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Brauer groups of Châtelet surfaces over local fields

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Abstract. A Châtelet surface over a field is a typical geometrically rational surface. Its Chow group of zero-cycles has been studied as an important birational invariant by many researchers since the 1970s. Recently, S. Saito and K. Sato obtained a duality between the Chow and Brauer groups from the Brauer–Manin pairing. For a Châtelet surface over a local field, we combine their result with the known calculation of the Chow group to determine the structure and generators of the Brauer group of a regular proper flat model of the surface over the integer ring of the base field.

Key words: Brauer groups, Chow groups, Châtelet surfaces, Local fields.

1. Introduction

Let p be a prime number, and K be a p-adic field (i.e. a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers). Let X be a smooth projective geometrically rational surface over K. Denote by $\mathrm{A}_0(X)$ the degree-zero part of the Chow group of zero-cycles on X modulo rational equivalence, and by $\mathrm{Br}\,(X)$ the cohomological Brauer group of X. They are birational invariants. Let $\mathfrak{O}$ be the integer ring of K. Fix a regular proper flat model $\mathscr{X}$ of X over $\mathfrak{O}$. Saito and Sato [16] recently proved that the Brauer–Manin pairing

$$\langle\ ,\ \rangle : \mathrm{A}_0(X) \times \mathrm{Br}\,(X)/\mathrm{Br}\,(K) \to \mathbb{Q}/\mathbb{Z} \tag{1.1}$$

induces the isomorphism

$$\mathrm{A}_0(X)^\vee \cong \mathrm{Br}\,(X)/(\mathrm{Br}\,(\mathscr{X}) + \mathrm{Br}\,(K)), \tag{1.2}$$

where the quotients denote the cokernels of the natural maps, and $\vee$ denotes the Pontryagin dual.

For a diagonal cubic surface X over K, a number of studies have been conducted on $\mathrm{Br}\,(X)$. Several successful examples in determining $\mathrm{Br}\,(\mathscr{X})$

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were found in the recent studies [16], [17], [18]. By combining with (1.2), they also succeeded in determining $\mathrm{A}_0(X)$ in such cases.

In this article, we will perform a similar argument for a cubic Châtelet surface X over K in the 'opposite' direction. That is, by applying the known results on $\mathrm{A}_0(X)$ to (1.2), we will determine the group structure and generators of $\mathrm{Br}(\mathscr{X})$, which were unknown yet. A *cubic Châtelet surface* $X_0 \subset \mathbb{A}^3_K$ is defined by

$$y^2 - dz^2 = f(x), \tag{1.3}$$

where $d \in K^\times$, and $f(x) \in K[x]$ is a monic cubic separable polynomial. From now on, let X be a smooth projective model of X_0 over K (see Proposition 2.4). We refer to X as a cubic Châtelet surface. By using the left-perfectness of the pairing (1.1) (see [1], [5], [6, Proposition 5]), the group structure of $\mathrm{A}_0(X)$ was determined in many cases by many researchers (see Theorem 1.1).

Denote by $K^{\times 2}$ the group of squares in $K^\times$. Let $v : K^\times \to \mathbb{Z}$ be the normalized valuation of K. Let us consider the following three cases under the assumption $d \notin K^{\times 2}$:

(i) $f(x)$ is irreducible over K.
(ii) $f(x) = x(x^2 - e)$ with $e \in K^\times$ and $d \not\equiv e \not\equiv 1 \pmod{K^{\times 2}}$.
(iii) $f(x) = x(x - e_1)(x - e_2)$ with $e_i \in K^\times$, $e_1 \neq e_2$ and $v(e_1) = v(e_2) =: r$.

Denote by $U^{(i)}$ the i-th unit group of K. Put $L = K(\sqrt{d})$. Let s be the conductor of L/K defined as the minimum integer $i > 0$ such that $U^{(i)} \subset N_{L/K}L^\times$. Put $s' = 2s - 1$ if $p = 2$, and $s' = 1$ if $p \neq 2$. The following results are known:

Theorem 1.1 *Let X be a cubic Châtelet surface over a p-adic field K.*

[14, Theorem 1.4]: *In the case* (i), *we have* $\mathrm{A}_0(X) \cong \{0\}$.

Suppose that we are in the case (ii).

(ii-a) [14, Theorem 1.2]: *If $p \neq 2$, then* $\mathrm{A}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$.
(ii-b) [12, Theorem 2 (1)], [14, Theorem 1.3 (1)]: *If $p = 2$ and L/K is unramified, then*

$$\mathrm{A}_0(X) \cong \begin{cases} \{0\} & \text{if } v(e) \equiv 0 \pmod 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v(e) \not\equiv 0 \pmod 4. \end{cases}$$

(ii-c) [14, Theorem 1.3 (2)]: *If* $K = \mathbb{Q}_2$ *and* L/K *is ramified, then* $\mathrm{A}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii-d) [12, Theorem 3]: *If* $K = \mathbb{Q}_2(\sqrt{2})$ *and* L/K *is ramified, then* $\mathrm{A}_0(X)$ *is isomorphic to either* $\{0\}$ *or* $\mathbb{Z}/2\mathbb{Z}$. *We can explicitly determine which group occurs by the values of* d *and* e *when* $v(d)$ *is even.*

Suppose that we are in the case (iii).

