Some classes of Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection

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Abstract

The object of the present paper is to study a quarter-symmetric metric connection in a Lorentzian α-Sasakian manifold. We study some curvature properties of Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. We investigate quasi-projectively flat, φ-symmetric, φ-projectively flat Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection. We also discuss Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S}=0$, where $\tilde{P}$ denote the projective curvature tensor with respect to quarter-symmetric metric connection.

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1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten ([4]). Further, Hayden ([6]), introduced the idea of metric connection with torsion on a Riemannian manifold. In ([13]), Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab ([5]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection ([5]) if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y)=\tilde{\nabla}_X Y-\tilde{\nabla}_Y X-[X,Y] \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y)=\eta(Y)\varphi X-\eta(X)\varphi Y, \tag{1.2}$$

where η is 1-form and φ is a tensor field of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z)=0 \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M^n, then $\tilde{\nabla}$ is said to be quarter-symmetric metric connection. In particular, if $\varphi X = X$ and $\varphi Y = Y$ $\forall X, Y \in \chi(M)$,

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then the quarter-symmetric connection reduces to a semi-symmetric connection ([4]).

In 1980, R. S. Mishra and S. N. Pandey ([7]) studied quarter-symmetric metric connection and in particular, Ricci quarter-symmetric symmetric metric connection on Riemannian, Sasakian and Kaehlerian manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form $(1, 2)$. Studies of various types of quarter-symmetric metric connection and their properties by various authors in (([10], [11]), ([1])) and ([14]) among others.

The notion of locally symmetry of Riemannian manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi ([12]) introduced the notion of locally φ-symmetry on Sasakian manifolds. In the context of contact geometry the notion of φ-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke ([2]) with several examples.

In 2005, Yildiz and Murathan ([15]) studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar ([16]) studied Lorentzian α-Sasakian manifolds. In 2015, Dey and Bhattacharyya [3] studied Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection.

Definition 1.1. *A Lorentzian α-Sasakian manifold M^n is said to be quasi-projectively flat if*

$$g(P(\varphi X, Y)Z, \varphi W) = 0 \tag{1.4}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 1.2. *A Lorentzian α-Sasakian manifold M^n is said to be φ-symmetric if*

$$\varphi^2((\nabla_W R)(X, Y)Z) = 0 \tag{1.5}$$

for arbitrary vector fields X, Y, Z, W.

The Projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each co-ordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor P is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y] \tag{1.6}$$

for $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of the manifold. In fact M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Definition 1.3. *A Lorentzian α-Sasakian manifold M^n is said to φ-projectively flat if*

$$\varphi^2(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{1.7}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P is the projective curvature tensor defined in (1.6).

In the present paper, we study Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Motivated by the above studies in this paper we give the relation between the Levi-Civita connection and the quarter-symmetric metric connection on a Lorentzian α-Sasakian manifold. We characterize quasi-projectively flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. Then we study φ-symmetric Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection. We also study φ-projectively flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. Next we cultivate Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S} = 0$. Finally we give an example of 3-dimensional Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection.

2 Preliminaries

A $n(= 2m+1)$-dimensional differentiable manifold M is said to be a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field φ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \varphi\xi = 0, \eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \varphi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \tag{2.5}$$

$\forall X, Y \in \chi(M)$ and for smooth functions α on M, ∇ denotes the covariant differentiation with respect to Lorentzian metric g ([9], [17]).

