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CERTAIN GEOMETRIC STRUCTURES OF Λ-SEQUENCE SPACES

ATANU MANNA

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Abstract. The Λ-sequence spaces Λ_p for $1 < p \leq \infty$ and their generalized forms $\Lambda_{\hat{p}}$ for $1 < \hat{p} < \infty$, $\hat{p} = (p_n)$, $n \in \mathbb{N}_0$ are introduced. The James constants and strong n-th James constants of Λ_p for $1 < p \leq \infty$ are determined. It is proved that the generalized Λ-sequence space $\Lambda_{\hat{p}}$ is a closed subspace of the Nakano sequence space $l_{\hat{p}}(\mathbb{R}^{n+1})$ of finite dimensional Euclidean space $\mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$. Hence it follows that sequence spaces Λ_p and $\Lambda_{\hat{p}}$ possess the uniform Opial property, (β)-property of Rolewicz, and weak uniform normal structure. Moreover, it is established that $\Lambda_{\hat{p}}$ possesses the coordinate wise uniform Kadec–Klee property. Further, necessary and sufficient conditions for element $x \in S(\Lambda_{\hat{p}})$ to be an extreme point of $B(\Lambda_{\hat{p}})$ are derived. Finally, estimation of von Neumann–Jordan and James constants of two dimensional Λ-sequence space $\Lambda_2^{(2)}$ are carried out. Upper bound for the Hausdorff matrix operator norm on the non-absolute type Λ-sequence spaces is also obtained.

1. Introduction

There are several important geometric constants of a Banach space such as von Neumann–Jordan constant, James constant, Dunkl–Williams constant, Khintchine constant, Zbăganu, Ptolemy constant which are very useful for studying the geometric theory of Banach spaces. The proximity (or remoteness) of the space to a Hilbert space was measured by using von Neumann–Jordan constant and Dunkl-Williams constant. The uniform non-squareness of a unit ball in a

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real Banach space was measured by James constant (or sometimes called *James non-square constant*). These constants, which have been investigated recently by many researchers, occupy a prominent place in the study of geometrical properties of Banach spaces.

Maligranda et al. [17] obtained the classical James constant and the n-th James constant for the Cesàro sequence spaces ces_p, $1 < p \leq \infty$ and Cesàro-Orlicz sequence spaces. Further, Kamińska and Kubiak [14] obtained these constants on Cesàro-Orlicz sequence spaces using an alternative approach. Let l^0 be the space of all real sequences and $\mathbb{N}_0$ be the set of all natural numbers $\mathbb{N}$ including 0, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $x = (x_n)_{n\in\mathbb{N}_0} \in l^0$. In order to avoid ambiguity $x = (x_n)_{n\in\mathbb{N}_0}$ will be replaced by $x = (x_n)$. The Cesàro sequence spaces ces_p, $1 < p \leq \infty$ were first introduced by Shiue [27] and studied by Leibowitz [15] later. They are defined as follows:

$$ces_p = \Big\{x \in l^0 : \Big(\sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\Big)^p\Big)^{\frac{1}{p}} < \infty\Big\} \text{ for } 1 < p < \infty,$$

$$ces_\infty = \Big\{x \in l^0 : \sup_{n\in\mathbb{N}_0}\frac{1}{n+1}\sum_{k=0}^{n}|x_k| < \infty\Big\}.$$

The Cesàro sequence space ces_p is generalized to $ces_{\hat{p}}$ for $\hat{p} = (p_n)$, $p_n > 1$, $n \in \mathbb{N}_0$ ([13]) and defined as

$$ces_{\hat{p}} = \Big\{x \in l^0 : \Big(\sum_{n=0}^{\infty}\Big(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\Big)^{p_n}\Big)^{\frac{1}{p_n}} < \infty\Big\}.$$

Several authors ([5], [6], [10], [18], [21], [22], [25], [28]) studied geometric properties such as Opial property, Kadec–Klee property, (β) property of Rolewicz, weak uniform normal structure etc. for the spaces ces_p and $ces_{\hat{p}}$. These constants play very important role in the study of fixed point theory. For example, Opial property has several applications in the Banach fixed point theory, differential equations, integral equations etc. On the other hand, Kadec–Klee property is applied to establish certain results in the ergodic theory (see [23]).

In line with the title of this paper, we report certain geometric structures of Λ-sequence spaces Λ_p, $1 < p \leq \infty$ and their generalizations $\Lambda_{\hat{p}}$, for $1 < \hat{p} < \infty$, $\hat{p} = (p_n)$. It may be noted that the spaces Λ_p and $\Lambda_{\hat{p}}$ are generalized form of the sequence spaces ces_p and $ces_{\hat{p}}$, respectively. For example, our generalization includes the following important results:

(i) ces_p and $ces_{\hat{p}}$ have the uniform Opial property (see [6] and [21], respectively),

(ii) ces_p and $ces_{\hat{p}}$ have the (β) property (see [5] and [25], respectively),

(iii) Both ces_p [6] and $ces_{\hat{p}}$ [25] possess the weak uniform normal structures,

(iv) $ces_{\hat{p}}$ possesses the uniform Kadec Klee property [22], and Kadec–Klee property ([28]),

(v) The James constants are given by $J(ces_p) = 2$, $J^s_n(ces_p) = n$ for $1 < p \leq \infty$ and $J(ces_2^{(2)}) = \sqrt{2 + \frac{2}{\sqrt{5}}}$ [17],

(vi) ces_p and $ces_{\hat{p}}$ have extreme points.

Therefore geometric structures of the spaces Λ_p and $\Lambda_{\hat{p}}$ are unified study of geometric structures for the known sequence spaces. The following subsection is begin with the definition of Λ-sequence spaces:

1.1. **Sequence spaces Λ_p, $1 < p \leq \infty$.** The Λ-sequence spaces stem from the notion of Λ-strong convergence coined by Móricz [19]. Let $\Lambda = \{\lambda_k : k = 0, 1, 2, \ldots\}$ be a non-decreasing sequence of positive numbers tending to ∞, i.e., $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_k \to \infty$ with $\frac{\lambda_{k+1}}{\lambda_k} \to 1$ as $k \to \infty$ and $x = (x_k) \in l^0$. Further, it is assumed that $\lambda_{-1} = 0$.
Define sequence spaces Λ_p, for $1 < p \leq \infty$ as

$$\Lambda_p = \big\{x = (x_k) \in l^0 : \|x\|_p < \infty\big\} \text{ for } 1 < p < \infty,$$
$$\Lambda_\infty = \big\{x = (x_k) \in l^0 : \|x\|_\infty < \infty\big\} \text{ for } p = \infty,$$

where $\|.\|_p$, $1 < p < \infty$ and $\|x\|_\infty$ are defined by

$$\|x\|_p = \Big(\sum_{n=0}^{\infty}\big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|\big)^p\Big)^{\frac{1}{p}}$$

and

$$\|x\|_\infty = \sup_{n\in\mathbb{N}_0}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|.$$

It is a routine work to establish that the spaces $(\Lambda_p, \|.\|_p)$ for $1 < p < \infty$ and $(\Lambda_\infty, \|.\|_\infty)$ are Banach spaces.

