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PROPERTY (aBw) AND PERTURBATIONS

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ABSTRACT. A bounded linear operator T acting on a Banach space X satisfies property (aBw), a strong version of a-Weyl's theorem, if the complement in the approximate point spectrum $\sigma_a(T)$ of the upper semi-B-Weyl spectrum $\sigma_{USBW}(T)$ is the set of all isolated points of approximate point spectrum which are eigenvalues of finite multiplicity. In this paper we investigate the property (aBw) in connection with Weyl type theorems. In particular, we show that T satisfies property (aBw) if and only if T satisfies a-Weyl's theorem and $\sigma_{USBW}(T) = \sigma_{USW}(T)$, where $\sigma_{USW}(T)$ is the upper semi-Weyl spectrum of T. The preservation of property (aBw) is also studied under commuting nilpotent, quasi-nilpotent, power finite rank or Riesz perturbations. The theoretical results are illustrated by some concrete examples.

1. INTRODUCTION

Throughout this paper, let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X, and $\mathcal{F}(X)$ denote its ideal of finite rank operators on X. For an operator $T \in \mathcal{B}(X)$, let T^* denote its dual, $\mathcal{N}(T)$ its kernel, $\alpha(T)$ its nullity, $\mathcal{R}(T)$ its range, $\beta(T)$ its defect, $\sigma(T)$ its spectrum, $\sigma_a(T)$ its approximate point spectrum and $\sigma_p(T)$ its point spectrum. If the range $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is said to be *upper semi-Fredholm* (resp. *lower semi-Fredholm*). If $T \in \mathcal{B}(X)$ is both upper and lower semi-Fredholm, then T is said to be *Fredholm*. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then T is said to be *semi-Fredholm*, and its index is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in \mathcal{B}(X)$ is called *Weyl* (resp. *upper semi-Weyl*) if T is Fredholm and $\text{ind}(T) = 0$ (resp. T is upper

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semi-Fredholm and $\mathrm{ind}(T) \leq 0$). An operator $T \in \mathcal{B}(X)$ is called *Riesz* if $T - \lambda$ is Fredholm for all non-zero $\lambda \in \mathbb{C}$. Note that if $T \in \mathcal{B}(X)$ is Weyl (resp. upper semi-Weyl) and $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T, then $T + R$ also is Weyl (resp. upper semi-Weyl); we refer the reader to [23] for a proof.

Recall that the *descent* and the *ascent* of $T \in \mathcal{B}(X)$ are $dsc(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$ and $asc(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$, respectively (the infimum of an empty set is defined to be ∞). Note that if T has finite ascent and descent, then $asc(T) = dsc(T)$ (see [1, Theorem 3.3]). It is well known that $0 < asc(T) = dsc(T) < \infty$ precisely when λ is a pole of the resolvent of T (see [16, Proposition 50.2]). An operator $T \in \mathcal{B}(X)$ is said to be *left* (resp. *right*) *Drazin invertible* if $asc(T) < \infty$ (resp. $dsc(T) < \infty$) and $\mathcal{R}(T^{asc(T)+1})$ (resp. $\mathcal{R}(T^{dsc(T)})$) is closed. Moreover, T is *Drazin invertible* if it has finite ascent and descent. Clearly, $T \in \mathcal{B}(X)$ is Drazin invertible if and only if it is both left and right Drazin invertible. An operator $T \in \mathcal{B}(X)$ is said to be *Browder* (resp. *upper semi-Browder*) if T is Fredholm and $asc(T) = dsc(T) < \infty$ (resp. T is upper semi-Fredholm and $asc(T) < \infty$). Note that if $T \in \mathcal{B}(X)$ is Browder (resp. upper semi-Browder) and $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T, then $T + R$ also is Browder (resp. upper semi-Browder); we refer the reader to [22] for a proof.

For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as the map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0 = T$). If there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is Fredholm (resp. upper semi-Fredholm), then T is called *B-Fredholm* (resp. *upper semi-B-Fredholm*). It follows from [8, Proposition 2.1] that if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n is upper semi-Fredholm, then $\mathcal{R}(T^m)$ is closed, T_m is upper semi-Fredholm and $\mathrm{ind}(T_m) = \mathrm{ind}(T_n)$ for all $m \geq n$. This enables us to define the index of a upper semi-B-Fredholm operator T as the index of the upper semi-Fredholm operator T_n, where n is an integer satisfying $\mathcal{R}(T^n)$ is closed and T_n is upper semi-Fredholm. An operator $T \in \mathcal{B}(X)$ is called *B-Weyl* (resp. *upper semi-B-Weyl*) if T is B-Fredholm and $\mathrm{ind}(T) = 0$ (resp. T is upper semi-B-Fredholm and $\mathrm{ind}(T) \leq 0$). It is established in [24] that if $T \in \mathcal{B}(X)$ is B-Weyl (resp. upper semi-B-Weyl) and $F \in \mathcal{B}(X)$ is an operator satisfying $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ that commutes with T, then $T + F$ also is B-Weyl (resp. upper semi-B-Weyl).