For a Lorentzian α-Sasakian manifold, it can be shown that ([9], [17]):

$$\nabla_X \xi = -\alpha\varphi X, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\varphi X, Y) \tag{2.7}$$

for all $X, Y \in \chi(M)$. Further on a Lorentzian α-Sasakian manifold, the following relations hold ([9])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$

$$R(\xi, X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi, X)\xi = \alpha^2[X + \eta(X)\xi], \tag{2.11}$$

$$S(X,\xi) = S(\xi, X) = (n-1)\alpha^2\eta(X), \tag{2.12}$$

$$S(\xi,\xi) = -(n-1)\alpha^2, \tag{2.13}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.14}$$

where Q is the Ricci operator, i.e.

$$g(QX,Y) = S(X,Y), \tag{2.15}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\alpha^2 g(Y,Z). \tag{2.16}$$

If ∇ is the Levi-Civita connection manifold M, then quarter-symmetric metric connection $\tilde{\nabla}$ in M is denoted by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi(X). \tag{2.17}$$

Now we will give the existence of the quarter-symmetric metric connection $\tilde{\nabla}$ on a Lorentzian α -Sasakian manifold M.

Let X, Y, Z be any vectors fields on a Lorentzian α-Sasakian manifold M and let a connection $\tilde{\nabla}$ is given by

$$\begin{aligned} 2g(\tilde{\nabla}_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) \\ &- g([Y,Z],X) + g([Z,X],Y) + g(\eta(Y)\varphi X \\ &- \eta(X)\varphi Y, Z) + g(\eta(X)\varphi Z - \eta(Z)\varphi X, Y) \\ &+ g(\eta(Y)\varphi Z - \eta(Z)\varphi Y, X). \end{aligned} \tag{2.18}$$

Then $\tilde{\nabla}$ is a quarter-symmetric metric connection on M. The proof of this has been discussed on [8].

3 Curvature tensor and Ricci tensor of Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

Let $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and with respect to the Riemannian connection ∇ respectively on a Lorentzian α-Sasakian manifold M. A relation between the curvature tensors $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ on M is given by

$$\begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[g(\varphi X, Z)\varphi Y - g(\varphi Y, Z)\varphi X] \\ &+ \alpha\eta(Z)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.1}$$

Also from (3.1), we obtain

$$\tilde{S}(X,Y) = S(X,Y) + \alpha[g(X,Y) + n\eta(X)\eta(Y)], \tag{3.2}$$

where $\tilde{S}$ and S are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively.
Contracting (3.2), we obtain,

$$\tilde{r} = r, \tag{3.3}$$

where $\tilde{r}$ and r are the scalar curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.
Also we have

$$\tilde{R}(\xi,X)Y = -\tilde{R}(X,\xi)Y = \alpha^2[g(X,Y))\xi - \eta(Y)X] + \alpha\eta(Y)[X + \eta(X)\xi], \tag{3.4}$$

$$\eta(\tilde{R}(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.5}$$

$$\tilde{R}(X,Y)\xi = (\alpha^2 - \alpha)[\eta(Y)X - \eta(X)Y], \tag{3.6}$$

$$\tilde{S}(X,\xi) = \tilde{S}(\xi,X) = (n-1)(\alpha^2 - \alpha)\eta(X), \tag{3.7}$$

$$\tilde{S}(\xi,\xi) = -(n-1)(\alpha^2 - \alpha), \tag{3.8}$$

$$\tilde{Q}X = QX - \alpha(n-1)X, \tag{3.9}$$

$$\tilde{Q}\xi = (n-1)(\alpha^2 - \alpha)\xi, \tag{3.10}$$

$$\tilde{R}(\xi,X)\xi = (\alpha^2 - \alpha)[X + \eta(X)\xi], \tag{3.11}$$

4 Quasi-projectively flat Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

The projective curvature tensor $\tilde{P}$ with respect to quarter-symmetric metric connection is defined by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \tag{4.1}$$

for $X, Y, Z \in \chi(M)$, where $\tilde{S}$ is the Ricci tensor of the manifold with respect to quarter-symmetric metric connection.