1.2. **Sequence spaces $\Lambda_{\hat{p}}$, $\hat{p} = (p_n)$, $p_n > 1$.** Let $\hat{p} = (p_n)$ be a bounded sequence of positive real numbers such that $p_n > 1$ for each $n \in \mathbb{N}_0$. Let $x = (x_k)$ be a sequence of real numbers defined on $\mathbb{N}_0$ and define a convex modular $\sigma(x)$ as $\sigma(x) = \sum_{n=0}^{\infty}\big(\Lambda x(n)\big)^{p_n}$, where $\Lambda x(n) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|$ for all $n \in \mathbb{N}_0$.
Then the following set is defined:

$$\Lambda_{\hat{p}} = \big\{x = (x_k) \in l^0 : \sigma(rx) < \infty \text{ for some } r > 0\big\},$$

which is a normed linear space equipped with the Luxemburg norm, defined as follows:

$$\|x\|_{\hat{p}} = \inf\Big\{r > 0 : \sigma\Big(\frac{x}{r}\Big) \leq 1\Big\}.$$

Indeed $(\Lambda_{\hat{p}}, \|x\|_{\hat{p}})$ becomes a Banach space.

Remark 1.1. In particular,
(i) substituting $\lambda_n = n + 1$, the sequence spaces Λ_p and $\Lambda_{\hat{p}}$ reduce to ces_p and $ces_{\hat{p}}$, respectively ([13], [15] and [27]).
(ii) by choosing $q = (q_k) = (\lambda_k - \lambda_{k-1})$ and $Q_n = \sum_{k=0}^{n} q_k = \lambda_n$, the sequence spaces Λ_p and $\Lambda_{\hat{p}}$ reduce to $ces[p, q]$ and $ces[\hat{p}, \hat{q}]$, respectively [13].

2. James constants of Λ_p for $1 < p \leq \infty$

The unit ball of a normed linear space is *uniformly non-square* if and only if there is a positive number δ such that there does not exist any member x and y of the unit ball for which $\|\frac{1}{2}(x+y)\| > 1-\delta$ and $\|\frac{1}{2}(x-y)\| > 1-\delta$ (see [12]). The *James constant* (or *measure of uniform non-squareness*) of a real Banach space $(X, \|\cdot\|)$ with $dim(X) \geq 2$ is denoted by $J(X)$ [8] and is defined as

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : \ x, y \in X, \ \|x\| = 1, \ \|y\| = 1\}.$$

The local versions of uniform non-squareness in Cesàro sequence spaces were studied by Cui and Płuciennik [7]. Now, the determination of James constant for the sequence spaces Λ_p for $1 < p \leq \infty$ will be presented next. An approach similar to Maligranda et al. [17] is considered to prove the following results.

Theorem 2.1. *The James constants of* Λ*-sequence spaces* Λ_p *for* $1 < p \leq \infty$ *is* 2*. Expressing by notation,* $J(\Lambda_p) = 2$*,* $1 < p \leq \infty$*.*

Proof. First consider the case $1 < p < \infty$. For each $m = 0, 1, 2, \ldots$, and denote $e_m = (e_{mk}) = (0, 0, \ldots, 0, 1, 0, \ldots)$, where 1 is at the m-th position. Note that

$$\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})e_{mk} = \begin{cases} 0 & \text{if } n < m, \\ \frac{\lambda_m - \lambda_{m-1}}{\lambda_n} & \text{if } n \geq m. \end{cases}$$

Let $x_m = (x_{mk})$ and $y_m = (y_{mk})$, where $x_m = \frac{e_m}{\|e_m\|_p}$ and $y_m = \frac{e_{m+1}}{\|e_{m+1}\|_p}$ for each $m = 0, 1, 2, \ldots$. Then $\|x_m\|_p = 1$ and $\|y_m\|_p = 1$.
Now the value of $\|x_m \pm y_m\|_p$ for $m = 0, 1, 2, \ldots$ is found as follows:

$$\begin{aligned} &\|x_m \pm y_m\|_p^p \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_{mk} \pm y_{mk}|\Big)^p \\ &= \sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Big|\frac{e_{mk}}{\|e_m\|_p} \pm \frac{e_{(m+1)k}}{\|e_{m+1}\|_p}\Big|\Big)^p \\ &= \Big(\frac{\lambda_m - \lambda_{m-1}}{\lambda_m\|e_m\|_p}\Big)^p + \sum_{n=m+1}^{\infty}\Big(\frac{1}{\lambda_n}\Big(\frac{\lambda_m - \lambda_{m-1}}{\|e_m\|_p} + \frac{\lambda_{m+1} - \lambda_m}{\|e_{m+1}\|_p}\Big)\Big)^p \\ &\geq \sum_{n=m+1}^{\infty}\Big(\frac{1}{\lambda_n}\Big(\frac{\lambda_m - \lambda_{m-1}}{\|e_m\|_p} + \frac{\lambda_{m+1} - \lambda_m}{\|e_{m+1}\|_p}\Big)\Big)^p \\ &= \Big(1 + \frac{\lambda_m - \lambda_{m-1}}{\lambda_{m+1} - \lambda_m}.\frac{\|e_{m+1}\|_p}{\|e_m\|_p}\Big)^p. \end{aligned} \tag{2.1}$$

Therefore by removing the power p, one gets $\|x_m \pm y_m\|_p \geq 1 + \frac{\lambda_m - \lambda_{m-1}}{\lambda_{m+1} - \lambda_m}.\frac{\|e_{m+1}\|_p}{\|e_m\|_p}$.
We claim that

$$\lim_{m\to\infty}\frac{\lambda_m - \lambda_{m-1}}{\lambda_{m+1} - \lambda_m}.\frac{\|e_{m+1}\|_p}{\|e_m\|_p} = 1.$$

Since

$$\Big(\frac{\lambda_m-\lambda_{m-1}}{\lambda_{m+1}-\lambda_m}\Big)^p.\frac{\|e_{m+1}\|_p^p}{\|e_m\|_p^p}=\frac{\sum\limits_{n=m+1}^{\infty}\frac{1}{\lambda_n^p}}{\sum\limits_{n=m}^{\infty}\frac{1}{\lambda_n^p}}=\frac{\sum\limits_{n=m+1}^{\infty}\frac{1}{\lambda_n^p}}{\frac{1}{\lambda_m^p}+\sum\limits_{n=m+1}^{\infty}\frac{1}{\lambda_n^p}}=\left(1+\frac{\frac{1}{\lambda_m^p}}{\sum\limits_{n=m+1}^{\infty}\frac{1}{\lambda_n^p}}\right)^{-1}.$$

By the discrete version of Bernoulli-de l' Hospital rule, it follows that

$$\lim_{m\to\infty}\frac{\frac{1}{\lambda_m^p}}{\sum\limits_{n=m+1}^{\infty}\frac{1}{\lambda_n^p}}=\lim_{m\to\infty}\frac{\frac{1}{\lambda_m^p}-\frac{1}{\lambda_{m+1}^p}}{\frac{1}{\lambda_{m+1}^p}}=\lim_{m\to\infty}\frac{\frac{1}{\lambda_m{}^p}}{\frac{1}{\lambda_{m+1}^p}}-1=\lim_{m\to\infty}\Big(\frac{\lambda_{m+1}}{\lambda_m}\Big)^p-1=0. \tag{2.2}$$

Hence, Eqn (2.2) proves the above claim and from inequality (2.1), it is found that $\|x_m\pm y_m\|_p\to 2$ as $m\to\infty$. Since $\|x_m\pm y_m\|_p\le 2$ is always true, it implies that $J(\Lambda_p)=2$ for $1<p<\infty$.
For $p=\infty$, $x_m=\frac{\lambda_m}{\lambda_m-\lambda_{m-1}}e_m$ and $y_m=\frac{\lambda_m}{\lambda_{m+1}-\lambda_m}e_{m+1}$ are chosen. Then it is easy to verify that $\|x_m\|_\infty=1$, $\|y_m\|_\infty=1$ and $\|x_m\pm y_m\|_\infty=1+\frac{\lambda_m}{\lambda_{m+1}}\to 2$ as $m\to\infty$. Hence $J(\Lambda_\infty)=2$. □

Corollary 2.2. *(i) Choose $\lambda_n=n+1$, then $J(ces_p)=2$ for $1<p\le\infty$ ([17]).*
(ii) If $Q_n=\sum\limits_{k=0}^{n}q_k=\lambda_n$, where $q=(q_k)=(\lambda_k-\lambda_{k-1})$, then $J(ces[p,q])=2$ for $1<p\le\infty$.

