hold by definition, the equivalence of (d) and (e) holds by (4.1). — As to (c) ⇒ (d) let $g \in L^p(\mu)$. Then $g' := g / \sqrt[p]{1 + |\varphi|^p} \in \mathcal{D}(M_\varphi)$ and hence for $\epsilon > 0$ there is $f \in \Pi(M_\varphi)Z$ satisfying $||(f, M_\varphi f) - (g', M_\varphi g')||_p < \epsilon$, which means $\epsilon^p > \int \big(|f - g'|^p + |\varphi f - \varphi g'|^p\big)\, \mathrm{d}\, \mu = \int \big| \sqrt[p]{1 + |\varphi|^p}\, f - g\big|^p \mathrm{d}\, \mu$ proving (d). — Finally assume (d) and let $f \in \mathcal{D}(M_\varphi)$. Then $f' := \sqrt[p]{1 + |\varphi|^p}\, f \in L^p(\mu)$ and for $\epsilon > 0$ there is $g \in \Pi(M_\varphi)Z$ satisfying

$||f' - \sqrt[p]{1+|\varphi|^p}\, g||_p < \epsilon$, which means $||(f, M_\varphi f) - (g, M_\varphi g)||_p < \epsilon$, thus proving (c). □

Proof of (3.2). Since $\Pi(M_\varphi)Z$ is dense in $L^p(\mu)$, by (4.2), we conclude that $\Pi(M_\varphi)Z\mathrm{e}^{-|\varphi|}\sqrt[p]{1+|\varphi|^p}$ is dense in $L^p(\mu)$. The result follows from (4.8). □

Proof of (3.5). Let $a = 1$ for convenience. — (i) Since X is locally compact there are compact sets $F_n \subset X$ with F_n contained in the interior of F_{n+1}, and functions $\eta_n \in C_c(X)$ satisfying $\eta_n|F_n = 1$, $\eta_n|(X \setminus F_{n+1}) = 0$, and $0 \le \eta_n \le 1_X$. Let $\alpha_n > 0$ with $\sum_n \alpha_n < \infty$. Then $\eta := \sum_n \alpha_n \eta_n \in C_0(X)$ and $\eta > 0$.

(ii) Let $f \in C_c(X)$. Extend f continuously, first onto the closure $\overline{X}$ by 0, and subsequently onto $\mathbb{C}$ by the Tietze–Urysohn extension theorem. Finally, multiplying the resulting function by a $j \in C_c(\mathbb{C})$ with $j|\operatorname{supp}(f) = 1$ one achieves an extension of f with compact support. — Now let $g \in C_c(\mathbb{C})$ extend $f/\eta \in C_c(X)$ with a positive $\eta \in C_0(X)$, see (i). Let $\epsilon > 0$. By (4.3) there is $p \in \Pi(z, \bar{z})$ with $||g - p\,\mathrm{e}^{-|z|^2}||_\infty < \epsilon/||\eta||_{\infty,X}$. This implies $||f - p\,\mathrm{e}^{-|z|^2}\eta||_{\infty,X} < \epsilon$. Thus $e^{-|z|^2}\eta$ is a $*$-cyclic vector.

(iii) Let $h \in C_0(X)$ be a cyclic vector. Since $C_0(X)$ vanishes nowhere, so does h. Let $K \subset X$ be compact. Let $\varphi \in C(K)$. By the Tietze–Urysohn extension theorem exists a bounded continuous ϕ on X with $\phi|K = \varphi$. Then $\phi h \in C_0(X)$. Set $c := \sup_{z\in K} |\frac{1}{h(z)}|$ and let $\epsilon > 0$. Then there is $p \in \Pi(z)$ satisfying $||\phi h - ph||_\infty < \epsilon/c$. This implies $||\varphi - p||_{\infty,K} < \epsilon$. Thus K is an α-set by definition.

