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# ON LINEAR MAPS PRESERVING CERTAIN PSEUDOSPECTRUM AND CONDITION SPECTRUM SUBSETS

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Abstract. We define two new types of spectrum, called the $\varepsilon$-left (or right) pseudospectrum and the $\varepsilon$-left (or right) condition spectrum, of an element $a$ in a complex unital Banach algebra $A$. We prove some basic properties among them the property that the $\varepsilon$-left (or right) condition spectrum is a particular case of Ransford spectrum. We study also the linear preserver problem for our defined functions and we establish the following:

(1) Let $A$ and $B$ be complex unital Banach algebras and $\varepsilon > 0$. Let $\phi : A \longrightarrow B$ be an $\varepsilon$-left (or right) pseudospectrum preserving onto linear map. Then $\phi$ preserves certain standard spectral functions.

(2) Let $A$ and $B$ be complex unital Banach algebras and $0 < \varepsilon < 1$. Let $\phi : A \longrightarrow B$ be a unital linear map. Then
(a) If $\phi$ is an $\varepsilon$-almost multiplicative map, then $\sigma^l(\phi(a)) \subseteq \sigma^l_\varepsilon(a)$ and $\sigma^r(\phi(a)) \subseteq \sigma^r_\varepsilon(a)$, for all $a \in A$.
(b)If $\phi$ is an $\varepsilon$-left (or right) condition spectrum preserving, then (i) if $A$ is semi-simple, then $\phi$ is injective; (ii) if B is spectrally normed, then $\phi$ is continuous.

## 1. Preliminaries

Let $A$ be a complex Banach algebra with unit 1. We shall identify $\lambda.1$ with $\lambda$. An element $a \in A$ is said to be left invertible if there exists $b \in A$ such that $ba = 1$, while it is said right invertible if there exists $b \in A$ such that $ab = 1$. An invertible element of $A$ is a right and left invertible element of $A$, while a semi-invertible element of $A$ is a right or left invertible element of $A$. We denote


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by $Inv(A), Inv^l(A)$ and $Inv^r(A)$ the sets of all invertible, left invertible and right invertible elements of $A$, respectively.
Let $a \in A$, the left spectrum, the right spectrum and the spectrum of $a$ are defined respectively as follow:

$$\sigma^l(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^l(A)\},$$

$$\sigma^r(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin Inv^r(A)\},$$

and

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin Inv(A)\}.$$

Since $Inv(A) = Inv^l(A) \cap Inv^r(A)$, then $\sigma(a) = \sigma^l(a) \cup \sigma^r(a)$. Note that $\partial\sigma(a)$: the boundary of $\sigma(a)$, is a subset of any one of the sets $\sigma^l(a)$ and $\sigma^r(a)$ (see [3]). In particular, $\sigma^l(a)$ and $\sigma^r(a)$ are non empty sets.
Given $\varepsilon > 0$ and $a \in A$. The $\varepsilon$-pseudospectrum of $a$ is denoted by $\Lambda_\varepsilon(a)$ and is defined to be the set

$$\Lambda_\varepsilon(a) := \{\lambda \in \mathbb{C} : \|(a - \lambda)^{-1}\| \geq \frac{1}{\varepsilon}\},$$

with the convention $\|(a - \lambda)^{-1}\| = \infty$ if $a - \lambda$ is not invertible.
For more information about the $\varepsilon$-pseudospectrum, we refer the reader to [1] and [13].

The $\varepsilon$-condition spectrum of $a \in A$ is denoted by $\sigma_\varepsilon(a)$ and is defined by

$$\sigma_\varepsilon(a) := \{\lambda \in \mathbb{C} : \|(a - \lambda)^{-1}\|\|a - \lambda\| \geq \frac{1}{\varepsilon}\},$$

with the convention $\|(a - \lambda)^{-1}\|\|a - \lambda\| = \infty$ if $a - \lambda$ is not invertible.

The spectral radius $r(a)$ and the $\varepsilon$-condition spectral radius $r_\varepsilon(a)$ of $a$ are defined respectively as follow:

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\} \quad \text{and} \quad r_\varepsilon(a) := \sup\{|\lambda| : \lambda \in \sigma_\varepsilon(a)\}.$$

The following property is proved by Kulkarni and Sukumar in [7, Theorem 2.9]:

$$r(a) \leqslant r_\varepsilon(a) \leqslant \frac{1 + \varepsilon}{1 - \varepsilon}\|a\| \ ; \text{ for all } \ 0 < \varepsilon < 1. \tag{1.1}$$

We may refer to [7] for more information about $\varepsilon$-condition spectrum and $\varepsilon$-condition spectral radius.