A Banach space is said to be *uniformly non-$l_n^{(1)}$* if there is $\delta\in(0,1)$ such that for any $x_0,x_1,x_2,\dots,x_{n-1}$ from the unit ball of X, we have $\min\limits_{\epsilon_k=\pm1}\big\|\sum\limits_{k=0}^{n-1}\epsilon_k x_k\big\|\le n(1-\delta)$.
This definition leads to the notion of *n-th James constant* (or the *measure of uniformly non-$l_n^{(1)}$*) $J_n(X)$, $n\in\mathbb{N}$ [8] of a Banach space X is defined as

$$J_n(X)=\sup\Big\{\min_{\varepsilon_k=\pm1}\big\|\sum_{k=0}^{n-1}\epsilon_k x_k\big\|:\ x_k\in X,\ \|x_k\|\le 1, k=0,1,2,\dots,n-1\Big\}.$$

If restricted to unit sphere of a real Banach space X then the James constants (or *n-th strong James constants*) are denoted by $J_n^s(X)$, $n\in\mathbb{N}$ and defined by

$$J_n^s(X)=\sup\Big\{\min_{\varepsilon_j=\pm1}\Big\|\sum_{j=0}^{n-1}\varepsilon_j x_j\Big\|:\|x_j\|=1, j=0,1,2,\dots,n-1\Big\}.$$

It is to be noted that $J_n^s(X)\le J_n(X)\le n$ and $J_2^s(X)=J_2(X)=J(X)$ [17].

Theorem 2.3. *The strong n-th James constant $J_n^s(\Lambda_p)=n$ for $1<p<\infty$, and $J_n^s(\Lambda_\infty)=n$.*

Proof. In the previous theorem, the results were established for the case where $n = 2$. Now the result for $n \geq 3$ is deduced hereunder. Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ such that $n \geq 3$ and put

$$x_{j,m} = \frac{e_{m+j}}{\|e_{m+j}\|_p}, j = 0, 1, 2, \dots, n-1.$$

With this setting, it is clear that $\|x_{j,m}\|_p = 1$ for each $j = 0, 1, 2, \dots, n-1$. Since $x_{j_1,m}$ and $x_{j_2,m}$ have disjoint supports for $j_1 \neq j_2$, it implies that

$$\min_{\varepsilon_j = \pm 1} \Big\| \sum_{j=0}^{n-1} \varepsilon_j x_{j,m} \Big\|_p = \Big\| \sum_{j=0}^{n-1} x_{j,m} \Big\|_p.$$

Choosing $b_{m,l} = \sum_{i=m}^{m+l} \frac{\lambda_i - \lambda_{i-1}}{\|e_i\|_p}$, $l = 0, 1, 2, \dots, n-1$, then one gets

$$\begin{aligned}
\Big\| \sum_{j=0}^{n-1} x_{j,m} \Big\|_p^p &= \Big\| \sum_{j=m}^{m+n-1} \frac{e_j}{\|e_j\|_p} \Big\|_p^p \\
&= \Big\| \Big(0, 0, \dots, 0, \frac{1}{\|e_m\|_p}, \frac{1}{\|e_{m+1}\|_p}, \dots, \frac{1}{\|e_{m+n-1}\|_p}, 0, 0, \dots \Big) \Big\|_p^p \\
&= \Big(\frac{b_{m,0}}{\lambda_m} \Big)^p + \Big(\frac{b_{m,1}}{\lambda_{m+1}} \Big)^p + \dots + \Big(\frac{b_{m,n-1}}{\lambda_{m+n-1}} \Big)^p + \Big(\frac{b_{m,n-1}}{\lambda_{m+n}} \Big)^p + \dots \\
&\geq \sum_{k=m+n-1}^{\infty} \Big(\frac{b_{m,n-1}}{\lambda_k} \Big)^p \\
&= (b_{m,n-1})^p \sum_{k=m+n-1}^{\infty} \Big(\frac{1}{\lambda_k} \Big)^p \\
&= \Big(\sum_{i=m}^{m+n-1} \frac{\lambda_i - \lambda_{i-1}}{\|e_i\|_p} \Big)^p . \Big(\frac{\|e_{m+n-1}\|_p}{\lambda_{m+n-1} - \lambda_{m+n-2}} \Big)^p \\
&\geq n^p \Big(\frac{\lambda_m - \lambda_{m-1}}{\|e_m\|_p} \Big)^p . \Big(\frac{\|e_{m+n-1}\|_p}{\lambda_{m+n-1} - \lambda_{m+n-2}} \Big)^p
\end{aligned}$$

Consequently,

$$n \geq \Big\| \sum_{j=0}^{n-1} x_{j,m} \Big\|_p \geq n \frac{\lambda_m - \lambda_{m-1}}{\lambda_{m+n-1} - \lambda_{m+n-2}} \frac{\|e_{m+n-1}\|_p}{\|e_m\|_p}. \tag{2.4}$$

We claim that $\lim_{m\to\infty}\frac{\lambda_m-\lambda_{m-1}}{\lambda_{m+n-1}-\lambda_{m+n-2}}.\frac{\|e_{m+n-1}\|_p}{\|e_m\|_p}=1$.

Denote $a_{m+n-1}:=\sum_{j=m+n-1}^{\infty}\frac{1}{\lambda_j^p}$. Then, the following holds:

$$\Big(\frac{\lambda_m-\lambda_{m-1}}{\lambda_{m+n-1}-\lambda_{m+n-2}}\Big)^p.\frac{\|e_{m+n-1}\|_p^p}{\|e_m\|_p^p}$$
$$=\frac{\sum_{j=m+n-1}^{\infty}\frac{1}{\lambda_j^p}}{\sum_{j=m}^{\infty}\frac{1}{\lambda_j^p}}$$
$$=\frac{a_{m+n-1}}{\frac{1}{\lambda_m^p}+\frac{1}{\lambda_{m+1}^p}+\ldots+\frac{1}{\lambda_{m+n-2}^p}+a_{m+n-1}}$$
$$=\Big(1+\frac{1}{\lambda_m^p a_{m+n-1}}+\frac{1}{\lambda_{m+1}^p a_{m+n-1}}+\ldots+\frac{1}{\lambda_{m+n-2}^p a_{m+n-1}}\Big)^{-1}.$$

Now the limits $\lim_{m\to\infty}\frac{1}{\lambda_{m+i}^p a_{m+n-1}}$ are found out for $i=0,1,\ldots,n-1$. Applying Bernoulli-de l' Hospital rule when $i=0$, one gets

$$\lim_{m\to\infty}\frac{\frac{1}{\lambda_m^p}}{\sum_{j=m+n-1}^{\infty}\frac{1}{\lambda_j^p}}=\lim_{m\to\infty}\frac{\frac{1}{\lambda_m^p}-\frac{1}{\lambda_{m+1}^p}}{\frac{1}{\lambda_{m+n-1}^p}}=\lim_{m\to\infty}\Big(\frac{\lambda_{m+n-1}}{\lambda_m}\Big)^p-\lim_{m\to\infty}\Big(\frac{\lambda_{m+n-1}}{\lambda_{m+1}}\Big)^p=0. \tag{2.5}$$

Hence, Eqn (2.5) proves the aforementioned claim. Similarly, $\lim_{m\to\infty}\frac{1}{\lambda_{m+i}^p a_{m+n-1}}=0$ for $i=1,\ldots,n-1$. Hence by inequality (2.4), we get $\Big\|\sum_{j=0}^{n-1}x_{j,m}\Big\|_p\to n$ as $m\to\infty$.