For $T \in \mathcal{B}(X)$, let us define the *upper semi-Weyl spectrum*, the *Weyl spectrum*, the *upper semi-B-Weyl spectrum* and the *B-Weyl spectrum* of T as follows respectively:

$$\sigma_{USW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a upper semi-Weyl operator}\};$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\};$$

$$\sigma_{USBW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a upper semi-B-Weyl operator}\};$$

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

Let $\Pi(T)$ denote the set of all poles of T. We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if $T - \lambda I$ is left Drazin invertible. Let $\Pi_a(T)$ denote the set of all left poles of T. We also set

$$\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\},$$

$$\Pi_a^0(T) = \{\lambda \in \Pi_a(T) : \alpha(T - \lambda I) < \infty\},$$
$$E(T) = \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(T - \lambda I)\},$$
$$E_a(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \alpha(T - \lambda I)\},$$
$$E^0(T) = \{\lambda \in E(T) : \alpha(T - \lambda I) < \infty\}$$

and

$$E_a^0(T) = \{\lambda \in E_a(T) : \alpha(T - \lambda I) < \infty\},$$

where $\operatorname{iso} K$ denotes the isolated points of $K \subseteq \mathbb{C}$. Evidently, we have the following inclusions: $\Pi(T) \subseteq E(T)$, $\Pi_a(T) \subseteq E_a(T)$, $\Pi^0(T) \subseteq E^0(T)$ and $\Pi_a^0(T) \subseteq E_a^0(T)$; $\Pi(T) \subseteq \Pi_a(T)$, $\Pi^0(T) \subseteq \Pi_a^0(T)$, $E(T) \subseteq E_a(T)$ and $E^0(T) \subseteq E_a^0(T)$. Moreover,

$$\Pi^0(T) = \{\lambda \in \operatorname{iso} \sigma(T) : T - \lambda I \text{ is Browder}\}$$

and

$$\Pi_a^0(T) = \{\lambda \in \operatorname{iso} \sigma_a(T) : T - \lambda I \text{ is upper semi-Browder}\}.$$

Let $T \in \mathcal{B}(X)$. Following Coburn [10], we say that T satisfies *Weyl's theorem* if $\sigma(T)\backslash\sigma_W(T) = E^0(T)$, while, according to Rakočević [21], we say that T satisfies *a-Weyl's theorem* if $\sigma_a(T)\backslash\sigma_{USW}(T) = E_a^0(T)$. Following Harte and Lee [15], T is said to satisfy *Browder's theorem* if $\sigma(T)\backslash\sigma_W(T) = \Pi^0(T)$, while, according to Djordjević and Han [11], T is said to satisfy *a-Browder's theorem* if $\sigma_a(T)\backslash\sigma_{USW}(T) = \Pi_a^0(T)$. Following Berkani and Koliha [7], we say that *generalized Browder's theorem* holds for T if $\sigma(T)\backslash\sigma_{BW}(T) = \Pi(T)$, and *generalized a-Browder's theorem* holds for T if $\sigma_a(T)\backslash\sigma_{USBW}(T) = \Pi_a(T)$. According to [3, 7, 4, 11, 21], we have the following implications:

$$\begin{array}{ccccc}
\text{Weyl's theorem} & \Longrightarrow & \text{Browder's theorem} & \Longleftrightarrow & \text{generalized Browder's theorem} \\
\Uparrow & & \Uparrow & & \\
\text{a-Weyl's theorem} & \Longrightarrow & \text{a-Browder's theorem} & \Longleftrightarrow & \text{generalized a-Browder's theorem}
\end{array}$$

Following Gupta and Kashyap [13], we say that $T \in \mathcal{B}(X)$ satisfies *property* (Bw) if $\sigma(T)\backslash\sigma_{BW}(T) = E^0(T)$. By [13, Theorem 2.5], $T \in \mathcal{B}(X)$ satisfies *property* (Bw) if and only if generalized Browder's theorem holds for T and $\Pi(T) = E^0(T)$. The following definition, which is introduced in [14] under another name and can be viewed as the approximate point spectrum version of property (Bw), describes the spectral property we will study in this paper.