Let $A$ and $B$ be unital Banach algebras over the complex field. Let $\phi : A \longrightarrow B$ be linear map and $\varepsilon > 0$. We say that

- $\phi$ is a Jordan homomorphism if $(\phi(a))^2 = (\phi(a))^2$ for all $a \in A$.
- $\phi$ is an $\varepsilon$-almost Jordan multiplicative map, if $\|\phi(a^2) - \phi(a)^2\| \leq \varepsilon \|a\|^2$ for all $a \in A$.
- $\phi$ is an $\varepsilon$-almost multiplicative map, if $\|\phi(ab) - \phi(a)\phi(b)\| \leq \varepsilon \|a\|\|b\|$ for all $a, b \in A$.
- $\phi$ is an $\varepsilon$-almost anti-multiplicative map, if $\|\phi(ab) - \phi(b)\phi(a)\| \leq \varepsilon \|a\|\|b\|$ for all $a, b \in A$.

It is obvious that $\varepsilon$-almost multiplicative and $\varepsilon$-almost anti-multiplicative maps are $\varepsilon$-almost Jordan multiplicative.

Over the last decade there has been a considerable interest in the so called linear preserver problems (see the survey articles [8], [9] and [10]). The objective is to study additive or linear maps between two Banach algebras preserving a given class of elements of algebras. The most famous problem is Kaplansky's problem [5] asking whether bijective unital linear maps between semi-simple Banach algebras preserving invertibility in both directions are Jordan isomorphisms. Many other linear preserver problems have attracted many researchers. Many mathematicians were interested to study certain quantities related on or extended the concept of spectrum. In [6], Krishna Kumar and Kulkarni gave several results about linear maps preserving pseudospectrum and condition spectrum. They proved that if $\phi$ is an $\varepsilon$-pseudospectrum preserving linear onto map between two Banach algebras $A$ and $B$, then $\phi$ preserves spectrum, and if in addition, $A$ and $B$ are uniform algebras, then $\phi$ is an isometric isomorphism. They proved also that if $\phi : A \longrightarrow B$ is an $\varepsilon$-condition spectrum preserving linear map, then $\phi$ is injective and unital and if in addition $A$, $B$ are uniform algebras, then $\phi$ is continuous and an $\varepsilon'$-almost multiplicative map, where $\varepsilon, \varepsilon'$ tend to zero simultaneously. In this paper we will prove some similar results for linear maps preserving $\varepsilon$-left (or right) pseudospectrum and condition spectrum.

## 2. Terminology and basic properties

In this section, we will introduce the concept of $\varepsilon$- left (or right) pseudospectrum and the concept of $\varepsilon$-left (or right) condition spectrum. We prove that the last one is a particular case of Ransford spectrum and that the relations connecting pseudospectrum and condition spectrum given in [6] remain true for left (or right) pseudospectrum and left (or right) condition spectrum.

**Definition 2.1.** ($\varepsilon$-left (or right) pseudospectrum) Let $A$ be a complex Banach algebra with unit 1 and let $\varepsilon > 0$. The $\varepsilon$-left pseudospectrum of an element $a \in A$ is denoted by $\Lambda^l_\varepsilon(a)$ and is defined as

$$\Lambda^l_\varepsilon(a) := \{\lambda \in \mathbb{C} : \inf\{\|b\| : b \text{ a left inverse of } a - \lambda\} \geq \frac{1}{\varepsilon}\},$$

with the convention $\inf\{\|b\| : b \text{ a left inverse of } a - \lambda\} = \infty$, if $a - \lambda$ is not left invertible. The $\varepsilon$-right pseudospectrum is denoted by $\Lambda^r_\varepsilon(a)$ and is defined in the obvious way.

It is clear that for all $a \in A$, we have

$$\sigma^l(a) \subseteq \Lambda^l_\varepsilon(a) \subseteq \Lambda_\varepsilon(a),$$

and

$$\sigma^r(a) \subseteq \Lambda^r_\varepsilon(a) \subseteq \Lambda_\varepsilon(a).$$

In particular $\Lambda^l_\varepsilon(a)$ and $\Lambda^r_\varepsilon(a)$ are non empty sets.