(iii-a) [7, Proposition 4.7], [8, Proposition 1]: *If* L/K *is unramified, then*

$$\mathrm{A}_0(X) \cong \begin{cases} \{0\} & \text{if } r \in 2\mathbb{Z} \text{ and } v(e_1 - e_2) = r, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } r \in 2\mathbb{Z} \text{ and } v(e_1 - e_2) > r, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{if } r \notin 2\mathbb{Z}. \end{cases}$$

(iii-b) [8, Proposition 2], [9, Proposition 3]: *If* L/K *is ramified, then*

$$\mathrm{A}_0(X) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } e_1/e_2 \in U^{(s')} \text{ and } e_1 \in N_{L/K}L^\times, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2} & \text{otherwise.} \end{cases}$$

(ii-b) is stated for $K = \mathbb{Q}_2$ by Pisolkar [14, Theorem 1.3 (1)], but her proof works under the assumption $p = 2$ when $v(e) \not\equiv 2 \pmod 4$. The case when $v(e) \equiv 2 \pmod 4$ is proven by the author [12, Theorem 2 (1)].

Let F be the function field of X. We regard $\mathrm{Br}(X)$ and $\mathrm{Br}(\mathscr{X})$ as subgroups of $\mathrm{Br}(F)$ by the natural injections. For any $a, b \in F^\times$, denote by $\{a, b\}_F$ the Brauer equivalence class of the quaternion algebra $\left(\frac{a,b}{F}\right)$ over F.

Theorem 1.2 *Let* X *be a cubic Châtelet surface over a* p*-adic field* K. *Let* $\mathscr{X}$ *be a regular proper flat model of* X *over the integer ring* $\mathfrak{O}$ *of* K.

In the case (i), *we have* $\mathrm{Br}(\mathscr{X}) = \{0\}$.

Suppose that we are in the case (ii). *Put* $B_0 = \{x, d\}_F$.

(ii-a) *If* $p \neq 2$, *then* $\mathrm{Br}(\mathscr{X}) = \{0\}$.

(ii-b) *If* $p = 2$ *and* L/K *is unramified, then*

$$\mathrm{Br}(\mathscr{X}) = \begin{cases} B_0 \cdot \mathbb{Z}/2\mathbb{Z} & \text{if } v(e) \equiv 0 \pmod 4, \\ \{0\} & \text{if } v(e) \not\equiv 0 \pmod 4. \end{cases}$$

(ii-c) *If* $K = \mathbb{Q}_2$ *and* L/K *is ramified, then* $\mathrm{Br}(\mathscr{X}) = \{0\}$.

(ii-d) *If* $K = \mathbb{Q}_2(\sqrt{2})$ *and* L/K *is ramified, then* $\mathrm{Br}\,(\mathscr{X})$ *is isomorphic to either* $\mathbb{Z}/2\mathbb{Z}$ *or* $\{0\}$. *We can explicitly determine which group occurs by the values of* d *and* e *when* $v(d)$ *is even.*

Suppose that we are in the case (iii). *Put* $B_0 = \{x, d\}_F$ *and* $B_1 = \{x - e_1, d\}_F$.

(iii-a) *If* L/K *is unramified, then*

$$\mathrm{Br}\,(\mathscr{X}) = \begin{cases} B_0 \cdot \mathbb{Z}/2\mathbb{Z} \oplus B_1 \cdot \mathbb{Z}/2\mathbb{Z} & \text{if } r \in 2\mathbb{Z} \text{ and } v(e_1 - e_2) = r, \\ B_0 \cdot \mathbb{Z}/2\mathbb{Z} & \text{if } r \in 2\mathbb{Z} \text{ and } v(e_1 - e_2) > r, \\ \{0\} & \text{if } r \notin 2\mathbb{Z}. \end{cases}$$

(iii-b) *If* L/K *is ramified, then*

$$\mathrm{Br}\,(\mathscr{X}) = \begin{cases} B_0 \cdot \mathbb{Z}/2\mathbb{Z} & \text{if } e_1/e_2 \in U^{(s')} \text{ and } e_1 \in N_{L/K}L^\times, \\ \{0\} & \text{otherwise.} \end{cases}$$

Remark 1.3 (1) As we will see in Section 4, the group $\mathrm{Br}\,(\mathscr{X})$ is isomorphic to the kernel of the homomorphism $\mathrm{Br}\,(X)/\mathrm{Br}\,(K) \to \mathrm{A}_0(X)^\vee$ induced by the pairing (1.1). This implies that the group structure of $\mathrm{Br}\,(\mathscr{X})$ depends only on the birational equivalence class of X over K, since $\mathrm{A}_0(X)$ and $\mathrm{Br}\,(X)/\mathrm{Br}\,(K)$ are birational invariants. The above results do not depend on the choice of $\mathscr{X}$.