Definition 1.1. An operator $T \in \mathcal{B}(X)$ is said to satisfy *property* (aBw) if

$$\sigma_a(T)\backslash\sigma_{USBW}(T) = E_a^0(T).$$

This paper is organized as follows. In Section 2, we study the property (aBw) in connection with Weyl type theorems. We show that an operator satisfying property (aBw) satisfies property (Bw), but not conversely. Moreover, we establish that property (aBw) holds for T if and only if a-Weyl's theorem holds for T and $\sigma_{USBW}(T) = \sigma_{USW}(T)$. In Section 3, the preservation of property (aBw) is investigated under commuting nilpotent, quasi-nilpotent, power finite rank or Riesz perturbations. Some concrete examples are also given in this paper to illustrate the theoretical results.

2. Property (aBw) and Weyl type theorems

Before stating our results, we need to introduce the following notations. For each $n \in \mathbb{N}$, we set

$$c_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1}),$$
$$c_n^{'}(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n)$$

and

$$k_n(T) = \dim(\mathcal{N}(T) \cap \mathcal{R}(T^n))/(\mathcal{N}(T) \cap \mathcal{R}(T^{n+1})).$$

Let $T \in \mathcal{B}(X)$ and let $d \in \mathbb{N}$. We say that T has *uniform descent* for $n \geq d$ if $k_n(T) = 0$ for all $n \geq d$. If in addition $\mathcal{N}(T^d) + \mathcal{R}(T)$ is closed in X, then we say that T has *eventual topological uniform descent*, and, more precisely, that T has *topological uniform descent for* $n \geq d$.

Operators with eventual topological uniform descent are introduced by Grabiner in [12]. It includes many classes of operators such as operators of Kato type, quasi-Fredholm operators, semi-B-Fredholm operators, operators with finite descent and operators with finite essential descent, and so on. One of the most important results for operators with eventual topological uniform descent is Grabiner's punctured neighbourhood theorem [12, Theorem 4.7]. Discussions of operators with eventual topological uniform descent may also be found in [6, 9, 12, 17, 18, 26].

The following result shows that property (aBw) entails property (Bw).

Theorem 2.1. *If $T \in \mathcal{B}(X)$ satisfies property (aBw), then it satisfies property (Bw).*

Proof. Suppose that T satisfies property (aBw). Let $\lambda \in \sigma(T) \backslash \sigma_{BW}(T)$. We claim that $\lambda \in \sigma_a(T)$. Otherwise, if $\lambda \notin \sigma_a(T)$, then $\alpha(T - \lambda I) = 0$. Since $T - \lambda I$ is a B-Weyl operator, there exists $m \in \mathbb{N}$ such that $c_m(T - \lambda I) = c_m^{'}(T - \lambda I) = 0$. From [19, Lemma 3.2] and the fact that $\mathcal{N}(T - \lambda I) = \{0\}$, it then follows that

$$\begin{aligned}
\beta(T - \lambda I) &= \dim X/\mathcal{R}(T - \lambda I) \\
&= \dim X/(\mathcal{R}(T - \lambda I) + \mathcal{N}((T - \lambda I)^m)) \\
&= \dim \mathcal{R}((T - \lambda I)^m)/\mathcal{R}((T - \lambda I)^{m+1}) \\
&= c_m(T - \lambda I) \\
&= 0.
\end{aligned}$$

Consequently, $\lambda \notin \sigma(T)$, which leads to a contradiction. Hence $\lambda \in \sigma_a(T)$. As T satisfies property (aBw), then $\lambda \in E_a^0(T)$. Since $T - \lambda I$ is a B-Weyl operator, there exists $d \in \mathbb{N}$ such that $T - \lambda I$ is an operator of topological uniform descent for $n \geq d$. Hence by [12, Theorem 4.7], there exists a punctured neighbourhood U_1 of λ such that $\alpha(T - \mu I) = c_n^{'}(T - \lambda I) = c_n(T - \lambda I) = \beta(T - \mu I)$ for all $n \geq d$ and $\mu \in U_1$. Since $\lambda \in E_a^0(T)$, there exists a punctured neighbourhood U_2 of λ such that $\alpha(T - \mu I) = 0$ for $\mu \in U_2$. Now if we let $U = U_1 \cap U_2$, we find that $U \cap \sigma(T) = \varnothing$, therefore $\lambda \in E^0(T)$.