**Definition 2.2.** ($\varepsilon$-left (or right) condition spectrum) Let $A$ be a complex Banach algebra with unit 1 and let $\varepsilon > 0$. The $\varepsilon$-left condition spectrum of an element $a \in A$ is denoted by $\sigma_\varepsilon^l(a)$ and is defined as

$$\sigma_\varepsilon^l(a) := \{\lambda \in \mathbb{C} : \inf\{\|b\|\|a-\lambda\| : \ b \ \text{ a left inverse of } \ a-\lambda\} \geq \frac{1}{\varepsilon}\},$$

with the convention $\inf\{\|b\|\|a-\lambda\| : \ b \ \text{ a left inverse of } \ a-\lambda\} = \infty$ if $a-\lambda$ is not left invertible.
The $\varepsilon$-right condition spectrum is denoted by $\sigma_\varepsilon^r(a)$ and is defined in the obvious way.

The following equations are immediate

$$\sigma^l(a) \subseteq \sigma_\varepsilon^l(a) \subseteq \sigma_\varepsilon(a),$$

and

$$\sigma^r(a) \subseteq \sigma_\varepsilon^r(a) \subseteq \sigma_\varepsilon(a).$$

In particular $\sigma_\varepsilon^l(a)$ and $\sigma_\varepsilon^r(a)$ are non empty sets.

*Remark* 2.3. The preceding inclusions are not equalities in general. Indeed, if we consider the right shift operator

$$\begin{array}{rccc} R: & \ell^2(\mathbb{C}) & \longrightarrow & \ell^2(\mathbb{C}) \\ & (e_i)_{i\in\mathbb{N}} & \longmapsto & (0, e_0, e_1, \dots, ) \end{array}$$

Then $R \in B(l^2(\mathbb{C}))$, the Banach algebra of all bounded linear maps on $l^2(\mathbb{C})$, $R$ is not surjective and hence $0 \in \sigma_\varepsilon(R)$. But, $R$ is left invertible and the left shift operator

$$\begin{array}{rccc} L: & \ell^2(\mathbb{C}) & \longrightarrow & \ell^2(\mathbb{C}) \\ & (e_i)_{i\in\mathbb{N}} & \longmapsto & (e_1, \dots, ) \end{array}$$

is a left inverse of $R$. It is easy to verify that $\|R\| = \|L\| = 1$. Thus $\|R\|\|L\| = 1 < \frac{1}{\varepsilon}$, for all $\ 0 < \varepsilon < 1$. So $0 \notin \sigma_\varepsilon^l(R)$. Hence $\sigma_\varepsilon^l(R) \subsetneq \sigma_\varepsilon(R)$.

Now, we will show that the $\varepsilon$-left (or right) condition spectrum is a particular case of Ransford spectrum.

**Definition 2.4.** (Ransford set) An open subset $\Omega$ of a complex unital Banach algebra $A$ satisfying the following assertions is called a Ransford set.

(1) $1 \in \Omega$.
(2) $0 \notin \Omega$.
(3) $\forall a \in \Omega, \forall \lambda \in \mathbb{C} \setminus \{0\}; \ \ \lambda a \in \Omega$.

**Definition 2.5.** (Ransford spectrum) Let $\Omega$ be a Ransford set of a complex unital Banach algebra $A$ and let $a \in A$. The Ransford spectrum of $a$ with respect to $\Omega$ is defined to be:

$$\sigma^\Omega(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \Omega\}.$$

An interested reader can find basic properties and more information about the Ransford spectrum in [11].

Let $A$ be a unital complex Banach algebra and $a \in A$. We will call left condition number of $a$, denoted by $k^l(a)$, the number

$$k^l(a) := \begin{cases} \inf\{\|a\|\|b\| : b \text{ a left inverse of } a\} & \text{if } a \text{ is left invertible,} \\ +\infty & \text{if } a \text{ is not left invertible.} \end{cases}$$

Let $0 < \varepsilon < 1$ and let $\Omega_\varepsilon^l = \{a \in Inv^l(A) : k^l(a) < \frac{1}{\varepsilon}\}$.