(2) If $d \in K^{\times 2}$, then X is birational to $\mathbb{P}^2_K$, and therefore we have $\mathrm{Br}\,(X) \cong \mathrm{Br}\,(\mathbb{P}^2_K) \cong \mathrm{Br}\,(K)$, which implies $\mathrm{Br}\,(\mathscr{X}) = \{0\}$ by the fact mentioned above.

(3) The computation of $\mathrm{Br}\,(\mathscr{X})$ in the case when $f(x)$ splits into a product $(x - e_0)(x - e_1)(x - e_2)$ of three linear factors over K can be reduced to the case (iii) by the above remark. Indeed, a permutation of e_i's and the translation $x - e_0 \mapsto x$ enable this reduction, since

$$\begin{aligned} & v(e_1 - e_0) > v(e_2 - e_0) \\ & \quad \Longrightarrow v(e_1 - e_2) = \min\{v(e_1 - e_0), v(e_0 - e_2)\} = v(e_0 - e_2). \end{aligned}$$

However, the case when $f(x)$ has only one linear factor over K cannot be reduced to the case (ii) in general. Remark that the cases (ii) and

(iii) are mutually exclusive.

2. Brauer group $\mathbf{Br}\,(X)/\mathbf{Br}\,(K)$

To prove Theorem 1.2, in Section 4 we will use the following fact:

Theorem 2.1 *Let X be a cubic Châtelet surface over a p-adic field K. With the same notation as in Theorem* 1.2*, put $\bar{B}_i = B_i \bmod \mathrm{Br}\,(K)$ in each case of* (ii) *and* (iii)*. Then we have*

$$\mathrm{Br}\,(X)/\mathrm{Br}\,(K) = \begin{cases} \{0\} & \textit{in the case (i)}, \\ \bar{B}_0 \cdot \mathbb{Z}/2\mathbb{Z} & \textit{in the case (ii)}, \\ \bar{B}_0 \cdot \mathbb{Z}/2\mathbb{Z} \oplus \bar{B}_1 \cdot \mathbb{Z}/2\mathbb{Z} & \textit{in the case (iii)}. \end{cases}$$

Remark 2.2 In the case (ii), the structure of the group $\mathrm{H}^1(K, \mathrm{Hom}\,(\mathrm{Pic}\,(\bar{X}), \bar{K}^\times))$ whose Pontryagin dual is isomorphic to $\mathrm{Br}\,(X)/\mathrm{Br}\,(K)$ is claimed in [12], [14] without proof, but the statement is not correct. Theorem 2.1 corrects these errors. Fortunately, the proofs of their main results in loc. cit. remain valid, since the criterion of the structure of $\mathrm{A}_0(X)$ in the case (ii) used there is not affected by the correction. The result for the case (i) [14, Theorem 1.4] follows correctly from that the target of the natural injection $\mathrm{A}_0(X) \to (\mathrm{Br}\,(X)/\mathrm{Br}\,(K))^\vee$ is trivial.

The proof of Theorem 2.1 will be completed in Section 2.2.

2.1. Group structure of $\mathbf{Br}\,(X)/\mathbf{Br}\,(K)$

It follows from the following two propositions that the both sides of the equation in Theorem 2.1 are isomorphic as abelian groups. For a given surface X over a field K, let $\bar{X}$ be the base change of X to an algebraic closure $\bar{K}$ of K.

Proposition 2.3 (see [2, Lemma 7.5]) *Let X be a smooth projective geometrically rational surface over a local field K. Then we have*

$$\mathrm{Br}\,(X)/\mathrm{Br}\,(K) \cong \mathrm{H}^1(K, \mathrm{Pic}\,(\bar{X})).$$

Proposition 2.4 (see [15, Proposition 7.1.1]) *Let K be a perfect field of characteristic $\neq 2$, and c, $d \in K^\times$. Let $f(x) \in K[x]$ be a monic separable polynomial of degree $n \geq 1$. Put $n' = n$ if n is even, and $n' = n + 1$ if n is* *odd. Define a smooth projective surface X by gluing the two surfaces*

$$X_1 : {Y_1}^2 - d{Z_1}^2 = cf(x_1){W_1}^2$$

in $\mathrm{Spec}\,(K[x_1]) \times \mathrm{Proj}\,(K[Y_1, Z_1, W_1])$ *and*

$$X_2 : {Y_2}^2 - d{Z_2}^2 = c{x_2}^{n'} f({x_2}^{-1}){W_2}^2$$

in $\mathrm{Spec}\,(K[x_2]) \times \mathrm{Proj}\,(K[Y_2, Z_2, W_2])$ *along the isomorphism*

$$X_1 \cap \{x_1 \neq 0\} \to X_2 \cap \{x_2 \neq 0\};$$
$$(x_1, (Y_1 : Z_1 : W_1)) \mapsto ({x_1}^{-1}, (Y_1 : Z_1 : {x_1}^{n'/2} W_1)).$$