Conversely, let $\lambda \in E^0(T)$. Then $\lambda \in E_a^0(T)$, and hence property (aBw) for T implies that $\lambda \notin \sigma_{USBW}(T)$. Thus, there exists $d \in \mathbb{N}$ such that $T - \lambda I$ is an operator of topological uniform descent for $n \geq d$. Hence by [12, Theorem 4.7], there exists a punctured neighbourhood V_1 of λ such that $\alpha(T-\mu I) = c_n^{'}(T-\lambda I)$ and $\beta(T-\mu I) = c_n(T-\lambda I)$ for all $n \geq d$ and $\mu \in V_1$. Since $\lambda \in E^0(T)$, there exists a punctured neighbourhood V_2 of λ such that $\alpha(T-\mu I) = \beta(T-\mu I) = 0$ for $\mu \in V_2$. Now if we let $V = V_1 \cap V_2$, we find that $c_n^{'}(T-\lambda I) = c_n(T-\lambda I) = 0$. So $T-\lambda I$ is Drazin invertible, and hence $\lambda \in \sigma(T)\backslash\sigma_{BW}(T)$. Therefore, property (Bw) holds for T. □

The converse of Theorem 2.1 does not hold in general, as we can see in the following example.

Example 2.2. We consider the operator $T = U \oplus 0$ defined on $l_2(\mathbb{N}) \oplus l_2(\mathbb{N})$, where $U : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ is the unilateral right shift operator defined by

$$U(x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \quad \text{for all } \ (x_n) \in l_2(\mathbb{N}).$$

Then we have,

$$\sigma(T) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\},$$
$$\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\},$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}$$

and

$$\sigma_{USBW}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

It then follows that $\sigma(T)\backslash\sigma_{BW}(T) = \varnothing = E^0(T)$, i.e. T satisfies property (Bw). But $\sigma_a(T)\backslash\sigma_{USBW}(T) = \{0\} \neq \varnothing = E_a^0(T)$, i.e. T does not satisfy property (aBw).

Definition 2.3. An operator $T \in \mathcal{B}(X)$ has the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0 for brevity), if for every open neighborhood U of λ_0 the only analytic solution $f : U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

It is easy to observe that property (aBw) for T is equivalent to property (Bw) for T, if T^* has SVEP at each $\lambda \notin \sigma_{USBW}(T)$. The following result shows that property (aBw) entails a-Weyl's theorem.

Theorem 2.4. *If $T \in \mathcal{B}(X)$ satisfies property (aBw), then it satisfies a-Weyl's theorem.*

Proof. Suppose that T satisfies property (aBw). Let $\lambda \in \sigma_a(T)\backslash\sigma_{USW}(T)$. Then $\lambda \in \sigma_a(T)\backslash\sigma_{USBW}(T)$ and hence property (aBw) ensures that $\lambda \in E_a^0(T)$. Conversely, let $\lambda \in E_a^0(T)$. Then $\alpha(T-\lambda I) < \infty$ and property (aBw) implies that $\lambda \in \sigma_a(T)\backslash\sigma_{USBW}(T)$. Thus, there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed, T_n is upper semi-Fredholm and $\text{ind}(T_n) \leq 0$. Since $\alpha(T-\lambda I) < \infty$, [5, Lemma 2.4] and [8, Proposition 2.1] ensure that $\mathcal{R}(T)$ is closed and $\text{ind}(T) = \text{ind}(T_n) \leq 0$, and this implies that $\lambda \in \sigma_a(T)\backslash\sigma_{USW}(T)$. Therefore, a-Weyl's theorem holds for T. □

The converse of Theorem 2.4 does not hold in general, as we can see in the following example.

Example 2.5. We consider T as in Example 2.2. Then T does not satisfy property (aBw). But we have

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\},$$
$$\sigma_{USW}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$$

and

$$E_a^0(T) = \varnothing,$$

hence a-Weyl's theorem holds for T.

The precise relationship between property (aBw) and a-Weyl's theorem is described by the following theorem.

Theorem 2.6. *An operator $T \in \mathcal{B}(X)$ satisfies property (aBw) if and only if it satisfies a-Weyl's theorem and $\sigma_{USBW}(T) = \sigma_{USW}(T)$.*

Proof. Suppose that T satisfies property (aBw). Then by Theorem 2.4, a-Weyl's theorem holds for T. And hence, $\sigma_a(T) \backslash \sigma_{USBW}(T) = E_a^0(T) = \sigma_a(T) \backslash \sigma_{USW}(T)$. Thus $\sigma_{USBW}(T) = \sigma_{USW}(T)$.

Conversely, suppose that T satisfies a-Weyl's theorem and $\sigma_{USBW}(T) = \sigma_{USW}(T)$. Then we have $\sigma_a(T) \backslash \sigma_{USBW}(T) = \sigma_a(T) \backslash \sigma_{USW}(T) = E_a^0(T)$. Therefore, property (aBw) holds for T. □

Noting that a-Weyl's theorem entails generalized a-Browder's theorem, by Theorem 2.4 and Example 2.5, we observe that property (aBw) entails generalized a-Browder's theorem, but not vice versa. The following characterization of property (aBw) is first established in [14, Theorem 2.5]. And we give here a different proof for completeness. Moreover, some useful information is also given for property (aBw).