**Proposition 2.6.** *$\Omega_\varepsilon^l$ is a Ransford set.*

*Proof.* Since 0 is not left invertible, then $0 \notin \Omega_\varepsilon^l$. But 1 is invertible and $k^l(1) = 1 < \frac{1}{\varepsilon}$, so $1 \in \Omega_\varepsilon^l$. Let $z \in \mathbb{C} \setminus \{0\}$ and $a \in \Omega_\varepsilon^l$. Then for all $b \in A$, we have

$$ba = 1 \Longleftrightarrow (\frac{1}{z}b)(za) = 1,$$

and

$$\|b\|\|a\| = \|\frac{1}{z}b\|\|za\|.$$

Thus $k^l(za) = k^l(a) < \frac{1}{\varepsilon}$ and so $za \in \Omega_\varepsilon^l$. To conclude we must prove that $\Omega_\varepsilon^l$ is an open subset of $A$. Let $a \in \Omega_\varepsilon^l$. Then $k^l(a) < \frac{1}{\varepsilon}$. Choose a scalar $k^l(a) < \eta < \frac{1}{\varepsilon}$ and let $b$ be a left inverse of $a$ such that $\|b\|\|a\| < \eta$. Let $c \in A$ satisfying $\|c\| < \frac{1}{\|b\|}$ and satisfying an other condition given later. Since $\|bc\| \leqslant \|b\|\|c\| < 1$, then $b(a - c) = 1 - bc$ is invertible. So $(1 - bc)^{-1}b$ is a left inverse of $a - c$ satisfying

$$\|(1 - bc)^{-1}b\|\|a - c\| \leqslant \|(1 - bc)^{-1}\|\|b\|\|a\|\|1 - cb\| \leqslant \eta\|(1 - bc)^{-1}\|\|1 - cb\|.$$

Since $\|(1-bc)^{-1}\|\|1-cb\|$ tends to 1 as $c$ tends to zero, then for a fixed $\varepsilon' > 0$ such that $(1+\varepsilon')\eta < \frac{1}{\varepsilon}$ there exists $\delta > 0$ satisfying $\|(1-bc)^{-1}\|\|(1-bc)\| < 1+\varepsilon'$, for all $d \in A$ with $\|d\| < \delta$. If we take $c$ such that $\|c\| < \delta$, we obtain $k^l(a - c) < \frac{1}{\varepsilon}$. So for all $c \in A$ satisfying $\|d\| < \min\{\frac{1}{\|b\|}, \delta\}$, we have $a - c \in \Omega_\varepsilon^l$. Hence, $\Omega_\varepsilon^l$ is an open subset of $A$ and so a Ransford set. □

**Proposition 2.7.** *For all $a \in A$, $\sigma_\varepsilon^l(a)$ is the Ransford spectrum of $a$ with respect to $\Omega_\varepsilon^l$. Similarly, $\sigma_\varepsilon^r(a)$ is the Ransford spectrum of $a$ with respect to $\Omega_\varepsilon^r$ the subset of $\mathbb{C}$ defined in the obvious way.*

The following properties follow from generalized properties of Ransford spectrum (see [11]) and from the fact that $\varepsilon$-left and right condition spectrums are non empty sets.

**Corollary 2.8.** *Let $A$ be a unital complex Banach algebra with unit 1, $a \in A$ and $0 < \varepsilon < 1$. Then*

(1) $\sigma_\varepsilon^l(0) = \{0\}$, $\sigma_\varepsilon^l(1) = \{1\}$, $\sigma_\varepsilon^r(0) = \{0\}$ *and* $\sigma_\varepsilon^r(1) = \{1\}$.
(2) *For all $a \in A, \sigma_\varepsilon^l(a)$ and $\sigma_\varepsilon^r(a)$ are non empty compact subsets of $\mathbb{C}$.*
(3) *The maps $a \longmapsto \sigma_\varepsilon^l(a)$ and $a \longmapsto \sigma_\varepsilon^r(a)$ are upper semicontinuous functions from $A$ to compact subsets of $\mathbb{C}$.*

The next two Propositions give some relations connecting $\varepsilon$-left condition spectrum and $\varepsilon$-left pseudospectrum of an element in a unital complex Banach algebra.

