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SOME CHARACTERIZATIONS OF HERZ-TYPE HARDY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper, the authors establish some real-variable characterizations of Herz-type Hardy spaces with variable exponent.

1. Introduction and preliminaries

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \to [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The L^p spaces with variable exponent are a special case of Musielak–Orlicz spaces.

For all compact subsets $E \subset \Omega$, the space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\text{loc}}(\Omega) := \{f : f \in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \to [1, \infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \Omega\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \Omega\} < \infty.$$

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Denote $p'(x) = p(x)/(p(x)-1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *([4]) Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, then fg is integrable on Ω and*

$$\int_\Omega |f(x)g(x)|dx \leq r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. *([1]) Given a set Ω with finite measure, and exponent functions $p(\cdot)$, $q(\cdot) : \Omega \to [1,\infty)$ such that $p(x) \leq q(x)$,*

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq C(1+|\Omega|)\|f\|_{L^{q(\cdot)}(\Omega)}.$$

Lemma 1.3. *([3]) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$ are constants.

Throughout this paper δ_1 and δ_2 are the same as in Lemma 1.3.

Lemma 1.4. *([3]) Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Firstly we give the definition of the Herz spaces with variable exponent. Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{A_k} is the characteristic function of A_k.

Definition 1.5. ([3]) Let $\alpha \in \mathbb{R}, 0 < p \leq \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

The non-homogeneous Herz space $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n) : \|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \left\{\sum_{k=0}^{\infty} 2^{k\alpha p}\|f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p}.$$

In [6], we establish the following boundedness theorem on the Herz spaces with variable exponent for a class of sublinear operators.

Lemma 1.6. *([6]) Let $0 < \alpha < n\delta_2, 0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If a sublinear operator T satisfies*

$$|Tf(x)| \le C\|f\|_1/|x|^n, \quad \text{if} \quad \text{dist}(x, \text{supp}f) > |x|/2, \tag{1.1}$$

for any integrable function f with a compact support and T is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then T is bounded on $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, respectively.

In [7], we gave the definition of Herz-type Hardy space with variable exponent $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathcal{S}(\mathbb{R}^n)$. Let $G_N f(x)$ be the grand maximal function of $f(x)$ defined by

$$G_N f(x) = \sup_{\phi \in \mathcal{A}_N} |\phi^*_\nabla(f)(x)|,$$

where $\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\le N} |x^\alpha D^\beta \phi(x)| \le 1\}$ and $N > n+1$, ϕ^*_∇ is the nontangential maximal operator defined by

$$\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$$

with $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 1.7. ([7]) Let $\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $N > n+1$.

(i)The homogeneous Herz-type Hardy space $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in \dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

(ii)The non-homogeneous Herz-type Hardy space $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : G_N f(x) \in K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\}$$

and we define $\|f\|_{HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}$.

Let us explain the outline of this paper. In Section 2 we will prove some properties for $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$. We will give our main result in Section 3, that is some real-variable characterizations of the Herz-type Hardy space with variable exponent.

2. Some properties for Herz spaces with variable exponent

We first give the following properties for $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then we have*

(1) *if $p_1 \le p_2$, then $\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n) \subset K^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$.*
(2) *if $0 < \alpha_2 \le \alpha_1$, then $K^{\alpha_1,p}_{q(\cdot)}(\mathbb{R}^n) \subset K^{\alpha_2,p}_{q(\cdot)}(\mathbb{R}^n)$.*
(3) *if $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$ and $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\Omega)$ such that $q_2(\cdot) \le q_1(\cdot)$, then*

$$\dot{K}^{\alpha,p}_{q_1(\cdot)}(\Omega) \subset \dot{K}^{\alpha,p}_{q_2(\cdot)}(\Omega),\ K^{\alpha,p}_{q_1(\cdot)}(\Omega) \subset K^{\alpha,p}_{q_2(\cdot)}(\Omega).$$