A Lorentzian α-Sasakian manifold M^n is said to be quasi-projectively flat with respect to the quarter-symmetric metric connection if

$$g(\tilde{P}(\varphi X, Y)Z, \varphi W) = 0 \tag{4.2}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

In view of the equation (4.1), we have

$$\begin{aligned} g(\tilde{P}(X,Y)Z,W) &= g(\tilde{R}(X,Y)Z,W) - \frac{1}{n-1}[\tilde{S}(Y,Z)g(X,W) \\ &- \tilde{S}(X,Z)g(Y,W)]. \end{aligned} \tag{4.3}$$

Putting $X = \varphi X$ and $W = \varphi W$ in the above equation, we get

$$\begin{aligned} g(\tilde{P}(\varphi X,Y)Z,\varphi W) &= g(\tilde{R}(\varphi X,Y)Z,\varphi W) - \frac{1}{n-1}[\tilde{S}(Y,Z)g(\varphi X,\varphi W) \\ &- \tilde{S}(\varphi X,Z)g(Y,\varphi W)]. \end{aligned} \tag{4.4}$$

Now assume that M^n is quasi-projectively flat with respect to quarter-symmetric metric connection. Then by virtue of equations (4.2) and (4.4), we have

$$g(\tilde{R}(\varphi X,Y)Z,\varphi W) = \frac{1}{n-1}[\tilde{S}(Y,Z)g(\varphi X,\varphi W) - \tilde{S}(\varphi X,Z)g(Y,\varphi W)], \tag{4.5}$$

Using equations (3.1) and (3.2) in above equation, we get

$$\begin{aligned} g(R(\varphi X,Y)Z,\varphi W) &= -\alpha[g(X,Z)g(Y,W) + g(X,Z)\eta(Y)\eta(W) \\ &+ g(Y,W)\eta(X)\eta(Z) + \eta(X)\eta(Y)\eta(Z)\eta(W) \\ &- g(\varphi Y,Z)g(\varphi W,X)] - \alpha g(X,W)\eta(Y)\eta(Z) \\ &- \alpha\eta(X)\eta(Y)\eta(Z)\eta(W) + \frac{1}{n-1}[S(Y,Z)g(\varphi X,\varphi W) \\ &+ \alpha\{g(Y,Z) + n\eta(Y)\eta(Z)\}g(\varphi X,\varphi W) \\ &- S(\varphi X,Z)g(\varphi W,Y) - \alpha g(\varphi X,Z)g(\varphi W,Y)]. \end{aligned} \tag{4.6}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis of M^n. Putting $X = W = e_i$ in (4.6) and taking summation over i, $1 \le i \le n-1$, we get

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) &= -\alpha \sum_{i=1}^{n-1} [g(e_i, Z)g(Y, e_i) + g(e_i, Z)\eta(Y)\eta(e_i) \\ &\quad + g(Y, e_i)\eta(e_i)\eta(Z) + \eta(e_i)\eta(Y)\eta(Z)\eta(e_i) \\ &\quad - g(\varphi Y, Z)g(\varphi e_i, e_i)] - \sum_{i=1}^{n-1} [\alpha g(e_i, e_i)\eta(Y)\eta(Z) \\ &\quad - \alpha\eta(e_i)\eta(Y)\eta(Z)\eta(e_i)] + \frac{1}{n-1} \sum_{i=1}^{n-1} [S(Y, Z) \\ &\quad g(\varphi e_i, \varphi e_i) + \alpha\{g(Y, Z) + n\eta(Y)\eta(Z)\}g(\varphi e_i, \varphi e_i) \\ &\quad - S(\varphi e_i, Z)g(\varphi e_i, Y) - \alpha g(\varphi e_i, Z)g(\varphi e_i, Y)]. \end{aligned} \tag{4.7}$$

Also,

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) = S(Y, Z) + g(Y, Z), \tag{4.8}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, Z)g(\varphi e_i, Y) = S(Y, Z), \tag{4.9}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = n - 1 \tag{4.10}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i, Y)g(\varphi e_i, Z) = g(Y, Z). \tag{4.11}$$