**Proposition 2.9.** *Let $A$ be a unital complex Banach algebra with unit $1$, $a \in A$ a non zero vector and $0 < \varepsilon < 1$. Then*

$$\sigma^l_\varepsilon(a) \subseteq \Lambda^l_{\frac{2\varepsilon\|a\|}{1-\varepsilon}}(a)\,.$$

*Proof.* Let $\lambda \in \sigma^l_\varepsilon(a)$. Then $\lambda \in \sigma_\varepsilon(a)$, so $|\lambda| \leq r_\varepsilon(a) \leq \dfrac{(1+\varepsilon)\|a\|}{1-\varepsilon}$. Hence

$$\|\lambda - a\| \;\leq\; |\lambda| + \|a\| \leq \frac{(1+\varepsilon)\|a\|}{1-\varepsilon} + \|a\| = \frac{2\|a\|}{1-\varepsilon}.$$

Since $\lambda \in \sigma^l_\varepsilon(a)$, then two cases occur. We can suppose first that $a - \lambda$ is left invertible. Then for every $b$ a left inverse of $a - \lambda$, we have

$$\|a - \lambda\|\|b\| \geq \frac{1}{\varepsilon},$$

that implies $\|b\| \geq \frac{1}{\varepsilon\|\lambda - a\|} \geq \frac{1-\varepsilon}{2\varepsilon\|a\|}$. Thus, $\lambda \in \Lambda^l_{\frac{2\varepsilon\|a\|}{1-\varepsilon}}(a)$.

Now, if $a - \lambda$ is not left invertible, then obviously $\lambda \in \Lambda^l_{\frac{2\varepsilon\|a\|}{1-\varepsilon}}(a)$. □

**Proposition 2.10.** *Let $A$ be a unital complex Banach algebra with unit $1$ and $\varepsilon > 0$. Suppose that $a \in A$ is not a scalar multiple of $1$ and let $M_a := \inf\{\|z-a\| : z \in \mathbb{C}\}$. Then $\Lambda^l_\varepsilon(a) \subseteq \sigma^l_{\frac{\varepsilon}{M_a}}(a)$.*

*Proof.* Assume that $\lambda \in \Lambda^l_\varepsilon(a)$ and suppose first that $a-\lambda$ is left invertible. Then for every $b$ a left inverse of $a - \lambda$, we have

$$\|b\| \geq \frac{1}{\varepsilon}.$$

Also

$$\|\lambda - a\| \geq \inf\{\|z - a\| : z \in \mathbb{C}\} = M_a > 0.$$

Hence

$$\|\lambda - a\|\|b\| \geq \frac{M_a}{\varepsilon}.$$

So $\lambda \in \sigma^l_{\frac{\varepsilon}{M_a}}(a)$. Now, if $a-\lambda$ is not left invertible, then obviously $\lambda \in \sigma^l_{\frac{\varepsilon}{M_a}}(a)$. □

*Remark* 2.11.
1. If $a = \mu.1$ for some $\mu \in \mathbb{C}$, then $\sigma^l_\varepsilon(a) = \sigma_\varepsilon(a) = \{\mu\}$ and $\Lambda^l_\varepsilon(a) = \Lambda_\varepsilon(a) = D(\mu, \varepsilon)$, the closed ball with center $\mu$ and radius $\varepsilon$. Thus the condition on $a$, to be not a scalar multiple of 1, can not be dropped from the above Proposition.
2. The Proposition 2.6 and the Proposition 2.9 remain true if we replace $\Lambda^l_\varepsilon(a)$ with $\Lambda^r_\varepsilon(a)$ and $\sigma^l_\varepsilon(a)$ with $\sigma^r_\varepsilon(a)$.

## 3. Main results

We begin this section by giving a sufficient condition for a map between Banach algebras to preserve the $\varepsilon$-left (or right) pseudospectrum and the $\varepsilon$-left (or right) condition spectrum. In the following, $\Delta^s_\varepsilon(.)$ will represent any one of the four sets $\Lambda^l_\varepsilon(.), \Lambda^r_\varepsilon(.), \sigma^l_\varepsilon(.)$ and $\sigma^r_\varepsilon(.)$.