Theorem 2.7. *An operator $T \in \mathcal{B}(X)$ satisfies property (aBw) if and only if it satisfies generalized a-Browder's theorem and $\Pi_a(T) = E_a^0(T)$. In this case, we have*

$$\Pi_a^0(T) = E_a^0(T)$$

and

$$\sigma_a(T) = \sigma_{USBW}(T) \cup \operatorname{iso} \sigma_a(T).$$

Proof. Suppose that T satisfies property (aBw). Then generalized a-Browder's theorem holds for T. And hence, $\Pi_a(T) = \sigma_a(T) \backslash \sigma_{USBW}(T) = E_a^0(T)$. Thus $\Pi_a(T) = E_a^0(T)$.

Conversely, suppose that T satisfies generalized a-Browder's theorem and $\Pi_a(T) = E_a^0(T)$. Then we have $\sigma_a(T) \backslash \sigma_{USBW}(T) = \Pi_a(T) = E_a^0(T)$. Therefore, property (aBw) holds for T.

Since $\Pi_a(T) = E_a^0(T)$, we have

$$E_a^0(T) = E_a^0(T) \cap \Pi_a(T) = \Pi_a^0(T).$$

To show the equality $\sigma_a(T) = \sigma_{USBW}(T) \cup \operatorname{iso} \sigma_a(T)$, observe first that $\sigma_{USBW}(T) \cup \operatorname{iso} \sigma_a(T) \subseteq \sigma_a(T)$ holds for every $T \in \mathcal{B}(X)$. To show the opposite inclusion, suppose that $\lambda \in \sigma_a(T)$ and $\lambda \notin \sigma_{USBW}(T)$. Then $\lambda \in E_a^0(T)$ because T satisfies property (aBw). Therefore $\lambda \in \operatorname{iso}\sigma_a(T)$, and so $\sigma_a(T) \subseteq \sigma_{USBW}(T) \cup \operatorname{iso}\sigma_a(T)$. This shows that $\sigma_a(T) = \sigma_{USBW}(T) \cup \operatorname{iso}\sigma_a(T)$. $\square$

Let $\mathcal{H}(\sigma(T))$ denote the set of functions f which are defined and analytic on an open neighbourhood of $\sigma(T)$.

Theorem 2.8. *Let $T \in \mathcal{B}(X)$. If $\sigma_p(T) = \varnothing$, then property (aBw) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. The hypothesis $\sigma_p(T) = \varnothing$ ensures from the proof of [2, Theorem 2.5] that $\sigma_p(f(T))$ is empty. This implies that $f(T)$ has SVEP, and in particular $f(T)$ has SVEP at every $\lambda \notin \sigma_{USBW}(f(T))$. Consequently, $f(T)$ satisfies generalized a-Browder's theorem. To prove property (aBw) holds for $f(T)$, by Theorem 2.7, it then suffices to prove that

$$\Pi_a(f(T)) = E_a^0(f(T)).$$

Evidently, the condition $\sigma_p(f(T)) = \varnothing$ entails that $E_a(f(T)) = E_a^0(f(T)) = \varnothing$. Because $\Pi_a(f(T)) \subseteq E_a(f(T))$ holds for every operator $T \in \mathcal{B}(X)$, we also have $\Pi_a(f(T)) = \varnothing$. By Theorem 2.7, it then follows that property (aBw) holds for $f(T)$. $\square$

Associated with $T \in \mathcal{B}(X)$, the *analytic core* of T is defined by

$$\begin{aligned} K(T) :=\{x \in X : &\text{ there exist a sequence } \{x_n\}_{n\geq 1} \text{ in } X \text{ and a constant } \delta > 0 \\ &\text{such that } Tx_1 = x, Tx_{n+1} = x_n \text{ and } ||x_n|| \leq \delta^n ||x|| \text{ for all } n \geq 1\}. \end{aligned}$$

Corollary 2.9. *Let $T \in \mathcal{B}(X)$. If there exists $\lambda_0 \in \mathbb{C}$ such that $K(T-\lambda_0) = \{0\}$ and $\mathcal{N}(T-\lambda_0) = \{0\}$, then property (aBw) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. Obviously, $\mathcal{N}(T-\lambda) \subseteq K(T-\lambda_0)$ for all complex $\lambda \neq \lambda_0$. Then $\mathcal{N}(T-\lambda) = \{0\}$ for all $\lambda \in \mathbb{C}$, and hence $\sigma_p(T) = \varnothing$. By Theorem 2.8, $f(T)$ satisfies property (aBw). $\square$

The conditions of Corollary 2.9 are satisfy by any injective operator $T \in \mathcal{B}(X)$ for which the *hyperrange* $\mathcal{R}(T^\infty) := \bigcap_{n=1}^\infty \mathcal{R}(T^n)$ is $\{0\}$. In fact, $K(T) \subseteq \mathcal{R}(T^\infty)$ for all $T \in \mathcal{B}(X)$, thus $K(T) = \{0\}$. In particular, the conditions of Corollary 2.9 are satisfy by the unilateral weighted right shift operators on the sequence space $l^p(\mathbb{N})$, for any $1 \leq p < \infty$.