Hence, by virtue of equations (4.8), (4.9), (4.10) and (4.11), the equation (4.7) takes the form

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + c\Phi(Y, Z), \tag{4.12}$$

where $a = (1-n)(\alpha^2 + 1), b = (n-1)\alpha, c = \alpha(n-1)\beta$ and

$$\beta = trace\Phi, \Phi(Y, Z) = g(\varphi Y, Z).$$

Hence we can state the following

Theorem 4.1. *A quasi-projectively flat Lorentzian α-Sasakian manifold with respect to a quarter-symmetric metric connection is an generalized η-Einstein manifold.*

5 φ-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n is said to be φ-symmetric with respect to the quarter-symmetric metric connection if

$$\varphi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0 \tag{5.1}$$

for arbitrary vector fields X, Y, Z, W.

From the equation (2.17) and (3.1), we have

$$(\tilde{\nabla}_W \tilde{R})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\varphi W. \tag{5.2}$$

Let us consider a φ-symmetric Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection. Then by virtue of (2.1) and (5.1) we have

$$((\tilde{\nabla}_W \tilde{R})(X,Y)Z) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi = 0. \tag{5.3}$$

From which it follows that

$$g((\tilde{\nabla}_W \tilde{R})(X,Y)Z,U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)g(\xi,U) = 0. \tag{5.4}$$

Let $\{e_i\}$, $i = 1, 2..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.4) and taking summation over i, $1 \leq i \leq n$, we have

$$(\tilde{\nabla}_W \tilde{S})(Y,Z) + \Sigma_{i=1}^{n} \eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)Z)\eta(e_i) = 0. \tag{5.5}$$

The second term of (5.5) by putting $Z = \xi$ takes the form

$$\eta((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi)\eta(e_i) = g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{5.6}$$

By using (2.16) and (5.2), we can write

$$g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) + \eta(\tilde{R}(e_i,Y)\xi)\varphi W. \tag{5.7}$$

After some calculations, from (5.7) we have

$$g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi). \tag{5.8}$$

In Lorentzian α-Sasakian manifold, we have

$$g((\nabla_W \tilde{R})(e_i,Y)\xi,\xi) = 0.$$

So from (5.8) we get

$$g((\tilde{\nabla}_W \tilde{R})(e_i,Y)\xi,\xi) = 0. \tag{5.9}$$

By replacing $Z = \xi$ in (5.5) and using (5.9), we get

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0. \tag{5.10}$$

We know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \tag{5.11}$$

Now using (2.6), (2.12), (2.17) and (3.7), we obtain

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y, \xi) &= -(n-1)(\alpha^2 - \alpha)\alpha g(Y, \varphi W) + (\alpha + 1)[S(Y, \varphi W) \\ &+ \alpha g(Y, \varphi W)].\end{aligned} \tag{5.12}$$

Applying (5.11) in (5.10), we obtain

$$S(Y, \varphi W) = g(Y, \varphi W)[(n-1)\alpha^2 \frac{\alpha - 1}{\alpha + 1} - \alpha]. \tag{5.13}$$

Replacing W by φW we get

$$\begin{aligned}S(Y, W) &= g(Y, W)[(n-1)\alpha^2 \frac{\alpha - 1}{\alpha + 1} - \alpha] \\ &- \frac{\alpha^3(n+1) + 3n\alpha^2 + \alpha}{\alpha + 1}\eta(Y)\eta(W).\end{aligned} \tag{5.14}$$

Contracting (5.14), we get

$$r = n[(n-1)\alpha^2 \frac{\alpha - 1}{\alpha + 1} - \alpha] + \frac{\alpha^3(n+1) + 3n\alpha^2 + \alpha}{\alpha + 1}. \tag{5.15}$$

This leads to the following theorem:

Theorem 5.1. *Let M be a φ-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold has a scalar curvature $r = n[(n-1)\alpha^2 \frac{\alpha-1}{\alpha+1} - \alpha] + \frac{\alpha^3(n+1)+3n\alpha^2+\alpha}{\alpha+1}$ with respect to Levi-Civita connection ∇ of M provided $\alpha \neq -1$.*

6 φ-projectively flat Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n is said to φ-projectively flat with respect to the quarter-symmetric metric connection if

$$\varphi^2(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\tilde{P}$ is the projective curvature tensor defined in (4.1).