Proof. We first consider (1). It suffices to prove the property for the homogeneous case. The non-homogeneous case can be proved in the same way. Note that $p_1 \le p_2$ and

$$\left(\sum_{k=1}^{\infty} |a_k|\right)^r \le \sum_{k=1}^{\infty} |a_k|^r,\ \ 0 < r \le 1. \tag{2.1}$$

So we have

$$\begin{aligned}
\|f\|_{\dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1\cdot\frac{p_2}{p_1}} \|f\chi_k\|^{p_1\cdot\frac{p_2}{p_1}}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p_1}\cdot\frac{p_1}{p_2}} \\
&\le \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|f\chi_k\|^{p_1}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p_1}} \\
&= \|f\|_{\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

That is $\dot{K}^{\alpha,p_1}_{q(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)$.

Now we see (2). Note that $0 < \alpha_2 \le \alpha_1$, so by the Hölder inequality we have

$$\begin{aligned}
\|f\|_{K^{\alpha_2,p}_{q(\cdot)}(\mathbb{R}^n)} &= \left\{\sum_{k=0}^{\infty} 2^{k\alpha_2 p} \|f\tilde{\chi}_k\|^{p\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_1-\alpha_2}{\alpha_1}\right)}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{1/p} \\
&\le \Bigg\{C\left(\sum_{k=0}^{\infty}\left(2^{k\alpha_2 p}\|f\tilde{\chi}_k\|^{p\frac{\alpha_2}{\alpha_1}}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{\alpha_1}{\alpha_2}}\right)^{\frac{\alpha_2}{\alpha_1}} \\
&\quad \times\left(\sum_{k=0}^{\infty}\|f\tilde{\chi}_k\|^{p\frac{\alpha_1-\alpha_2}{\alpha_1}\left(\frac{\alpha_1}{\alpha_2}\right)'}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{\left(\frac{\alpha_1}{\alpha_2}\right)'}}\Bigg\}^{1/p} \\
&\le \left\{C\left(\sum_{k=0}^{\infty} 2^{k\alpha_1 p}\|f\tilde{\chi}_k\|^{p}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{\alpha_2}{\alpha_1}}\left(\sum_{k=0}^{\infty}\|f\tilde{\chi}_k\|^{p}_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^{\frac{\alpha_1-\alpha_2}{\alpha_1}}\right\}^{1/p} \\
&\le C\|f\|_{K^{\alpha_1,p}_{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

That is $K^{\alpha_1,p}_{q(\cdot)}(\mathbb{R}^n) \subset K^{\alpha_2,p}_{q(\cdot)}(\mathbb{R}^n)$.

Next we estimate (3). It suffices to prove the property for the homogeneous case. By Lemma 1.2 we have

$$\begin{aligned}\|f\|_{\dot K^{\alpha,p}_{q_2(\cdot)}(\Omega)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q_2(\cdot)}(\Omega)}\right\}^{1/p}\\ &\le C(1+|\Omega|)\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q_1(\cdot)}(\Omega)}\right\}^{1/p}\\ &\le C\|f\|_{\dot K^{\alpha,p}_{q_1(\cdot)}(\Omega)}.\end{aligned}$$

That is $\dot K^{\alpha,p}_{q_1(\cdot)}(\Omega) \subset \dot K^{\alpha,p}_{q_2(\cdot)}(\Omega)$.

Thus we complete the proof of Theorem 2.1. □

Theorem 2.2. *Let* $0<\alpha<\infty$, $0<p\le\infty$ *and* $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. *Then*

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \supset \dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\cap L^{q(\cdot)}(\mathbb{R}^n)$$

and for $f\in \dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\cap L^{q(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le \|f\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Proof. If $f\in \dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\cap L^{q(\cdot)}(\mathbb{R}^n)$, then

$$\begin{aligned}\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &= \|f\|^p_{L^{q(\cdot)}(|x|\le 1)} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\le \|f\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \|f\|^p_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

This finishes the proof of Theorem 2.2. □

3. Some real-variable characterizations for Herz-type Hardy spaces with variable exponent

By Theorem 2.2 and the $L^{q(\cdot)}(\mathbb{R}^n)$-boundedness ($q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$) of the grand maximal operator G_N, it is easy to deduce the following conclusion.