**Theorem 3.1.** *Let $A, B$ be unital complex Banach algebras and $\varepsilon > 0$. Suppose that $\phi : A \longrightarrow B$ is a bijective linear and multiplicative isometry. Then for all $a \in A$, we have*

$$\Delta_\varepsilon^s(a) = \Delta_\varepsilon^s(\phi(a)).$$

*Proof.* Assume that $\Delta_\varepsilon^s(.) = \Lambda_\varepsilon^l(.)$. Let $a \in A$. If $\lambda \in \Lambda_\varepsilon^l(a)$, then for all $b$ a left inverse of $a - \lambda$ we have

$$\|b\| \geq \frac{1}{\varepsilon}.$$

Since $\phi$ is an isometry, then

$$\|\phi(b)\| \geq \frac{1}{\varepsilon}.$$

On the other hand, since $\phi$ is a bijective multiplicative map then it is unital. Indeed, let $u \in A$ such that $\phi(u) = 1$ then

$$1 = \phi(u) = \phi(u.1) = \phi(u)\phi(1) = 1.\phi(1) = \phi(1).$$

Hence $\phi$ is a unital bijective linear and multiplicative map. Thus for every $c \in A$, we have

$$c \text{ is a left inverse of } a - \lambda \Longleftrightarrow \phi(c) \text{ is a left inverse of } \phi(a) - \lambda.$$

It follows that for every $d$ a left inverse of $\phi(a) - \lambda$,

$$\|d\| \geq \frac{1}{\varepsilon}.$$

Hence $\lambda \in \Lambda_\varepsilon^l(\phi(a))$ and so $\Lambda_\varepsilon^l(a) \subseteq \Lambda_\varepsilon^l(\phi(a))$. The same argument shows that $\Lambda_\varepsilon^l(\phi(a)) \subseteq \Lambda_l^\varepsilon(a)$, so $\Lambda_\varepsilon^l(\phi(a)) = \Lambda_\varepsilon^l(a)$. Similarly we can prove the result if $\Delta_\varepsilon^s(.)$ takes any one of the other sets. □

In the next two Theorems, we prove that if a surjective linear map between Banach algebras preserves $\varepsilon$-left or (respectively right) pseudospectrum, then it preserves left (respectively right) spectrum and norms of all invertible elements.

**Theorem 3.2.** *Let $A, B$ be complex unital Banach algebras and $\varepsilon > 0$. Let $\phi : A \longrightarrow B$ be an $\varepsilon$-left (respectively right) pseudospectrum preserving linear onto map. Then $\phi$ preserves left (respectively right) spectra of elements.*

*Proof.* Suppose that $\Lambda_\varepsilon^l(a) = \Lambda_\varepsilon^l(\phi(a))$, for all $a \in A$. Let $a \in A$ and $\lambda \in \mathbb{C} \setminus \sigma^l(a)$. Choose $t > \varepsilon\|b\|$, where $b$ is a left inverse of $a - \lambda$. Then

$$\|\frac{1}{t}b\| < \frac{1}{\varepsilon}.$$

Since $\frac{1}{t}b$ is a left inverse of $ta - t\lambda$, then

$$t\lambda \notin \Lambda_\varepsilon^l(ta) = \Lambda_\varepsilon^l(\phi(ta)) \supseteq \sigma^l(\phi(ta)) = t\sigma^l(\phi(a)).$$

So

$$\lambda \notin \sigma^l(\phi(a)).$$

Therefore

$$\sigma^l(\phi(a)) \subseteq \sigma^l(a).$$

In a similar way we can prove that

$$\sigma^l(a) \subseteq \sigma^l(\phi(a)).$$

Hence

$$\sigma^l(\phi(a)) = \sigma^l(a).$$

□

**Theorem 3.3.** *Let $A, B$ be complex unital Banach algebras and $\varepsilon > 0$. let $\phi : A \longrightarrow B$ be an $\varepsilon$-left (or right) pseudospectrum preserving linear onto map. Suppose that $\phi$ is multiplicative or anti-multiplicative. Then $\phi$ preserves norms of all invertible elements of $A$.*