Since $\Big\|\sum_{j=0}^{n-1}x_{j,m}\Big\|_p\le n$, so by definition $J_n^s(\Lambda_p)=n$ for $1<p<\infty$.

For the case $p=\infty$, the following is chosen

$$x_{j,m}=\frac{\lambda_{m+j}}{\lambda_{m+j}-\lambda_{m+j-1}}e_{m+j},\ \text{for } j=0,1,\ldots,n-1.$$

It is easy to find that $\|x_{j,m}\|_\infty=1$ and for any fixed $n\in\mathbb{N}$

$$\Big\|\sum_{j=0}^{n-1}\varepsilon_j x_{j,m}\Big\|_\infty=\Big\|\sum_{j=0}^{n-1}x_{j,m}\Big\|_\infty=1+\frac{\lambda_m}{\lambda_{m+n-1}}+\frac{\lambda_{m+1}}{\lambda_{m+n-1}}+\ldots+\frac{\lambda_{m+n-2}}{\lambda_{m+n-1}}$$
$$\to n \text{ as } m\to\infty$$
$$\Big(\text{Since } \lim_{m\to\infty}\frac{\lambda_{m+i}}{\lambda_{m+n-1}}=1 \text{ for each } i=0,1,2,\ldots,n-2\Big).$$

Again by definition of n-th strong James constant, $J_n^s(\Lambda_\infty) = n$ and thus the theorem is proved. □

Corollary 2.4. *(i) Choose $\lambda_n = n+1$, then $J_n^s(ces_p) = n$ for $1 < p \leq \infty$ ([17]).*
(ii) If $Q_n = \sum_{k=0}^{n} q_k = \lambda_n$, where $q = (q_k) = (\lambda_k - \lambda_{k-1})$, then $J_n^s(ces[p,q]) = n$ for $1 < p \leq \infty$.

3. Geometric properties of $\Lambda_{\hat{p}}$

Let $(X, \|.\|)$ be a Banach space which is a subspace of l^0. As usual, $S(X)$ and $B(X)$ are denoted for the unit sphere and closed unit ball of X, respectively. A point $x \in S(X)$ is said to be an extreme point of $B(X)$ if there does not exist two distinct points y, $z \in B(X)$ such that $2x = y + z$. The concept of extreme point plays an important role in the study of Krein–Milman theorem, Choquet integral representation theorem etc.
Foralewski [9] introduced the notion of *coordinatewise Kadec–Klee property* of a Banach space which is denoted by *(H_c)*. X is said to possess the property *(H_c)*, if $x \in X$ and every sequence $(x_l) \subset X$ such that

$$\|x_l\| \to \|x\| \text{ and } x_{il} \to x_i \text{ as } l \to \infty \text{ for each } i, \text{ then } \|x_l - x\| \to 0.$$

If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(x_l) \subset B(X),\ sep(x_l) \geq \varepsilon,\ \|x_l\| \to \|x\| \text{ and } x_{li} \to x_i \text{ for each } i \text{ implies } \|x\| \leq 1 - \delta,$$

where $sep(x_l) = \inf\{\|x_l - x_m\| : l \neq m\}$, then X is said to have the *coordinatewise uniformly Kadec–Klee property* and is denoted by $X \in (UKK_c)$ [29]. For any Banach space X, $(UKK_c) \Rightarrow (H_c)$.
X is said to have the *uniform Opial property* (abbreviated as (UOP)) if for each $\varepsilon > 0$ there exists $\mu > 0$ such that

$$1 + \mu \leq \liminf_{l \to \infty} \|x_l + x\|$$

for any weakly null sequence (x_l) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$ [23].
A Banach space X has the property (β) if and only if, there exists $\delta > 0$ for every $\epsilon > 0$ such that, for each element $x \in B(X)$ and each sequence $(x_l) \in B(X)$ with $sep(x_l) \geq \epsilon$, there is an index k such that

$$\left\| \tfrac{x + x_k}{2} \right\| \leq 1 - \delta \text{ [5]}.$$

The *weakly uniform normal structure* of a Banach space X (abbreviated as $WUNS(X)$) is determined by the *weakly convergent sequence coefficient* of X (abbreviated as $WCS(X)$) ([1], [3]) is defined as

$$WCS(X) = \inf \left\{ \frac{\lim_k \sup_{n,m \geq k} \|x_n - x_m\|}{\inf\{\limsup_n \|x_n - y\| : y \in \text{Conv}(\ x_n)\}} \right\},$$

where infimum is taken over all weakly convergent sequence (x_n) which is not norm convergent. If $WCS(X) > 1$ then Banach space X has $WUNS$. A Banach space X has $WUNS$ if it possesses (UOP) [16].

Let $(X_n, \|.\|_n)$ be Banach spaces for each $n \in \mathbb{N}_0$. Then the Nakano sequence spaces $l_{\hat{p}}(X_n)$ is defined as

$$l_{\hat{p}}(X_n) = \Big\{x : x_n \in X_n \text{ for each } n \in \mathbb{N}_0 \text{ and } \rho(rx) < \infty \text{ for some } r > 0\Big\},$$

where convex modular ρ is defined as $\rho(x) = \sum_{n=0}^{\infty} \|x_n\|_n^{p_n}$. It is easy to show that the sequence space $l_{\hat{p}}(X_n)$ is a Banach space equipped with the Luxemburg norm

$$\|x\| = \inf\Big\{r > 0 : \rho(\frac{x}{r}) \le 1\Big\}. \tag{3.1}$$

In particular, the Nakano sequence space $l_{\hat{p}}(\mathbb{R}^{n+1})$, $n \in \mathbb{N}_0$, defined as

$$l_{\hat{p}}(\mathbb{R}^{n+1}) = \Big\{x : \rho(rx) = \sum_{n=0}^{\infty} \|rx_n\|_{\mathbb{R}^{n+1}}^{p_n} < \infty \text{ for some } r > 0\Big\}$$

is a Banach space equipped with the norm defined in Eqn. (3.1) and $\mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$ is a $(n+1)$-dimensional Euclidean space equipped with the following norm:

$$\|(\alpha_0, \alpha_1, \dots, \alpha_n)\| = \sum_{i=0}^{n} |\alpha_i| \text{ for } (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1}.$$

Saejung [25] proved that Cesàro sequence spaces ces_p for $1 < p < \infty$ are isometrically embedded in the infinite l_p-sum $l_p(\mathbb{R}^n)$ of finite dimensional spaces $\mathbb{R}^n$. Here similar result for the sequence space $\Lambda_{\hat{p}}$ is presented below:

Lemma 3.1. *The sequence space $\Lambda_{\hat{p}}$ is a closed subspace in the Nakano sequence space $l_{\hat{p}}(\mathbb{R}^{n+1})$, $n \in \mathbb{N}_0$.*

Proof. For all $x = (x_i) \in \Lambda_{\hat{p}}$, the following linear isometry $T : \Lambda_{\hat{p}} \to l_{\hat{p}}(\mathbb{R}^{n+1})$ is defined by

$$T((x_i)) = \Big(x_0, \Big(\tfrac{\lambda_0}{\lambda_1}x_0, \tfrac{(\lambda_1-\lambda_0)}{\lambda_1}x_1\Big), \dots, \Big(\tfrac{\lambda_0}{\lambda_n}x_0, \tfrac{(\lambda_1-\lambda_0)}{\lambda_n}x_1, \dots, \tfrac{(\lambda_n-\lambda_{n-1})}{\lambda_n}x_n\Big), \dots\Big).$$

Then

$$\begin{aligned}
&\|T((x_i))\| \\
&= \|T(x_0, x_1, \dots, x_i, \dots)\| \\
&= \Big\|\Big(x_0, \Big(\frac{\lambda_0}{\lambda_1}x_0, \frac{(\lambda_1-\lambda_0)}{\lambda_1}x_1\Big), \dots, \Big(\frac{\lambda_0}{\lambda_n}x_0, \dots, \frac{(\lambda_n-\lambda_{n-1})}{\lambda_n}x_n\Big), \dots\Big)\Big\| \\
&= \inf\Big\{r > 0 : \sum_{n=0}^{\infty}\Big(\frac{1}{r\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|\Big)^{p_n} \le 1\Big\} \\
&= \inf\Big\{r > 0 : \sigma\Big(\frac{x}{r}\Big) \le 1\Big\} \\
&= \|(x_i)\|_{\hat{p}}
\end{aligned}$$

Therefore $\Lambda_{\hat{p}}$ is a closed subspace in $l_{\hat{p}}(\mathbb{R}^{n+1})$, $n \in \mathbb{N}_0$. □

Instead of studying geometric properties of $\Lambda_{\hat{p}}$ it is enough to study geometric properties of $l_{\hat{p}}(\mathbb{R}^{n+1})$. If such geometric properties are inherited by subspaces then $\Lambda_{\hat{p}}$ will have the same properties. Certain geometric structure of $l_p(X_n)$,

$p > 1$ of finite dimensional spaces X_n has also been investigated by Rolewicz [24]. Saejung in his recent work ([25], Theorem 11 (2), p.535) established the following important result.

Proposition 3.2. *Suppose each X_n is finite dimensional. Then the space $l_{\hat{p}}(X_n)$ has property (β) and uniform Opial property if and only if $\limsup_{n\to\infty} p_n < \infty$.*

Now we proceed for the following new result:

Theorem 3.3. *The sequence space $\Lambda_{\hat{p}}$ has property (β) and uniform Opial property if and only if $\limsup_{n\to\infty} p_n < \infty$.*

Proof. Since $\mathbb{R}^{n+1}$, $n \in \mathbb{N}_0$ is finite dimensional, proposition 3.2 implies that the space $l_{\hat{p}}(\mathbb{R}^{n+1})$ possesses the property (β) and uniform Opial property if and only if $\limsup_{n\to\infty} p_n < \infty$. Since the property (β) and uniform Opial property are inherited by subspaces [25], it follows from Lemma 3.1 that the sequence space $\Lambda_{\hat{p}}$ has property (β) and uniform Opial property. □

Corollary 3.4. (i) *The space $\Lambda_{\hat{p}}$ has WUNS if $\limsup_{n\to\infty} p_n < \infty$.*
(ii) *If $\limsup_{n\to\infty} p_n < \infty$, and $\lambda_n = n + 1$, then $ces_{\hat{p}}$ has property (β) and uniform Opial property ([25], Theorem 11 (2)).*

Theorem 3.5. *Suppose $p^* = \sup_n p_n < \infty$. Then the sequence space $\Lambda_{\hat{p}}$ possesses coordinate-wise uniform Kadec–Klee property.*

Proof. Let $\varepsilon \in (0, 1)$ and take $\eta = (\frac{\varepsilon}{4})^{p^*}$, where $p^* < \infty$ as (p_n) is bounded. The $\delta \in (0, 1)$ is chosen such that $(1-\delta)^{p^*} > 1-\eta$. Suppose $(x_l) \subset B(\Lambda_{\hat{p}})$, $sep(x_l) \geq \epsilon$, $\|x_l\|_{\hat{p}} \to \|x\|_{\hat{p}}$, $x_{il} \to x_i$ as $l \to \infty$ and for all $i \in \mathbb{N}_0$. It is shown that there exists a $\delta > 0$ such that $\|x\|_{\hat{p}} \leq 1-\delta$. In a hypothetical situation, suppose $\|x\|_{\hat{p}} > 1-\delta$ is true for all $\delta > 0$. Then one can select a finite set $I = \{0, 1, 2, \ldots, N-1\}$ on which $\|x\chi_I\|_{\hat{p}} > 1-\delta$, where $x\chi_I = x_i$ for $i \in I$ and $= 0$ for $i \notin I$. Since $x_{il} \to x_i$ for each $i \in \mathbb{N}_0$, therefore $x_l \to x$ uniformly on I. Since $\|x_l\|_{\hat{p}} \to \|x\|_{\hat{p}}$, there exists $d_N \in \mathbb{N}$ such that

$$\|x_l\chi_I\|_{\hat{p}} > 1-\delta \text{ and } \|(x_l - x_m)\chi_I\|_{\hat{p}} \leq \tfrac{\epsilon}{2} \text{ for all } l, m \geq d_N.$$

The first inequality implies that $\sigma(x_l\chi_I) \geq \|x_l\chi_I\|_{\hat{p}}^{p^*} > (1-\delta)^{p^*} > 1-\eta$ for $l \geq d_N$. Since $sep(x_l) \geq \epsilon$, i.e., $\|x_l - x_m\|_{\hat{p}} \geq \epsilon$, the second inequality implies that $\|(x_l - x_m)\chi_{\mathbb{N}-I}\|_{\hat{p}} \geq \frac{\epsilon}{2}$ for $l, m \geq d_N, l \neq m$. Hence, for $N \in \mathbb{N}$ there exists a d_N such that $\|x_{l_N}\chi_{\mathbb{N}-I}\|_{\hat{p}} \geq \frac{\epsilon}{4}$. It may be assumed without losing generality that $\|x_l\chi_{\mathbb{N}-I}\|_{\hat{p}} \geq \frac{\epsilon}{4}$ for all $l, N \in \mathbb{N}$. Therefore from the relation between norm and modular, one obtains $\sigma(x_l\chi_{\mathbb{N}-I}) \geq \|x_l\chi_{\mathbb{N}-I}\|_{\hat{p}}^{p^*} \geq (\frac{\epsilon}{4})^{p^*} = \eta$.
The convexity of the function $f(t) = |t|^{p_n}$ for each $n \in \mathbb{N}_0$ affords $f(\gamma u) = f(\gamma u + (1-\gamma)0) \leq \gamma f(u)$ for any $\gamma \in [0, 1]$ and $u \in \mathbb{R}$. Therefore, if $0 \leq u < v < \infty$, then $f(u) = f(\frac{u}{v}v) \leq \frac{u}{v}f(v)$, which means that $\frac{f(u)}{u} \leq \frac{f(v)}{v}$. Assuming now that $0 \leq u, v < \infty$, $u + v > 0$, one gets

$$f(u+v) = u\tfrac{f(u+v)}{u+v} + v\tfrac{f(u+v)}{u+v} \geq u\tfrac{f(u)}{u} + v\tfrac{f(v)}{v} = f(u) + f(v).$$