Theorem 3.1. *Let* $0<\alpha<\infty$, $0<p<\infty$ *and* $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. *Then*

$$HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n) \supset H\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\cap L^{q(\cdot)}(\mathbb{R}^n)$$

To give some real-variable characterizations for $H\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ and $HK^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, we first introduce some maximal operator.

Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ with integral 1. For $t>0$, set $\phi_t(x)=t^{-n}\phi(x/t)$. For $f\in\mathcal{S}'(\mathbb{R}^n)$, define the maximal operator ϕ^*_+ by

$$\phi^*_+(f)(x)=\sup_{t>0}|(f*\phi_t)(x)|.$$

Also, we define the maximal operator $\phi^*_{\nabla,N}$ (with $N>1$) and ϕ^{**}_M (with $M\in\mathbb{Z}_+$) by

$$\phi^*_{\nabla,N}(f)(x)=\sup_{t>0}\sup_{|x-y|<Nt}|(f*\phi_t)(y)|$$

and

$$\phi_M^{**}(f)(x) = \sup_{(y,t)\in\mathbb{R}_+^{n+1}} |(f*\phi_t)(y)| \left(\frac{t}{|x-y|+t}\right)^M .$$

About the relation of these operators, we first have

Lemma 3.2. *([2]) If $N \geq M+n+1$, then there exists a constant C such that*

$$G_N(f)(x) \leq C\phi_M^{**}(f)(x).$$

Next we give the following characterization theorem.

Theorem 3.3. *Let $0<\alpha<\infty$, $0<p<\infty$ and $q(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. For $f\in\mathcal{S}'(\mathbb{R}^n)$, the following statements are equivalent:*

(i) *$f \in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$).*

(ii) *For some $N>1$, $\phi_{\nabla,N}^*(f)\in\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$).*

(iii) *$\phi_\nabla^*(f)\in\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$).*

(iv) *$\phi_+^*(f)\in\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$).*

Proof. We only prove the homogeneous case. The non-homogeneous case is similar. Note that

$$\phi_{\nabla,N}^*(f)(x) \geq \phi_\nabla^*(f)(x) \geq \phi_+^*(f)(x)$$

and that for any $N>n+1$,

$$\phi_\nabla^*(f)(x) \leq CG_N(f)(x).$$

It is obvious that (ii)⇒(iii)⇒(iv) and (i)⇒(iii). Thus, it suffices to prove that (iv)⇒(ii) and (iv)⇒(i).

We first prove (iv)⇒(ii). For $l,N\in\mathbb{Z}_+$, define

$$u_{\varepsilon,l,N}^*(x) = \sup_{|x-y|<Nt<1/\varepsilon} |(f*\phi_t)(y)| \left(\frac{Nt}{Nt+\varepsilon}\right)^l (1+\varepsilon N|y|)^{-l}.$$

By the Fatou lemma of series and integration, we need only to show that for any $r\in(0,1)$,

$$\|u_{\varepsilon,l,N}^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq CN^{n/r}\|\phi_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

Let

$$U_{\varepsilon,l,N}^*(x) = \sup_{|x-y|<Nt<1/\varepsilon} t|\nabla_y(f*\phi_t)(y)| \left(\frac{Nt}{Nt+\varepsilon}\right)^l (1+\varepsilon N|y|)^{-l}.$$