*Proof.* Suppose that $\phi$ preserves $\varepsilon$-left pseudospectrum and suppose that there exists $a \in Inv(A)$ such that $\|\phi(a^{-1})\| \neq \|a^{-1}\|$, for example $\|\phi(a^{-1})\| > \|a^{-1}\|$. Let $t > 0$ such that $\varepsilon\|\phi(a^{-1})\| < t < \varepsilon\|a^{-1}\|$. Then $\|(ta)^{-1}\| \geq \frac{1}{\varepsilon}$. But, $(ta)^{-1}$ is the unique left inverse of $ta$, so $0 \in \Lambda^l_\varepsilon(ta)$. On the other hand, since $\phi$ is an onto multiplicative or anti-multiplicative map, then it is unital. So $\phi$ preserves invertibility. Hence, $(\phi(ta))^{-1}$ is the inverse of $\phi(ta)$ and so a left inverse of $\phi(ta)$. But $\|(\phi(ta))^{-1}\| < \frac{1}{\varepsilon}$, then $0 \notin \Lambda^l_\varepsilon(\phi(ta)) = \Lambda^l_\varepsilon(ta)$, this is a contradiction. So $\phi$ preserves norms of all invertible elements of $A$. Similarly, we prove that if $\phi$ preserves $\varepsilon$-right pseudospectrum, then $\phi$ preserves norms of all invertible elements of $A$. □

**Corollary 3.4.** *Let $X$ and $Y$ be complex Banach spaces. Let $\phi : B(X) \longrightarrow B(Y)$ be a unital bijective linear map. Then, the following assertions are equivalent:*

(1) *$\phi$ preserves $\varepsilon$-left pseudospectrum for some $\varepsilon > 0$.*
(2) *$\phi$ preserves $\varepsilon$-right pseudospectrum for some $\varepsilon > 0$.*
(3) *$\phi$ preserves $\varepsilon$-pseudospectrum for some $\varepsilon > 0$.*
(4) *$\phi$ preserves $\varepsilon$-left pseudospectrum for every $\varepsilon > 0$.*
(5) *$\phi$ preserves $\varepsilon$-right pseudospectrum for every $\varepsilon > 0$.*
(6) *$\phi$ preserves $\varepsilon$-pseudospectrum for every $\varepsilon > 0$.*
(7) *Either there exists an isometry $U \in B(X, Y)$ such that $\phi(T) = UTU^{-1}$, for every $T \in B(X)$, or there exists an isometry $V \in B(X^*, Y)$ such that $\phi(T) = VT^*V^{-1}$, for every $T \in B(X)$. The last case can not occur if $X$ or $Y$ is not reflexive, or if there exists a semi-invertible but not invertible element in $B(X)$.*

*Proof.* • Assume that (7) hold. Suppose first that there exists an isometry $U \in B(X, Y)$ such that $\phi(T) = UTU^{-1}$, for every $T \in B(X)$. Then $\phi$ verify the hypothesis of 3.1 and so $(1), (2), (3), (4), (5)$ and $(6)$ hold. Now, suppose that there exists an isometry $V \in B(X^*, Y)$ such that $\phi(T) = VT^*V^{-1}$, for every $T \in B(X)$. Then $X$ and $Y$ are reflexive and every semi-invertible element of $B(X)$ is invertible. Let $\varepsilon > 0$, then

$$\Lambda^l_\varepsilon(T) = \Lambda^r_\varepsilon(T) = \Lambda_\varepsilon(T) = \Lambda_\varepsilon(T^*) = \Lambda_\varepsilon(\phi(T)) = \Lambda^l_\varepsilon(\phi(T)) = \Lambda^r_\varepsilon(\phi(T)).$$

Hence $(1), (2), (3), (4), (5)$ and $(6)$ hold.
• "(1) $\Longrightarrow$ (7)" Let $\varepsilon > 0$ such that $\Lambda^l_\varepsilon(T) = \Lambda^l_\varepsilon(\phi(T))$, for every $T \in B(X)$. Then by Theorem 3.2, $\phi$ preserves left spectrum. It follows, by [3, Corollary 4.5], that

either there exists an invertible operator $A \in B(X, Y)$ such that $\phi(T) = ATA^{-1}$, for every $T \in B(X)$ or there exists an invertible operator $C \in B(X^*, Y)$ such that $\phi(T) = CT^*C^{-1}$, for every $T \in B(X)$, and the last case can not occur if $X$ or $Y$ is not reflexive, or if there exists a semi-invertible but not invertible element in $B(X)$. But, by Theorem 3.3, $\phi$ preserves the norm of all invertible elements of $B(X)$. Hence, by [12, Theorem 3.1, p 141], there exists a bijective isometry $U \in B(X, Y)$ or a bijective isometry $V \in B(X^*, Y)$, and $\lambda \in \mathbb{C}^*$ such that $A = \lambda U$ or $C = \lambda V$. So $\phi$ has the desired forms.
The other implications can be proved similarly. □

In the following, we give some results about the $\varepsilon$-left (or right) condition spectrum preserver.