Let M^n be a φ-projectively flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. It is easy to see that $\varphi^2(\tilde{P}(\varphi X, \varphi Y)\varphi Z,$
$\varphi W) = 0$ holds if and only if

$$g(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0 \tag{6.2}$$

for any $X, Y, Z, W \in \chi(M)$.

Putting $Y = \varphi Y$ and $Z = \varphi Z$ in (4.4), we get

$$\begin{aligned} g(\tilde{P}(\varphi X, \varphi Y)\varphi Z, \varphi W) &= g(\tilde{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) - \frac{1}{n-1}[\tilde{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- \tilde{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{6.3}$$

Using the equation (6.3) in the equation (6.2), we obtain

$$\begin{aligned} g(\tilde{R}(\varphi X, \varphi Y)\varphi Z, \varphi W) &= \frac{1}{n-1}[\tilde{S}(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- \tilde{S}(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{6.4}$$

Now using the equation (3.1) and (3.2), we obtain

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= -\alpha[g(X, \varphi Z)g(Y, \varphi W) - g(Y, \varphi Z)g(X, \varphi W)] \\ &+ \frac{1}{n-1}[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) + \alpha g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &- S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) - \alpha g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{6.5}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Then $\{\varphi e_1, \varphi e_2, ..., \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis of M_n. Putting $X = W = e_i$ in (6.5) and taking summation over i, $1 \leq i \leq n-1$, we get

$$\begin{aligned} \sum_{i=1}^{n-1} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) &= -\alpha \sum_{i=1}^{n-1}[g(X, \varphi Z)g(Y, \varphi W) - g(Y, \varphi Z)g(X, \varphi W)] \\ &+ \frac{1}{n-1}\sum_{i=1}^{n-1}[S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) \\ &+ \alpha g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) \\ &- \alpha g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \end{aligned} \tag{6.6}$$

Hence, by virtue of equations (4.8), (4.9), (4.10) and (4.11), the equation (6.6) takes the form

$$S(\varphi Y, \varphi Z) = g(\varphi Y, \varphi Z)[\alpha - 1 - \frac{\alpha}{n-1}] + \alpha(\beta - 1)\Phi(Y, Z), \tag{6.7}$$

where $\beta = trace\Phi, \Phi(Y, Z) = g(\varphi Y, Z)$. Using the equation (2.3) and (2.16), we obtain

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z) + c\Phi(Y, Z), \tag{6.8}$$

where $a = [\alpha - 1 - \frac{\alpha}{n-1} - (n-1)\alpha^2]$, $b = [\alpha - 1 - \frac{\alpha}{n-1}]$ and $c = \alpha(\beta - 1)$. Hence we can state the following

Theorem 6.1. *A φ-projectively flat Lorentzian α-Sasakian manifold with respect to a quarter-symmetric metric connection is an generalized η-Einstein manifold.*

7 Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S} = 0$

A Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection satisfying

$$(\tilde{P}(X,Y).\tilde{S})(Z,U) = 0, \tag{7.1}$$

where $\tilde{S}$ is the Ricci tensor with respect to a quarter-symmetric metric connection. Then, we have

$$\tilde{S}(\tilde{P}(X,Y)Z,U) + \tilde{S}(Z,\tilde{P}(X,Y)U) = 0. \tag{7.2}$$

Putting $X = \xi$ in the equation (7.2), we have

$$\tilde{S}(\tilde{P}(\xi,Y)Z,U) + \tilde{S}(Z,\tilde{P}(\xi,Y)U) = 0. \tag{7.3}$$