As in [5], if l is large enough, then for any $p_1\in(0,1)$ we have

$$U_{\varepsilon,l,N}^*(x) \leq C\left(M[(u_{\varepsilon,l,N}^*)^{p_1}](x)\right)^{1/p_1},$$

where M is the Hardy–Littlewood maximal operator, and C is independent of ε, N and f. Set $E_\varepsilon = \{x : U_{\varepsilon,l,N}^*(x) \leq C_0 u_{\varepsilon,l,N}^*(x)\}$ and $E_\varepsilon^c = \mathbb{R}^n\backslash E_\varepsilon$, where

C_0 is a positive constant which will be chosen later. Take $p_1 \in (0,1)$ such that $0 < p_1\alpha < n\delta_2$, then by Lemma 1.6 we have

$$\begin{aligned}\|u^*_{\varepsilon,l,N}\chi_{E^c_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\le C_0^{-1}\|U^*_{\varepsilon,l,N}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\le CC_0^{-1}\left\|M[(u^*_{\varepsilon,l,N})^{p_1}]\right\|^{1/p_1}_{\dot K^{\alpha p_1,p/p_1}_{q(\cdot)/p_1}(\mathbb{R}^n)}\\ &\le CC_0^{-1}\left\|(u^*_{\varepsilon,l,N})^{p_1}\right\|^{1/p_1}_{\dot K^{\alpha p_1,p/p_1}_{q(\cdot)/p_1}(\mathbb{R}^n)}\\ &= CC_0^{-1}\left\|u^*_{\varepsilon,l,N}\right\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u^*_{\varepsilon,l,N}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} &\le \|u^*_{\varepsilon,l,N}\chi_{E_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + \|u^*_{\varepsilon,l,N}\chi_{E^c_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\le \|u^*_{\varepsilon,l,N}\chi_{E_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} + CC_0^{-1}\left\|u^*_{\varepsilon,l,N}\right\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\\ &\le 2\|u^*_{\varepsilon,l,N}\chi_{E_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

if we choose C_0 large enough. Thus, the proof that (iv)⇒(ii) can be reduced to prove that

$$\|u^*_{\varepsilon,l,N}\chi_{E_\varepsilon}\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \le CN^{n/r}\|\phi^*_+(f)\|_{\dot K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)} \tag{3.1}$$

for any $r \in (0,1)$.

To prove (3.1) we first show that if $x \in E_\varepsilon$, then

$$u^*_{\varepsilon,l,N}(x) \le CN^{n/r}\left(M[\phi^*_+(f)]^r(x)\right)^{1/r}. \tag{3.2}$$

Note that for each fixed $x \in E_\varepsilon$, there exists $(y,t) \in \mathbb{R}^{n+1}_+$ such that $|x-y| < Nt < 1/\varepsilon$ and

$$|(f*\phi_t)(x)| \ge |(f*\phi_t)(y)|\left(\frac{Nt}{Nt+\varepsilon}\right)^l(1+\varepsilon N|y|)^{-l} > u^*_{\varepsilon,l,N}(x)/2.$$

On the other hand, we know by the definition of E_ε that if $x \in E_\varepsilon$ and $|x-z| < Nt$, then

$$t|\nabla_z(f*\phi_t)(z)| \le C_0\left(\frac{Nt}{Nt+\varepsilon}\right)^{-l}(1+\varepsilon l|y|)^l u^*_{\varepsilon,l,N}(x).$$

Therefore, if $x \in E_\varepsilon$, $|x-y| < Nt$ and $|x-z| < Nt$, then $t|\nabla_z(f*\phi_t)(z)| \le C_1|(f*\phi_t)(y)|$. Applying the mean value theorem, we have that for $w \in B(x,Nt)\cap B(y,t/(2C_1))$,

$$|(f*\phi_t)(w)-(f*\phi_t)(y)| \le |\nabla_z(f*\phi_t)(z)||w-y| \le |(f*\phi_t)(y)|/2,$$

where $z = \theta w + (1-\theta)y$ and $\theta \in (0,1)$. This shows that if $x \in E_\varepsilon$ and $w \in B(x,Nt)\cap B(y,t/(2C_1))$, then