(iv) Now let every compact $K \subset X$ be an α-set. Set $k(z) := \bar{z}$. There are $q_n \in \Pi(z)$ satisfying $||1_{F_n}(k - q_n)||_\infty < \frac{1}{n}$. Then set $M_n := \max\{1, ||1_{F_{n+1}}k||_\infty, ||1_{F_{n+1}}q_1||_\infty, \dots, ||1_{F_{n+1}}q_n||_\infty\}$ and $\alpha_n := 2^{-n}/M_n$ in (i). For $j \ge n$ one has $||\alpha_j\eta_j(k - q_n)||_{\infty,X} \le 2 \cdot 2^{-j}$, whence $||1_{\complement F_n}\eta(k - q_n)||_{\infty,X} \to 0$. It follows $||\eta(k - q_n)||_{\infty,X} \to 0$. — Now we show that $h := \mathrm{e}^{-|z|^2}\eta$ is a cyclic vector. By (ii), $A := \Pi(z, \bar{z})h$ is dense in $C_0(X)$. We conclude the proof showing $A \subset \overline{\Pi(z)h}$ by the method used in (4.3). Let $q \in \Pi(z)$. Induction occurs on $m = 0, 1, 2 \dots$ Then $||q\,\bar{z}^{m+1}h - q\,q_n\bar{z}^m h||_{\infty,X} \le C\,||\eta(k - q_n)||_{\infty,X}$ with $C := ||q\,\bar{z}^m\,\mathrm{e}^{-|z|^2}||_\infty$ vanishes for $n \to \infty$. □

5. Further results

Let $p \in]0, \infty[$. We know by (3.4), (3.3) that, for every finite Borel measure μ, M_z in $L^p(\mu)$ is $*$-cyclic by the continuous vector $\mathrm{e}^{-a|z|^2}$ and that M_z is cyclic. The question is whether there are continuous cyclic vectors.

Example 5.1. Let $\mu := 1_{\mathbb{D}}\lambda$ with λ the Lebesgue measure on $\mathbb{C}$ and $\mathbb{D}$ the open unit disc. Let h be a cyclic vector for M_z in $L^2(\mu)$. Then $\{h = 0\}$ is a μ-null set containing all continuity points of h.

Proof. $\{h = 0\}$ is a μ-null set, since $L^2(\mu) = \overline{\Pi(z)h} \subset \{f \in L^2(\mu) : f = 1_{\{h\neq 0\}}f\}$. — Let h be continuous at $x \in \mathbb{D}$. Assume $h(x) \neq 0$. Then there are an open disc D with center x and $\delta > 0$ such that $\delta 1_D \le |h|$. Hence, by (4.2), $\Pi(z)1_D$ is dense in $L^2(1_D\lambda)$. This contradicts e.g. 3.22. (c) in [6]. □

In particular, there is no cyclic vector for M_z in $L^2(1_{\mathbb{D}}\lambda)$ that is continuous on $\mathbb{D}$, thus answering a question about continuity of cyclic vectors posed by Shields [22]. If, however, $\bar{z} \in \overline{\Pi(z)}$ holds then we have

Proposition 5.2. *Let $p \in]0,\infty[$. Let μ be a finite Borel measure on $\mathbb{C}$ such that $\Pi(z) \subset L^p(\mu)$ and $\bar{z} \in \overline{\Pi(z)}$, then $\mathrm{e}^{-a|z|^2}$ for $a > 0$ is a cyclic vector for M_z in $L^p(\mu)$.*

Proof. Apply (5.3) below with $A := \Pi(z)$, $b := \bar{z}$, and $c := \mathrm{e}^{-a|z|^2}$. The result follows from (3.4). □

The following is a useful tool in establishing as in (5.2) that the closure of a coset of a given algebra contains the coset of some larger algebra.

Lemma 5.3. *Let $p \in]0,\infty[$ and let μ be a finite Borel measure on $\mathbb{C}$. Let $A \subset L^p(\mu)$ be an algebra, let $b \in \overline{A}$, and let $c \in L^p(\mu)$. Suppose $Ab^n c \subset L^\infty(\mu)$ for $n = 0, 1, 2 \dots$. Then $\Pi(A, b)c \subset \overline{Ac}$.*

Proof. It suffices to show $Ab^n c \subset \overline{Ac}$ by induction on n. Let (a_k) be a sequence in A converging to b. As to the step $n \to n+1$ note $||ab^{n+1}c - aa_k b^n c||_p \leq ||ab^n c||_\infty ||b - a_k||_p \to 0$ for $k \to \infty$. Since by assumption $aa_k b^n c \in \overline{Ac}$ the result follows. □

For $z \in \mathbb{C}$ and $\Delta \subset \Omega$ let $\mathrm{n}_\Delta(z) := |\{\omega \in \Delta : \varphi(\omega) = z\}|$ denote the number in $\mathbb{N} \cup \{\infty\}$ of preimages in Δ of z under φ. For a Rohlin decomposition (π, ν) of (φ, μ), $\varphi(\mu) = \mu_c + \sum_n \mu_n$ holds. Set $P_n := \{\frac{\mathrm{d}\,\mu_n}{\mathrm{d}\,\varphi(\mu)} > 0\}$ for $n \in \{c\} \cup \mathbb{N}$ and define the *local multiplicity* by

$$\mathrm{m}_\varphi(z) := \infty 1_{P_c}(z) + \sup\left(\{0\} \cup \{n \in \mathbb{N} : z \in P_n\}\right).$$

We will keep in mind that P_n is unique up to a $\varphi(\mu)$-null set.