Since $x_l = x_l\chi_I + x_l\chi_{\mathbb{N}-I}$, one gets $\sigma(x_l\chi_I)+\sigma(x_l\chi_{\mathbb{N}-I}) \leq \sigma(x_l) \leq 1$ by applying the above mentioned fact. It implies that $\sigma(x_l\chi_{\mathbb{N}-I}) \leq 1-\sigma(x_l\chi_I) < 1-(1-\eta) = \eta$, i.e., $\sigma(x_l\chi_{\mathbb{N}-I}) < \eta$, which contradicts the inequality $\sigma_\Phi(x_l\chi_{\mathbb{N}-I}) \geq \eta$ and this contradiction completes the proof. □

Theorem 3.6. *Let $\limsup_{n\to\infty} p_n < \infty$. Then a point $x \in S(\Lambda_{\hat{p}})$ is an extreme point of $B(\Lambda_{\hat{p}})$ if and only if $\sigma(x) = 1$.*

Proof. Necessity: Let $x \in S(\Lambda_{\hat{p}})$ be an extreme point and n_0 be a natural number. Assume that $\sigma(x) \neq 1$. Putting $\varepsilon = 1-\sigma(x) > 0$ and considering the following two sequences:

$$y = (y_k) = (x_0, x_1, \ldots, x_{n_0}, 0, 0, \ldots),$$
$$z = (z_k) = (x_0, x_1, \ldots, x_{n_0}, 2x_{n_0+1}, 2x_{n_0+2}, \ldots).$$

It is clear that $2x = y+z$ and $y \neq z$. But

$$\sigma(y) = \sum_{n=0}^{n_0}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|\Big)^{p_n} \leq \sigma(x) = 1-\varepsilon < 1, \text{ and}$$

$$\sigma(z) \leq \sum_{n=0}^{n_0}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|\Big)^{p_n} + \sum_{n=n_0+1}^{\infty} 2^{p_n}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|\Big)^{p_n} \tag{3.2}$$

Since $\limsup_{n\to\infty} p_n < \infty$ and $x \in \Lambda_{\hat{p}}$, there exists a natural number n_0 and a constant $M > 0$ such that one has $2^{p_n} \leq 2^M$ for all $n > n_0$ and for every $\varepsilon > 0$, $\sum_{n=n_0+1}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|x_k|\Big)^{p_n} < \frac{\varepsilon}{2^M}$. Hence from inequality (3.2), one obtains $\sigma(z) < \sigma(x)+\varepsilon = 1$. The relation between norm and modular implies that $\|y\|_{\hat{p}} < 1$ and $\|z\|_{\hat{p}} < 1$, which contradict the assumption that x is an extreme point. Therefore $\sigma(x) = 1$ must holds.

Sufficiency: Let it be assumed that x satisfies condition $\sigma(x) = 1$. Suppose $2x = y+z$ for some $y, z \in B(\Lambda_{\hat{p}})$. Then convexity of modular σ implies that

$$1 = \sigma(x) = \sigma\Big(\frac{y+z}{2}\Big) \leq \frac{1}{2}\big(\sigma(y)+\sigma(z)\big) \leq 1.$$

Therefore $\sigma(y) = 1$ and $\sigma(z) = 1$ and for each $n \in \mathbb{N}_0$, one gets

$$\begin{aligned}&\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})\Big|\frac{y_k+z_k}{2}\Big|\Big)^{p_n}\\ &= \frac{1}{2}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|y_k|\Big)^{p_n} + \frac{1}{2}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})|z_k|\Big)^{p_n}.\end{aligned} \tag{3.3}$$

Since the functions $f(t) = |t|^{p_n}$, $p_n > 1$, $n \in \mathbb{N}_0$ are strictly convex so from Eqn. (3.3) it follows that $\Lambda x(n) = \Lambda y(n) = \Lambda z(n)$ for each $n \in \mathbb{N}_0$, which in turn, implies that

$$|x_k| = |y_k| = |z_k| \text{ for each } k \in \mathbb{N}_0.$$

If there exists $i_0 \in \mathbb{N}_0$ such that $y_{i_0} + z_{i_0} = 0$, then one gets

$$\begin{aligned}
1 = \sigma(x) = \sigma\Big(\frac{y+z}{2}\Big) &= \sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\Big|\frac{y_k + z_k}{2}\Big|\Big)^{p_n} \\
&= \sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k\in\mathbb{N}_0\setminus\{i_0\}}(\lambda_k - \lambda_{k-1})\Big|\frac{y_k + z_k}{2}\Big|\Big)^{p_n} \\
&\le \frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k\in\mathbb{N}_0\setminus\{i_0\}}(\lambda_k - \lambda_{k-1})\big|y_k\big|\Big)^{p_n} + \frac{1}{2}\sum_{n=0}^{\infty}\Big(\frac{1}{\lambda_n}\sum_{k\in\mathbb{N}_0\ \setminus\{i_0\}}(\lambda_k - \lambda_{k-1})\big|z_k\big|\Big)^{p_n} \\
&< \frac{1}{2}\sigma(y) + \frac{1}{2}\sigma(z) = 1,
\end{aligned}$$

which leads to a contradiction. This means $x_k = y_k = z_k$ for every $k \in \mathbb{N}_0$, i.e., $x = y = z$. □

Remark 3.7. The statement of the Theorem 3.6 also says that the space $\Lambda_{\hat{p}}$ is strictly convex, i.e., if $\|y\|_{\hat{p}} = 1$, $\|z\|_{\hat{p}} = 1$ and $y \neq z$ implies that $\Big\|\frac{y+z}{2}\Big\|_{\hat{p}} < 1$. The result is evident from the proof (sufficiency) of the last theorem. Indeed, if $\sigma(x) = 1$ then we have $\sigma(y) = 1$ and $\sigma(z) = 1$ which gives $\|y\|_{\hat{p}} = 1$ and $\|z\|_{\hat{p}} = 1$, respectively. Further, assume that $\|y + z\|_{\hat{p}} = 2$. Then proceeds similarly as above, one gets $|y_k| = |z_k|$ for each $k \in \mathbb{N}_0$. We show that this implies $y_k = z_k$ for each $k \in \mathbb{N}_0$. Assume on the contrary that $y \neq z$. Then there exists $i_0 \in \mathbb{N}_0$ such that $y_{i_0} + z_{i_0} = 0$. Using the similar steps as above, we arrived at a contradiction. Hence $y = z$. Therefore the space $\Lambda_{\hat{p}}$ is strictly convex.

4. Von Neumann–Jordan constant of $\Lambda_2^{(2)}$

For two dimensional sequence space $\Lambda_p^{(2)}$, norm $\|(u,v)\|_p$ is given by

$$\|(u,v)\|_p = \Big(|u|^p + \Big(\frac{\lambda_0|u| + (\lambda_1 - \lambda_0)|v|}{\lambda_1}\Big)^p\Big)^{\frac{1}{p}}.$$

The *von Neumann–Jordan constant* $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [4], which is defined as follows:

$$C_{NJ}(X) = \sup\Big\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and } \|x\| + \|y\| \neq 0\Big\}.$$

Now the following theorem is taken as the starting point:

Theorem 4.1. *The von Neumann–Jordan constant is given by:* $C_{NJ}(\Lambda_2^{(2)}) = 1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}$.