In view of the equation (4.1), we have

$$\tilde{P}(\xi,Y)Z = \tilde{R}(\xi,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)\xi - \tilde{S}(\xi,Z)Y] \tag{7.4}$$

for $X, Y, Z \in \chi(M)$.
Using equations (3.4) and (3.7) in the equation (7.4), we get

$$\begin{aligned} \tilde{P}(\xi,Y)Z &= \alpha^2[g(Y,Z)\xi - \eta(Z)Y] + \alpha\eta(Z)[Y + \eta(Y)\xi] \\ &- \frac{1}{n-1}[S(Y,Z)\xi + \alpha\{g(Y,Z) + n\eta(Y)\eta(Z)\}\xi \\ &- (\alpha^2 - \alpha)(n-1)\eta(Z)Y]. \end{aligned} \tag{7.5}$$

Now using the equation (7.5) and putting $U = \xi$ in the equation (7.3) and using the equations (2.2), (2.12), we get

$$\begin{aligned} S(Y,Z)[\alpha^2 &+ \frac{\alpha}{n-1} - \frac{\alpha n}{n-1}] + g(Y,Z)[-\alpha^4(n-1) + \frac{\alpha^2}{n-1} + n\alpha^3 - \frac{\alpha^2 n}{n-1}] \\ &+ \eta(Y)\eta(Z)[(\alpha^2 - \alpha)(n-1)\alpha^2 + \alpha(\alpha^2 - \alpha^2) \\ &+ 2\alpha^3 n - n\alpha^4] = 0. \end{aligned} \tag{7.6}$$

This gives

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z), \tag{7.7}$$

where $a = \alpha[n\alpha - (\alpha + 1)]$ and $b = 2\alpha^2 + \frac{n\alpha^2}{n-1}$. In view of above discussions we can state the following theorem:

Theorem 7.1. *A n-dimensional Lorentzian α-Sasakian manifold with a quarter-symmetric metric connection satisfying $\tilde{P}.\tilde{S} = 0$. is an η-Einstein manifold.*

8 Example of 3-dimensional Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

We consider a 3-dimensional manifold $M = \{(x, y, u) \in R^3\}$, where (x, y, u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = e^u \frac{\partial}{\partial x},\ e_2 = e^u \frac{\partial}{\partial y},\ e_3 = e^u \frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1,1)$ tensor field φ is defined by

$$\varphi e_1 = -e_1,\ \varphi e_2 = -e_2,\ \varphi e_3 = 0.$$

From the linearity of φ and g, we have

$$\eta(e_3) = -1,$$
$$\varphi^2 X = X + \eta(X)e_3$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (φ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = e_1 e^{-u},\ [e_2, e_3] = e_2 e^{-u}.$$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

Then from above formula we can calculate the followings:

$$\nabla_{e_1}e_1=-e_3e^u,\ \nabla_{e_1}e_2=0,\ \nabla_{e_1}e_3=-e_1e^u,$$
$$\nabla_{e_2}e_1=0,\ \nabla_{e_2}e_2=-e_3e^u,\ \nabla_{e_2}e_3=-e_2e^u,$$
$$\nabla_{e_3}e_1=0,\ \nabla_{e_3}e_2=0,\ \nabla_{e_3}e_3=0.$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi)=-1$ and $\nabla_X\xi=-\alpha\varphi X$ for $\alpha=-e^u$.

Hence the structure (φ,ξ,η,g) is a Lorentzian α-Sasakian structure.