Theorem 5.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space with $(\Omega, \mathcal{A})$ a standard measurable space. Let $\varphi : \Omega \to \mathbb{C}$ be measurable. Then there is a μ-null set N such that $\mathrm{n}_{\Omega\setminus N}$ is measurable and such that*

$$\mathrm{m}_\varphi = \mathrm{n}_{\Omega\setminus N} \leq \mathrm{n}_{\Omega\setminus N'} \quad \varphi(\mu)\text{–}a.e.$$

for every μ-null set N'. Furthermore $\mathrm{mp}(M_\varphi) = \sup \mathrm{m}_\varphi$ holds for M_φ in $L^p(\mu)$, $p \in]0,\infty[$.

Proof. As $(\Omega, \mathcal{A})$ is standard and μ is finite there is a Rohlin decomposition (π, ν) of (φ, μ) by a measure space isomorphism ϑ onto the complement of a ν-null set of $[0,1] \times \mathbb{C}$. Because of $\mu_{n+1} \ll \mu_n \;\forall\, n$, $(P_n)_n$ is $\varphi(\mu)$-almost decreasing. Therefore $\mathrm{m}_\varphi = \infty 1_{P_c} + \sum_n 1_{P_n}$ $\varphi(\mu)$-a.e. holds. $S := \big([0,1] \times P_c\big) \cup \bigcup_n \big(\{\frac{1}{n}\} \times P_n\big)$ is the complement of a ν-null set since $\mu_n(\complement P_n) = 0$ for $n = c, 1, 2, \dots$, and $|S_z| = \infty 1_{P_c}(z) + \sum_n 1_{P_n}(z) \;\forall\; z \in \mathbb{C}$ for $S_z := \{t \in [0,1] : (t, z) \in S\}$ holds. — Now let R be the complement of any ν-null set. Then $B_c := \{z \in \mathbb{C} : \lambda(R_z) = 1\}$ and $B_n := \{z \in \mathbb{C} : \frac{1}{n} \in R_z\}$ satisfy $\mu_n(\mathbb{C} \setminus B_n) = 0$, whence $\varphi(\mu)(P_n \setminus B_n) = 0$ for $n = c, 1, 2, \dots$ This implies $|R_z| \geq |S_z|$ $\varphi(\mu)$-a.e. Moreover, we may choose without restriction $P_n \subset B_n$, $n = c, 1, 2, \dots$ for $R := \vartheta(\Omega)$. Then $|S_z| = |S_z \cap \vartheta(\Omega)|$. Since generally $\mathrm{n}_\Delta(z) = \vartheta(\Delta)_z$ by $\varphi = \pi \circ \vartheta$, we obtain

$n_{\Omega\setminus N}(z) = |S_z|$ for $N := \complement\,\vartheta^{-1}(S)$, whence $n_{\Omega\setminus N}$ is measurable, and $n_{\Omega\setminus N}(z) = |S_z| \leq |\vartheta(\Omega \setminus N')_z| = n_{\Omega\setminus N'}(z)$ $\varphi(\mu)$-a.e. — The last assertion is obvious. □

Obviously, in (5.4), $n_{\Omega\setminus(N\cup N')} = n_{\Omega\setminus N}$, whence $m_\varphi = n_{\Omega\setminus M}$ holds $\varphi(\mu)$-a.e., if the ν-null set M is large enough. Finally we mention that in the Hilbert space case m_φ is a complete invariant. This means that normal operators $T \simeq M_\varphi$ in $L^2(\mu)$ and $T' \simeq M_{\varphi'}$ in $L^2(\mu')$ with μ and μ' Borel measures on $\mathbb{C}$ are isomorphic if and only if $\varphi(\mu) \sim \varphi'(\mu')$ and $m_\varphi = m_{\varphi'}$ a.e. In other words m_φ is the usual local multiplicity derived from the spectral theorem. Results relating local multiplicity to the number of preimages can be found in [3, 13, 14, 15, 16].

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