$$|(f*\phi_t)(w)| \ge |(f*\phi_t)(y)|/2 \ge u^*_{\varepsilon,l,N}(x)/4.$$

Thus, for any $r \in (0,1)$ and $x \in E_\varepsilon$, we have

$$\begin{aligned} M((\phi_+^*(f))^r)(x) &\geq \frac{1}{|B(x,Nt)|}\int_{B(x,Nt)} (\phi_+^*(f)(w))^r dw \\ &\geq \frac{1}{|B(x,Nt)|}\int_{B(x,Nt)\cap B(y,t/(2C_1))} |f*\phi_t(w)|^r dw \\ &\geq CN^{-n}(u_{\varepsilon,l,N}^*(x))^r, \end{aligned}$$

and so (3.2) is true. Now choosing r sufficiently small so that $0 < r\alpha < n\delta_2$, then by (3.2) and Lemma 1.6 we have

$$\begin{aligned} \|u_{\varepsilon,l,N}^*\chi_{E_\varepsilon}\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &\leq CN^{n/r}\|(M(\phi_+^*(f))^r)^{1/r}\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \\ &= CN^{n/r}\|M(\phi_+^*(f))^r\|_{\dot{K}_{q(\cdot)/r}^{\alpha r,p/r}(\mathbb{R}^n)}^{1/r} \\ &\leq CN^{n/r}\|\phi_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \end{aligned}$$

This completes the proof of (iv)⇒(ii). Moreover,

$$\|\phi_{\nabla,N}^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq CN^{n/r}\|\phi_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}. \tag{3.3}$$

Now we consider (iv)⇒(i). By a simple computation, we know that

$$\phi_M^{**}(x) \leq \phi_\nabla^*(f)(x) + \sum_{k=0}^{\infty} 2^{-kM}\phi_{\nabla,2^{k+1}}^*(f)(x).$$

This via Lemma 3.2 and (3.3) gives that if N is large enough, then

$$\begin{aligned} \|G_N(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &\leq C\|\phi_\nabla^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} + C\sum_{k=0}^{\infty} 2^{-kM}\|\phi_{\nabla,2^{k+1}}^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \\ &\leq C\sum_{k=0}^{\infty} 2^{-k(M-n/r)}\|\phi_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \\ &\leq C\|\phi_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}, \end{aligned}$$

where $M > n/r$. Thus (iv)⇒(i) holds and the proof of Theorem 3.3 is completed. □

Remark 3.4. From the proof of Theorem 3.3 we can see that for any N_1, $N_2 > n+1$, the set

$$\{f \in \mathcal{S}(\mathbb{R}^n) : G_{N_1}(f) \in \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)\}$$

coincide with the set

$$\{f \in \mathcal{S}(\mathbb{R}^n) : G_{N_2}(f) \in \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)\}.$$

Moreover

$$\|G_{N_1}(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \approx \|G_{N_2}(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

The same conclusions are also true for non-homogeneous space.

Now we will give another characterization of spaces $H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ and $HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$. Given $s \in (0,1) \cup \mathbb{Z}_+$, define $\mathcal{T}_s$ to be the space of C^∞ functions on $\mathbb{R}^n$ with support contained in $B(0,1)$ such that $|\varphi(x) - \varphi(y)| \leq |x-y|^s$, for all $x, y \in \mathbb{R}^n$, when $s \in (0,1)$ and $\sum_{j=1}^{s} \|\nabla^j \varphi\|_\infty \le 1$, when $s \in \mathbb{Z}_+$. Let $f_s^* = \sup_{t>0} \sup_{\varphi \in \mathcal{T}_s} |\varphi_t * f(x)|$, for $s \in (0,1) \cup \mathbb{Z}_+$, and set $\mathcal{T}_1 = \mathcal{T}$ and $f_1^*(x) = f^*(x)$.