3. Property (aBw) under perturbations

In this section, we consider the stability of property (aBw) (resp. property (Bw)) under commuting nilpotent, quasi-nilpotent, power finite rank or Riesz perturbations.

Theorem 3.1. (1) *If $T \in \mathcal{B}(X)$ satisfies property (aBw) and $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T, then $T+N$ satisfies property (aBw).*

(2) *If $T \in \mathcal{B}(X)$ satisfies property (Bw) and $N \in \mathcal{B}(X)$ is a nilpotent operator commuting with T, then $T+N$ satisfies property (Bw).*

Proof. It is well known that

$$\sigma(T+N)=\sigma(T), \quad \sigma_a(T+N)=\sigma_a(T).$$

Moreover, we know from [24] that

$$\sigma_{BW}(T+N)=\sigma_{BW}(T), \quad \sigma_{USBW}(T+N)=\sigma_{USBW}(T);$$
$$E^0(T+N)=E^0(T), \quad E^0_a(T+N)=E^0_a(T).$$

Therefore, property (aBw) and property (Bw) are transmitted from T to $T+N$. $\square$

As a-Weyl's theorem (resp. Weyl's theorem), property (aBw) (resp. property (Bw)) is not preserved under commuting quasi-nilpotent perturbations.

Example 3.2. We consider the operator $T=V\oplus I$ defined on $X\oplus Y$, where $V: X \longrightarrow X$ is an injective quasi-nilpotent operator which is not nilpotent and Y is a non-zero finite-dimensional Banach space. Let $Q=-V\oplus 0$. Then Q is quasi-nilpotent and $TQ=QT$. Moreover,

$$\sigma_a(T)=\sigma(T)=\{0,1\},$$
$$\sigma_{USBW}(T)=\sigma_{BW}(T)=\{0\},$$
$$\sigma_a(T+Q)=\sigma(T+Q)=\{0,1\}$$

and

$$\sigma_{USBW}(T+Q)=\sigma_{BW}(T+Q)=\varnothing.$$

It follows that $E^0_a(T)=E^0(T)=\{1\}$ and $E^0_a(T+Q)=E^0(T+Q)=\{1\}$. Therefore, $\sigma_a(T)\backslash\sigma_{USBW}(T)=\{1\}=E^0_a(T)$ and $\sigma(T)\backslash\sigma_{BW}(T)=\{1\}=E^0(T)$, i.e. property (aBw) and property (Bw) hold for T. But $\sigma_a(T+Q)\backslash\sigma_{USBW}(T+Q)=\{0,1\}\neq\{1\}=E^0_a(T+Q)$ (resp. $\sigma(T+Q)\backslash\sigma_{BW}(T+Q)=\{0,1\}\neq\{1\}=E^0(T+Q)$), i.e. $T+Q$ does not satisfy property (aBw) (resp. property (Bw)).

As a-Weyl's theorem (resp. Weyl's theorem), property (aBw) (resp. property (Bw)) is not preserved under commuting finite rank perturbations as the following example shows.

Example 3.3. We also consider the operator $T=V\oplus I$ defined on $X\oplus Y$ as in Example 3.2. Take any non-zero (finite rank) projection $P\in\mathcal{B}(Y)$ and let $F=0\oplus -P$. Then F is of finite rank, $TF=FT$ and property (aBw) and property (Bw) hold for T by Example 3.2. But property (aBw) (resp. property (Bw)) does not hold for $T+F$ because $0\in E^0_a(T+F)\cap\sigma_{USBW}(T+F)$ (resp. $0\in E^0(T+F)\cap\sigma_{BW}(T+F)$).