**Definition 3.5.** (Spectrally normed algebra) Let $A$ be a complex Banach algebra with unit 1. $A$ is said to be a spectrally normed algebra, if there exists a scalar $k \geq 1$ such that $\|a\| \leq k\, r(a)$, for all $a \in A$.

**Theorem 3.6.** *Let $A, B$ be unital Banach algebras and $0 < \varepsilon < 1$. Let $\phi : A \longrightarrow B$ be a unital linear map. Then*

(1) *If $\phi$ is an $\varepsilon$-almost multiplicative map, then $\sigma^l(\phi(a)) \subseteq \sigma^l_\varepsilon(a)$ and $\sigma^r(\phi(a)) \subseteq \sigma^r_\varepsilon(a)$, for all $a \in A$.*
(2) *If $\phi$ is an $\varepsilon$-left (or right) condition spectrum preserving map, then*
   (a) *If $A$ is semi-simple then $\phi$ is injective.*
   (b) *if $B$ is spectrally normed, then $\phi$ is continuous and $\|\phi\| \leq k\dfrac{1+\varepsilon}{1-\varepsilon}$ for some constant $k > 0$. In particular if $B$ is a uniform algebra, then $\phi$ is continuous and $\|\phi\| \leq \dfrac{1+\varepsilon}{1-\varepsilon}$.*

*Proof.* (1) Let $\lambda \notin \sigma^l_\varepsilon(a)$. Then $\lambda - a$ is left invertible and there exists $b$ a left inverse of $\lambda - a$ such that

$$\|(\lambda - a)\|\|b\| < \frac{1}{\varepsilon}.$$

Thus,

$$\begin{aligned}
\|1 - \phi(b)\phi(\lambda - a)\| &= \|\phi(1) - \phi(b)\phi(\lambda - a)\| \\
&= \|\phi(b(\lambda - a)) - \phi(b)\phi(\lambda - a)\| \\
&\leq \varepsilon\|\lambda - a\|\|b\| \\
&< 1.
\end{aligned}$$

Hence $\phi(b)\phi(\lambda - a)$ is invertible and therefore, $\phi(\lambda - a) = \lambda - \phi(a)$ is left invertible, this implies that $\lambda \notin \sigma^l(\phi(a))$. Hence $\sigma^l(\phi(a)) \subseteq \sigma^l_\varepsilon(a)$. Similarly, we prove $\sigma^r(\phi(a)) \subseteq \sigma^r_\varepsilon(a)$.

(2) Suppose that $\phi$ preserves $\varepsilon$-left condition spectrum, then

$$\sigma^l(\phi(a)) \subseteq \sigma^l_\varepsilon(\phi(a)) = \sigma^l_\varepsilon(a), \text{ for all } a \in A.$$

In particular,

$$r(\phi(a)) \leq r_\varepsilon(a) \leq \frac{1+\varepsilon}{1-\varepsilon}\|a\| \text{ for all } a \in A.$$

$(a)$ Suppose that $A$ is semi-simple and let $a \in N(\phi)$, the kernel of $\phi$, then for all $x \in A$ we have:

$$\sigma^l(x) \subseteq \sigma^l_\varepsilon(x) = \sigma^l_\varepsilon(\phi(x)) = \sigma^l_\varepsilon(\phi(x-a)) = \sigma^l_\varepsilon(x-a).$$

So

$$r(x) \leq r_\varepsilon(x-a) \leq \frac{1+\varepsilon}{1-\varepsilon}\ \|x-a\|$$

Hence, by J. Zemánek Theorem (see [2, Theorem 5.31, p 95]), we get $a \in Rad(A)$. But $A$ is semi-simple, so $a = 0$ and thus $\phi$ is injective.
$(b)$ If $B$ is spectrally normed, then for some $k > 0$ we get

$$\|\phi(a)\| \leq kr(\phi(a)) \leq k\frac{1+\varepsilon}{1-\varepsilon}\|a\|,\ \forall\ a \in A.$$

So $\phi$ is continuous and $\|\phi\| \leq k\frac{1+\varepsilon}{1-\varepsilon}$. Similarly we get similar results if $\phi$ preserves $\varepsilon$-right condition spectrum. □

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