Theorem 3.5. *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose $s \in (0,1) \cup \mathbb{Z}_+$ and $s > \alpha/\delta_2 - n$. Then $f \in H\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $HK_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$) if and only if $f_s^* \in \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ (or $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$).*

The proof of Theorem 3.5 is based on Theorem 3.3 and the following two lemmas.

Lemma 3.6. *Let $n\delta_2 \le \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If $\sigma, s \in (0,1) \cup \mathbb{Z}_+$ and $\alpha/\delta_2 - n < \sigma < s$, then there are constants $C_2, C_3 > 0$ such that*

$$C_2^{-1}\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le \|f_s^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_2\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

and

$$C_3^{-1}\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le \|f_s^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_3\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

for all distributions f on $\mathbb{R}^n$.

Proof. We only prove the homogeneous case. Note that $\sigma > \alpha/\delta_2 - n$. We can choose q_1 to satisfy $\frac{n}{\sigma+n} < q_1 < \frac{n\delta_2}{\alpha}$. Since $n\delta_2 \le \alpha < \infty$, we have $n\delta_2/\alpha \le 1$. So $0 < q_1 < 1$. Setting $\varphi \in \mathcal{T}_\sigma$, from [5] we know $f_\sigma^* \le CM((f_s^*)^{q_1})^{1/q_1}$. Therefore, by Lemma 1.6 we have

$$\begin{aligned}
\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_\sigma^*\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|M((f_s^*)^{q_1})\chi_k\|_{L^{q(\cdot)/q_1}(\mathbb{R}^n)}^{p/q_1}\right\}^{1/p} \\
&= C\|M((f_s^*)^{q_1})\chi_k\|_{\dot{K}_{q(\cdot)/q_1}^{\alpha q_1, p/q_1}(\mathbb{R}^n)}^{q_1} \le C\|f_s^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}}. \qquad (3.4)
\end{aligned}$$

On the other hand, it follows from the definition of f_s^* that $f_s^*(x) \le f_\sigma^*(x)$. From this, we deduce the conclusion of Lemma 3.6. □

Remark 3.7. Let p, $q(\cdot)$ be as in Lemma 3.6. If $\alpha = n\delta_2$, $\sigma \in (0,1)$ and $s \in \mathbb{Z}_+$, then there are constants $C_4, C_5 > 0$ such that

$$C_4^{-1}\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le \|f_s^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_4\|f^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_4\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

and

$$C_5^{-1}\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le \|f_s^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_5\|f^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \le C_5\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

for all distributions f on $\mathbb{R}^n$.

Let $\theta(z)$ be a bump function which satisfies $\theta \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp}\theta \subset B(0,1)$, and $\int_{\mathbb{R}^n} \theta(x)dx = 1$.

Lemma 3.8. *Let $n\delta_2 \leq \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Suppose $f \in \mathcal{S}'(\mathbb{R}^n)$, θ is above and $\theta_+^*(f)$ is as in Theorem 3.3. If $s \in \mathbb{Z}_+$, $\sigma \in (0,1) \cup \mathbb{Z}_+$ and $\alpha/\delta_2 - n < \sigma$, then there are constants $C_6, C_7 > 0$ such that*

$$C_6^{-1}\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq \|\theta_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq C_6\|f_s^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

and

$$C_7^{-1}\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq \|\theta_+^*(f)\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq C_7\|f_s^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

Proof. The method of proof is similar to [5, Lemma 2.2]. Here we omit it. □

Remark 3.9. Let p, $q(\cdot)$, θ and $\theta_+^*(f)$ be as in Lemma 3.3. If $\alpha = n\delta_2$, $\sigma \in (0,1)$ and $s \in \mathbb{Z}_+$, then there are constants $C_8, C_9 > 0$ such that

$$C_8^{-1}\|f_\sigma^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq \|\theta_+^*(f)\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq C_8\|f_s^*\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}$$

and

$$C_9^{-1}\|f_\sigma^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq \|\theta_+^*(f)\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} \leq C_9\|f_s^*\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)}.$$

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