Proof. Choose (a,b), $(c,d) \in \Lambda_2^{(2)}$. Then

$$\begin{aligned}
&\|(a,b) \pm (c,d)\|^2 \\
&\quad = \|(a \pm c, b \pm d)\|^2 \\
&\quad = \Big(|a \pm c|^2 + \Big(\frac{\lambda_0|a \pm c| + (\lambda_1 - \lambda_0)|b \pm d|}{\lambda_1}\Big)^2\Big) \\
&\quad = \Big(1 + \frac{\lambda_0^2}{\lambda_1^2}\Big)|a \pm c|^2 + \frac{2\lambda_0(\lambda_1 - \lambda_0)}{\lambda_1^2}|a \pm c||b \pm d| + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}|b \pm d|^2.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{2\lambda_0(\lambda_1 - \lambda_0)}{\lambda_1^2}|a \pm c||b \pm d| &= \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big\{2\frac{\sqrt{\lambda_0^2 + \lambda_1^2}}{\lambda_1}|a \pm c|\frac{(\lambda_1 - \lambda_0)}{\lambda_1}|b \pm d|\Big\} \\
&\le \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big\{\frac{\lambda_0^2 + \lambda_1^2}{\lambda_1^2}|a \pm c|^2 + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}|b \pm d|^2\Big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(a,b) + (c,d)\|^2 + \|(a,b) - (c,d)\|^2 \\
&\le \Big(1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big)\Big\{\frac{\lambda_0^2 + \lambda_1^2}{\lambda_1^2}(|a + c|^2 + |a - c|^2) + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}(|b + d|^2 + |b - d|^2)\Big\} \\
&= 2\Big(1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big)\Big\{\frac{\lambda_0^2 + \lambda_1^2}{\lambda_1^2}(|a|^2 + |c|^2) + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}(|b|^2 + |d|^2)\Big\} \\
&\le 2\Big(1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big)\Big\{\frac{\lambda_0^2 + \lambda_1^2}{\lambda_1^2}|a|^2 + \frac{2\lambda_0(\lambda_1 - \lambda_0)}{\lambda_1^2}|a||b| + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}|b|^2 \\
&\quad + \frac{\lambda_0^2 + \lambda_1^2}{\lambda_1^2}|c|^2 + \frac{2\lambda_0(\lambda_1 - \lambda_0)}{\lambda_1^2}|c||d| + \frac{(\lambda_1 - \lambda_0)^2}{\lambda_1^2}|d|^2\Big\} \\
&= \Big(1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}\Big)\big\{2(\|(a,b)\|^2 + \|(c,d)\|^2)\big\}.
\end{aligned}$$

Hence $C_{NJ}(\Lambda_2^{(2)}) \le 1 + \frac{\lambda_0}{\sqrt{\lambda_0^2+\lambda_1^2}}$.

Consider $(a,b) = (\lambda_1 - \lambda_0, 0)$ and $(c,d) = (0, \sqrt{\lambda_0^2 + \lambda_1^2})$. Then by definition, one gets

$$C_{NJ}(\Lambda_2^{(2)}) \ge \frac{\|(a,b) + (c,d)\|^2 + \|(a,b) - (c,d)\|^2}{2(\|(a,b)\|^2 + \|(c,d)\|^2)} = 1 + \frac{\lambda_0}{\sqrt{\lambda_0^2 + \lambda_1^2}}.$$

Combining the last two inequalities, one gets $C_{NJ}(\Lambda_2^{(2)}) = 1 + \frac{\lambda_0}{\sqrt{\lambda_0^2+\lambda_1^2}}$. □

Corollary 4.2. *The James constant is given by:* $J(\Lambda_2^{(2)}) = \sqrt{2 + \frac{2\lambda_0}{\sqrt{\lambda_0^2+\lambda_1^2}}}$.

Proof. It is well-known that $\frac{1}{2}\big(J(\Lambda_2^{(2)})\big)^2 \leq C_{NJ}(\Lambda_2^{(2)}) = 1 + \frac{\lambda_0}{\sqrt{\lambda_0^2+\lambda_1^2}}$. Therefore $J(\Lambda_2^{(2)}) \leq \sqrt{2 + \frac{2\lambda_0}{\sqrt{\lambda_0^2+\lambda_1^2}}}$. Equality occurs when $x = \Big(\frac{\lambda_1}{\sqrt{\lambda_0^2+\lambda_1^2}}, 0\Big)$ and $y = \Big(0, \frac{\lambda_1}{\lambda_1-\lambda_0}\Big)$. □

Corollary 4.3. *If* $\lambda_0 = 1$ *and* $\lambda_1 = 2$*, then* $C_{NJ}(ces_2^{(2)}) = 1 + \frac{1}{\sqrt{5}}$ *and* $J(ces_2^{(2)}) = \sqrt{2 + \frac{2}{\sqrt{5}}}$ *(see* [17], [25]*).*

Corollary 4.4. *If* $\lambda_0 = q_0$ *and* $\lambda_1 = q_0 + q_1$*, then* $C_{NJ}(ces[2,q]^{(2)}) = 1 + \frac{q_0}{\sqrt{2q_0^2+2q_0q_1+q_1^2}}$ *and* $J(ces[2,q]^{(2)}) = \sqrt{2 + \frac{2q_0}{\sqrt{2q_0^2+2q_0q_1+q_1^2}}}$.

The following theorem is the counter part of a theorem presented by Saejung ([25], Theorem 15, p.536) for the sequence space $\Lambda_p^{(2)}$. We repeat a similar treatment here for the sake of completeness.

Theorem 4.5. *The von Neumann–Jordan constant* $C_{NJ}(\Lambda_p^{(2)}) = \Big(\sup_{0\leq t\leq 1} \frac{\psi(t)}{\psi_2(t)}\Big)^2$, $1 < p \leq 2$ *where*
$\psi(t) = \bigg(\frac{\lambda_1^p(1-t)^p}{\lambda_0^p+\lambda_1^p} + \Big(\frac{\lambda_0(1-t)}{(\lambda_0^p+\lambda_1^p)^{1/p}} + t\Big)^p\bigg)^{\frac{1}{p}}$ *and* $\psi_2(t) = \sqrt{\big\{(1-t)^2 + t^2\big\}}$.