Recall that an operator $T\in\mathcal{B}(X)$ is called *isoloid* if $\operatorname{iso}\sigma(T)=E(T)$, and *a-isoloid* if $\operatorname{iso}\sigma_a(T)=E_a(T)$. It is established in [25] that if $F\in\mathcal{B}(X)$ satisfies $F^n\in\mathcal{F}(X)$ for some $n\in\mathbb{N}$ and $T\in\mathcal{B}(X)$ is an isoloid operator commuting with F, then property (Bw) is transmitted from T to $T+F$. Similarly, we have:

Theorem 3.4. *Suppose that $F\in\mathcal{B}(X)$ satisfies $F^n\in\mathcal{F}(X)$ for some $n\in\mathbb{N}$ and that $T\in\mathcal{B}(X)$ is an a-isoloid operator commuting with F. If T satisfies property (aBw), then $T+F$ satisfies property (aBw).*

Proof. By Theorem 2.6, we know that an operator $T \in \mathcal{B}(X)$ satisfies property (aBw) if and only if a-Weyl's theorem holds for T and $\sigma_{USBW}(T) = \sigma_{USW}(T)$. Since F satisfies $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$, $T + F$ satisfies a-Weyl's theorem by [20, Theorem 2.6]. Hence it suffices to establish that

$$\sigma_{USBW}(T + F) = \sigma_{USW}(T + F).$$

By [24, Theorem 2.11] and [23, Proposition 5], we have that

$$\sigma_{USBW}(T + F) = \sigma_{USBW}(T) = \sigma_{USW}(T) = \sigma_{USW}(T + F).$$

Consequently, $T + F$ satisfies property (aBw). □

Corollary 3.5. *Suppose that $F \in \mathcal{F}(X)$ and that $T \in \mathcal{B}(X)$ is an a-isoloid operator commuting with F. If T satisfies property (aBw), then $T + F$ satisfies property (aBw).*

Recall that an operator $T \in \mathcal{B}(X)$ is called *finite-isoloid* if $\operatorname{iso}\sigma(T) = E^0(T)$, and *finite-a-isoloid* if $\operatorname{iso}\sigma_a(T) = E^0_a(T)$. In the following, we denote by $\operatorname{acc} K$ the accumulation points of $K \subseteq \mathbb{C}$.

Theorem 3.6. (1) *Suppose that $T \in \mathcal{B}(X)$ is finite-a-isoloid, $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T and $\operatorname{acc}\sigma_a(T) = \operatorname{acc}\sigma_a(T + R)$. If T satisfies property (aBw), then $T + R$ satisfies property (aBw).*

(2) *Suppose that $T \in \mathcal{B}(X)$ is finite-isoloid, $R \in \mathcal{B}(X)$ is a Riesz operator commuting with T and $\operatorname{acc}\sigma(T) = \operatorname{acc}\sigma(T + R)$. If T satisfies property (Bw), then $T + R$ satisfies property (Bw).*

Proof. (1) By Theorem 2.7, we know that an operator $T \in \mathcal{B}(X)$ satisfies property (aBw) if and only if generalized a-Browder's theorem holds for T and $E^0_a(T) = \Pi_a(T)$. Since R is a Riesz operator, $T+R$ satisfies generalized a-Browder's theorem by [24]. Hence it suffices to establish that

$$E^0_a(T + R) = \Pi_a(T + R).$$

Let $\lambda \in E^0_a(T + R)$. If $\lambda \notin \sigma_a(T)$, then $T + R - \lambda$ is upper semi-Browder by [22, Theorem 1], and hence $\lambda \in \Pi_a(T + R)$. Suppose that $\lambda \in \sigma_a(T)$. Then from the assumption

$$\operatorname{acc}\sigma_a(T) = \operatorname{acc}\sigma_a(T + R)$$

it follows that $\lambda \in \operatorname{iso}\sigma_a(T)$, and so $\lambda \in E^0_a(T)$ because T is finite-a-isoloid. Thus, Theorem 2.7 implies that $\lambda \in \Pi^0_a(T)$. Consequently, $T - \lambda$ is upper semi-Browder, and hence $T + R - \lambda$ also is upper semi-Browder by [22, Theorem 1]. So, $\lambda \in \Pi_a(T + R)$.

Conversely, let $\lambda \in \Pi_a(T + R)$. In order to show that $\lambda \in E^0_a(T + R)$, it need only to show that $\alpha(T + R - \lambda) < \infty$. If $\lambda \notin \sigma_a(T)$, then $T + R - \lambda$ is upper semi-Browder by [22, Theorem 1], and hence $\alpha(T + R - \lambda) < \infty$. Suppose that $\lambda \in \sigma_a(T)$. Then from the assumption

$$\operatorname{acc}\sigma_a(T) = \operatorname{acc}\sigma_a(T + R)$$

it follows that $\lambda \in \operatorname{iso}\sigma_a(T)$, and so $\lambda \in E^0_a(T)$ because T is finite-a-isoloid, hence $\lambda \in \Pi^0_a(T)$ by Theorem 2.7. Thus $T - \lambda$ is upper semi-Browder, and so is

$T+R-\lambda$ by [22, Theorem 1], hence $\alpha(T+R-\lambda)<\infty$. Consequently, $T+R$ satisfies property (aBw).