Using (2.17), we find $\tilde{\nabla}$, the quarter-symmetric metric connection on M following:

$$\tilde{\nabla}_{e_1}e_1=-e_3e^u,\ \tilde{\nabla}_{e_1}e_2=0,\ \tilde{\nabla}_{e_1}e_3=e_1(1-e^u),$$
$$\tilde{\nabla}_{e_2}e_1=0,\ \tilde{\nabla}_{e_2}e_2=-e_3e^u,\ \tilde{\nabla}_{e_2}e_3=e_2(1-e^u),$$
$$\tilde{\nabla}_{e_3}e_1=0,\ \tilde{\nabla}_{e_3}e_2=0,\ \tilde{\nabla}_{e_3}e_3=0.$$

Using (1.2), the torson tensor T, with respect to quarter-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i,e_i)=0,\forall i=1,2,3,$$
$$\tilde{T}(e_1,e_2)=0,\ \tilde{T}(e_1,e_3)=e_1,\ \tilde{T}(e_2,e_3)=e_2.$$

Also,

$$(\tilde{\nabla}_{e_1}g)(e_2,e_3)=0,\ (\tilde{\nabla}_{e_2}g)(e_3,e_1)=0,\ (\tilde{\nabla}_{e_3}g)(e_1,e_2)=0.$$

Thus M is Lorentzian α-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$.

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1,e_3)e_3=-e_1\alpha^2,\ R(e_2,e_1)e_1=e_2\alpha^2,\ R(e_2,e_3)e_3=-e_2\alpha^2,$$
$$R(e_3,e_1)e_1=e_3\alpha^2,\ R(e_3,e_2)e_2=e_3\alpha^2,\ R(e_1,e_2)e_3=0,$$
$$R(e_2,e_3)e_2=-e_3\alpha^2,\ R(e_1,e_2)e_2=e_1\alpha^2$$

and

$$\tilde{R}(e_1,e_3)e_3=e_1(\alpha-\alpha^2),\ \tilde{R}(e_2,e_1)e_1=e_2(\alpha^2-\alpha),$$
$$\tilde{R}(e_2,e_3)e_3=e_2(\alpha-\alpha^2),\ \tilde{R}(e_3,e_1)e_1=e_3\alpha^2,$$
$$\tilde{R}(e_3,e_2)e_2=e_3\alpha^2,\ \tilde{R}(e_1,e_2)e_3=0,\ \tilde{R}(e_2,e_3)e_2=-e_3\alpha^2,$$

$$\tilde{R}(e_1, e_2)e_2 = e_1(\alpha^2 - \alpha).$$

Using the expressions of the curvature tensors, we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 0,\ S(e_2, e_2) = 0,\ S(e_3, e_3) = -2\alpha^2,$$
$$S(e_1, e_2) = 0,\ S(e_2, e_3) = 0,\ S(e_1, e_3) = 0$$

and

$$\tilde{S}(e_1, e_1) = \alpha,\ \tilde{S}(e_2, e_2) = \alpha,\ \tilde{S}(e_3, e_3) = 2(\alpha - \alpha^2),$$
$$\tilde{S}(e_1, e_2) = 0,\ \tilde{S}(e_2, e_3) = 0,\ \tilde{S}(e_1, e_3) = 0.$$

Let $\Phi(X, Y)$ be a $(0, 2)$ tensor. Then we have,

$$\Phi(X, Y) = g(X, \varphi Y).$$

Then

$$\Phi(e_1, e_1) = g(e_1, \varphi e_1) = g(e_1, -e_1) = -1,$$
$$\Phi(e_2, e_2) = g(e_2, \varphi e_2) = g(e_2, -e_2) = -1,$$

and

$$\Phi(e_3, e_3) = g(e_3, \varphi e_3) = g(e_1, 0) = 0.$$

If we take the scalars a_1, b_1 as follows:

$$a_1 = \alpha \textit{ and } b_1 = 3\alpha - 2\alpha^2,$$

then we have

$$\tilde{S}(e_1, e_1) = a_1 g(e_1, e_1) + b_1 \eta(e_1)\eta(e_1),$$
$$\tilde{S}(e_2, e_2) = a_1 g(e_2, e_2) + b_1 \eta(e_2)\eta(e_2),$$
$$\tilde{S}(e_3, e_3) = a_1 g(e_3, e_3) + b_1 \eta(e_3)\eta(e_3).$$