Proof. For $(x, y) \in \mathbb{R}^2$, a norm on $\mathbb{R}^2$ is defined as

$$|(x,y)| = \bigg\|\Big(\frac{\lambda_1 x}{(\lambda_0^p+\lambda_1^p)^{1/p}}, \frac{\lambda_1 y}{(\lambda_1-\lambda_0)}\Big)\bigg\|_{\Lambda_p^{(2)}}.$$

It may be easily verified that $|(x,y)| = |(|x|,|y|)|$ and $|(1,0)| = 1 = |(0,1)|$. In other words, $|(x,y)|$ defines an absolute and normalized norm. Additionally, a map $T : (\mathbb{R}^2, |(,)|) \to \Lambda_p^{(2)}$ is chosen which is defined as

$$T\big((x,y)\big) = \Big(\frac{\lambda_1 x}{(\lambda_0^p+\lambda_1^p)^{1/p}}, \frac{\lambda_1 y}{(\lambda_1-\lambda_0)}\Big).$$

Again, it may be easily shown that T is an isometric isomorphism, i.e., sequence spaces $\Lambda_p^{(2)}$ are isometrically isomorphic to $(\mathbb{R}^2, |(,)|)$. It suffices to prove that $\psi \geq \psi_2$ by using derivation by Saito et al. ([26], Theorem 1, p. 521) and deduce that

$$\Big(\frac{\lambda_0(1-t)}{(\lambda_0^p+\lambda_1^p)^{1/p}} + t\Big)^p \geq \frac{\lambda_0^p(1-t)^p}{\lambda_0^p+\lambda_1^p} + t^p,$$

which implies that $\psi(t) \geq \big\{(1-t)^p + t^p\big\}^{\frac{1}{p}} \geq \big\{(1-t)^2 + t^2\big\}^{\frac{1}{2}} = \psi_2(t)$. Hence, the result. □

5. Upper bounds for Hausdorff matrix operators

As usual, l_p for $1 < p < \infty$ denotes the p-summable sequence spaces. Denote $\widetilde{\Lambda}x = \big(\widetilde{\Lambda}x(n)\big)$, where $\widetilde{\Lambda}x(n) = \Big|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k\Big|$. Then the non-absolute type Λ-sequence spaces $\widetilde{\Lambda}_p$, $1 < p \leq \infty$ are introduced which were studied by Mursaleen and Noman [20] who defined them as

$$\widetilde{\Lambda}_p = \Big\{x = (x_k) \in l^0 : \sum_{n=0}^{\infty} \big(\widetilde{\Lambda}x(n)\big)^p < \infty\Big\}.$$

This is a Banach space equipped with the norm $\|x\|_{\widetilde{\Lambda}_p} = \|\widetilde{\Lambda}x\|_{l_p}$. The inclusion relation between l_p and $\widetilde{\Lambda}_p$ is given in the following lemma:

Lemma 5.1. *([20], Corollary 4.11. & 4.13.) If* $\frac{1}{\lambda} = \left(\frac{1}{\lambda_n}\right) \in l_1$, *then* $\sup_k \Big\{(\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n}\Big\} < \infty$ *and the inclusion* $l_p \subset \widetilde{\Lambda}_p$ *holds for* $1 \leq p < \infty$.

In this section, it is contemplated to establish a Hardy's type formula as an upper bound for $\|H_\theta\|_{l_p, \widetilde{\Lambda}_p}$, where $H_\theta : l_p \to \widetilde{\Lambda}_p$. Firstly, the definition of Hausdorff matrix [2] is recalled for the sake of convenience. Hausdorff matrix is denoted by $H_\theta = (h_{n,k})$, $n, k = 0, 1, 2, \ldots$ and is defined by

$$h_{n,k} = \begin{cases} 0 & \text{if } k > n, \\ \binom{n}{k}\Delta^{n-k}\mu_k & \text{if } k \leq n, \end{cases}$$

where Δ is the difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\mu = (\mu_0, \mu_1, \ldots)$ is a sequence of real numbers with $\mu_0 = 1$ and

$$\mu_k = \int_0^1 \theta^k d\mu(\theta),$$

where $d\mu(\theta)$ is a Borel probability measure on $[0, 1]$. Therefore the equivalent form of the matrix $H_\theta = (h_{n,k})$ is

$$h_{n,k} = \begin{cases} 0 & \text{if } k > n, \\ \binom{n}{k} \displaystyle\int_0^1 \theta^k (1-\theta)^{n-k} d\mu(\theta) & \text{if } k \leq n. \end{cases}$$

If $d\theta$ is chosen as the Lebesgue measure, then the Hausdorff matrix includes four famous classes of matrices as given below:
(a) By putting $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$, H_θ leads to (C, α), the Cesàro matrix of order α;
(b) By putting $d\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$, H_θ reduces to (H, α), the Hölder matrix of order α;
(c) By putting $d\mu(\theta) =$ point evaluation at $\theta = \alpha$, H_θ reduces to (E, α), the Euler matrices of order α;
(d) By putting $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$, H_θ becomes (Γ, α), the Gamma matrices of order α. The following lemma is required to establish the desired result.

Lemma 5.2. *([11], Theorem 216) Consider two non-negative sequences* $x = (x_k)$ *and* $\mu = (\mu_k)$ *of real numbers with* $\mu_0 = 1$. *Then for* $1 < p < \infty$

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} h_{n,k} x_k\right)^p < \left(\int_0^1 \theta^{-1/p} d\mu(\theta)\right)^p \sum_{k=0}^{\infty} x_k^p.$$

Theorem 5.3. *Suppose $1 < p < \infty$, $\frac{1}{\lambda} \in l_1$ and $x = (x_k)$ be a non-negative sequence of real numbers. Then for the Hausdorff matrix $H_\theta : l_p \to \widetilde{\Lambda}_p$, one gets*

$$\|H_\theta\|_{l_p,\widetilde{\Lambda}_p} \leq \left(\sup_k \left\{ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right\} \right)^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Proof. Let $x = (x_k)$ be a non-negative sequence of real numbers in l_p and $q = \frac{p}{p-1}$. Applying the Hölder inequality and the Lemma 5.2, the following result is obtained:

$$\begin{aligned}
\|H_\theta x\|^p_{\widetilde{\Lambda}_p} &= \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \sum_{j=0}^{k} h_{k,j} x_j \right)^p \\
&\leq \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\sum_{j=0}^{k} h_{k,j} x_j \right)^p \left(\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \right)^{p/q} \\
&= \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \left(\sum_{j=0}^{k} h_{k,j} x_j \right)^p \\
&= \sum_{k=0}^{\infty} (\lambda_k - \lambda_{k-1}) \left(\sum_{j=0}^{k} h_{k,j} x_j \right)^p \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \\
&\leq \sup_k \left\{ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right\} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} h_{k,j} x_j \right)^p \\
&\leq \left(\sup_k \left\{ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right\} \right) \left(\int_0^1 \theta^{-1/p} d\mu(\theta) \right)^p \|x\|^p_{l_p}.
\end{aligned}$$

Hence, we have

$$\|H_\theta\|_{l_p,\widetilde{\Lambda}_p} \leq \left(\sup_k \left\{ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right\} \right)^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta),$$

and this completes the proof. □

Corollary 5.4. *By denoting $M = \sup_k \left\{ (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \right\}$ and imposing similar assumptions as those of Theorem 5.3, the following results are obtained:*

(a) $\|(C,\alpha)\|_{l_p,\widetilde{\Lambda}_p} \leq M^{1/p} \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)}, \alpha > 0;$

(b) $\|(H,\alpha)\|_{l_p,\widetilde{\Lambda}_p} \leq M^{1/p} \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{-1/p} |\log \theta|^{\alpha-1} d\theta, \alpha > 0;$

(c) $\|(E,\alpha)\|_{l_p,\widetilde{\Lambda}_p} \leq M^{1/p} \alpha^{-1/p}, 0 < \alpha < 1;$

(d) $\|(\Gamma,\alpha)\|_{l_p,\widetilde{\Lambda}_p} \leq M^{1/p} \frac{\alpha p}{\alpha p - 1}, \alpha p > 1.$

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Faculty of Mathematics, Indian Institute of Carpet Technology, Chauri Road, Bhadohi- 221401, Uttar Pradesh, India.
E-mail address: atanu.manna@iict.ac.in or atanuiitkgp86@gmail.com