(2) By [13, Theorem 2.5], we know that an operator $T\in\mathcal{B}(X)$ satisfies property (Bw) if and only if generalized Browder's theorem holds for T and $E^0(T)=\Pi(T)$. Since R is a Riesz operator, $T+R$ satisfies generalized Browder's theorem by [24]. Hence it suffices to establish that

$$E^0(T+R)=\Pi(T+R).$$

Let $\lambda\in E^0(T+R)$. If $\lambda\notin\sigma(T)$, then $T+R-\lambda$ is Browder by [22, Theorem 1], and hence $\lambda\in\Pi(T+R)$. Suppose that $\lambda\in\sigma(T)$. Then from the assumption

$$\operatorname{acc}\sigma(T)=\operatorname{acc}\sigma(T+R)$$

it follows that $\lambda\in\operatorname{iso}\sigma(T)$, and so $\lambda\in E^0(T)$ because T is finite-isoloid. Thus, $\lambda\in E^0(T)\cap\Pi(T)=\Pi^0(T)$ by [13, Theorem 2.5]. Consequently, $T-\lambda$ is Browder, and hence $T+R-\lambda$ also is Browder by [22, Theorem 1]. So, $\lambda\in\Pi(T+R)$.

Conversely, let $\lambda\in\Pi(T+R)$. In order to show that $\lambda\in E^0(T+R)$, it need only to show that $\alpha(T+R-\lambda)<\infty$. If $\lambda\notin\sigma(T)$, then $T+R-\lambda$ is Browder by [22, Theorem 1], and hence $\alpha(T+R-\lambda)<\infty$. Suppose that $\lambda\in\sigma(T)$. Then from the assumption

$$\operatorname{acc}\sigma(T)=\operatorname{acc}\sigma(T+R)$$

it follows that $\lambda\in\operatorname{iso}\sigma(T)$, and so $\lambda\in E^0(T)$ because T is finite-isoloid, hence $\lambda\in E^0(T)\cap\Pi(T)=\Pi^0(T)$ by [13, Theorem 2.5]. Thus $T-\lambda$ is Browder, and so is $T+R-\lambda$ by [22, Theorem 1], hence $\alpha(T+R-\lambda)<\infty$. Consequently, $T+R$ satisfies property (Bw). □

Corollary 3.7. (1) *Suppose that $T\in\mathcal{B}(X)$ is finite-a-isoloid, $K\in\mathcal{B}(X)$ is a compact operator commuting with T and $\operatorname{acc}\sigma_a(T)=\operatorname{acc}\sigma_a(T+K)$. If T satisfies property (aBw), then $T+K$ satisfies property (aBw).*

(2) *Suppose that $T\in\mathcal{B}(X)$ is finite-isoloid, $K\in\mathcal{B}(X)$ is a compact operator commuting with T and $\operatorname{acc}\sigma(T)=\operatorname{acc}\sigma(T+K)$. If T satisfies property (Bw), then $T+K$ satisfies property (Bw).*

Without the assumption $\operatorname{acc}\sigma_a(T)=\operatorname{acc}\sigma_a(T+K)$ (resp. $\operatorname{acc}\sigma(T)=\operatorname{acc}\sigma(T+K)$), the result of Corollary 3.7(1) (resp. Corollary 3.7(2)) will not be true.

Example 3.8. Let $T:l_2(\mathbb{N})\longrightarrow l_2(\mathbb{N})$ be a compact operator defined by

$$T(x_1,x_2,\cdots)=(x_1,\frac{1}{2}x_2,\frac{1}{3}x_3,\cdots),\quad \text{for all}\ \ (x_n)\in l_2(\mathbb{N}).$$

Let $K=-T$.

(1) Clearly, T is finite-a-isoloid and T satisfies property (aBw). But $T+K=0$ does not satisfy property (aBw).

(2) Clearly, T is finite-isoloid and T satisfies property (Bw). But $T+K=0$ does not satisfy property (Bw).

For quasi-nilpotent perturbations, we have:

Theorem 3.9. (1) *Suppose that $T\in\mathcal{B}(X)$ is finite-a-isoloid. If T satisfies property (aBw), then so does $T+Q$ for every quasi-nilpotent operator Q commuting with T.*

(2) *Suppose that $T \in \mathcal{B}(X)$ is finite-isoloid. If T satisfies property (Bw), then so does $T + Q$ for every quasi-nilpotent operator Q commuting with T.*

Proof. Since $\sigma_a(T) = \sigma_a(T + Q)$ and $\sigma(T) = \sigma(T + Q)$, the conclusions follow immediately from Theorem 3.6. $\square$

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