Thus the manifold under consideration is a η-Einstein Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Again, if we take another scalars

$$a_2 = \alpha + 1,\ b_2 = 3\alpha - 2\alpha^2 + 1,\ f = 1,$$

then we have

$$\tilde{S}(e_1, e_1) = a_2 g(e_1, e_1) + b_2 \eta(e_1)\eta(e_1) + f\Phi(e_1, e_1),$$

$$\tilde{S}(e_2, e_2) = a_2 g(e_2, e_2) + b_2 \eta(e_2)\eta(e_2) + f\Phi(e_2, e_2),$$
$$\tilde{S}(e_3, e_3) = a_2 g(e_3, e_3) + b_2 \eta(e_3)\eta(e_3) + f\Phi(e_3, e_3).$$

So, the manifold becomes generalized η-Einstein Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.

References

[1] C. S. Bagewadi, D. G. Prakasha and Venkatesha., *A Study of Ricci quarter-symmetric metric connection on a Riemannian manifold,* Indian J. Math., 50 (2008), no. 3, 607-615.

[2] E. Boeckx, P. Buecken, and L. Vanhecke, *φ-symmetric contact metric spaces,* Glasgow Math.J., 41 (1999), 409-416.

[3] S. Dey and A. Bhattacharyya, *Some Properties of Lorentzian a-Sasakian Manifolds with Respect to Quarter-symmetric Metric Connection,* Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 54 (2015), No. 2, 21-40.

[4] A. Friedmann and J. A. Schouten. *Uber die Geometrie der halbsymmetrischen Uber-tragung.* Math. Zeitschr, 21:211-223, 1924.

[5] S. Golab. *On semi-symmetric and quarter-symmetric linear connections,* Tensor, N.S. 29 (1975) 249-254.

[6] H. A. Hayden. *Subspaces of a space with torsion.* Proc. London Math. Soc., 34:27-50, 1932.

[7] R. S. Mishra and S. N. Pandey, *On quarter-symmetric metric F-connection,* Tensor, N.S. 34 (1980) 1-7.

[8] C. Patra and A. Bhattacharyya, *Quarter-symmetric metric connection on pseudosymmetric Lorentzian α-Sasakian manifolds.* International J. Math. Combin. Vol.1 (2013), 46-59.

[9] D. G. Prakasha, C. S. Bagewadi and N. S. Basavarajappa, *On pseudosymmetric Lorentzian α-Sasakian manifolds,* IJPAM, Vol. 48, No.1, 2008, 57-65.

[10] S. C. Rastogi, *On quarter-symmetric connection*, C.R. Acad. Sci. Bulgar 31 (1978) 811-814.

[11] S. C. Rastogi, *On quarter-symmetric metric connection,* Tensor 44 (1987) 133-141.

[12] T. Takahashi, *Sasakian φ-symmetric spaces,* Tohoku Math. J. 29 (1977), 91-113.

[13] K. Yano. *On semi-symmetric metric connections.* Rev. Roumaine Math. Pures Appl., 15:1579-1586, 1970.

[14] K. Yano and T. Imai, *Quarter-symmetric metric connections and their curvature tensors,* Tensor, N.S. 38 (1982) 13-18.

[15] A. Yildiz and C. Murathan, *On Lorentzian α-Sasakian manifolds.* Kyungpook Math. J. 45 (2005), 95-103.

[16] S. Yadav and D. L. Suthar, *Certain derivation on Lorentzian α-Sasakian manifolds.* Mathematics and Decision Science 12 (2012).

[17] A. Yildiz, M. Turan and B. F. Acet, *On three dimensional Lorentzian α-Sasakian manifolds,* Bulletin of Mathematical Analysis and Applications, Vol. 1, Issue 3(2009), pp. 90-98.