Put l the number of irreducible factors of $f(x)$ over K, and l_1 the number of those of odd degree. If $d \notin K^{\times 2}$ and $f(\sqrt{d}) \neq 0$, then

$$\mathrm{H}^1(K, \mathrm{Pic}\,(\bar{X})) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^{\oplus(l-1)} & \text{if } l_1 = 0, \text{ or if } l_1 \notin 2\mathbb{Z}, \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus(l-2)} & \text{if } l_1 > 0 \text{ and } l_1 \in 2\mathbb{Z}. \end{cases}$$

Proposition 2.4 is stated only for even n by Skorobogatov in loc. cit., but his proof works independently of the parity of n. The above statement is an explicit version of his original statement.

2.2. Generators of $\mathrm{Br}\,(X)/\mathrm{Br}\,(K)$

In this subsection, we will finish the proof of Theorem 2.1. From now on, we take a smooth projective model X of X_0 as in Proposition 2.4 with $c = 1$ and $n = 3$. Put

$$x = x_1, \quad y = Y_1{W_1}^{-1}, \quad z = Z_1{W_1}^{-1},$$
$$x' = x_2, \quad y' = Y_2{W_2}^{-1} \text{ and } z' = Z_2{W_2}^{-1}.$$

Then we have $x' = x^{-1}$, $y' = yx^{-2}$ and $z' = zx^{-2}$ in the function field F of X. Furthermore, (1.3) and the following equation hold:

$$y'^2 - dz'^2 = x'^4 f(x'^{-1}). \tag{2.1}$$

We will prove that $\mathrm{Br}\,(X)/\mathrm{Br}\,(K)$ is generated by $\bar{B}_0 = \{x, d\}_F$ mod

$\mathrm{Br}(K)$ in the case (ii). It suffices to show $B_0 \in \mathrm{Br}(X)$ and $B_0 \notin \mathrm{Br}(K)$, since $\mathrm{Br}(X)/\mathrm{Br}(K) \cong \mathbb{Z}/2\mathbb{Z}$. We use the commutative diagram

$$\begin{array}{ccccccc} \{0\} & \longrightarrow & {}_2\mathrm{Br}(X) & \longrightarrow & {}_2\mathrm{Br}(F) & \xrightarrow{\oplus \mathrm{res}_C} & \bigoplus_{C \subset X} K(C)^\times/K(C)^{\times 2} \\ & & \uparrow & & \uparrow & & \\ \{0\} & \longrightarrow & {}_2\mathrm{Br}(K) & \longrightarrow & {}_2\mathrm{Br}(K(x)) & \xrightarrow{\oplus \mathrm{res}_P} & \bigoplus_{P \in \mathbb{P}^1_K} K(P)^\times/K(P)^{\times 2} \end{array} \tag{2.2}$$

obtained by combining the two exact sequences due to Grothendieck [10] and Faddeev (see [11, Theorem 6.4.5]), and by passing to the 2-torsion part ${}_2A$ of each term A. Here res_C and res_P are the residue maps, C varies over all the irreducible curves on X, and P varies over all the closed points on $\mathbb{P}^1_K$.

In order to prove $B_0 \in \mathrm{Br}(X)$, it suffices to show $\mathrm{res}_C(B_0) = 1$ for any irreducible curve C on X by the top exact sequence in (2.2). The value of $\mathrm{res}_C(B_0)$ is determined by the tame symbol:

$$\begin{aligned} \mathrm{res}_C(B_0) &= (-1)^{\mathrm{ord}_C(x)\mathrm{ord}_C(d)} x^{\mathrm{ord}_C(d)} d^{-\mathrm{ord}_C(x)} \bmod K(C)^{\times 2} \\ &= d^{-\mathrm{ord}_C(x)} \bmod K(C)^{\times 2}. \end{aligned}$$

We have only to show $d \in K(C)^{\times 2}$ under the assumption $\mathrm{ord}_C(x) \neq 0$. On such a curve C, we have $x = 0$ or $x' = 0$, and therefore $y^2 = dz^2$ or $y'^2 = dz'^2$ by (1.3) and (2.1), which implies $d \in K(C)^{\times 2}$.

Let B_0' be the Brauer equivalence class of the quaternion algebra $\left(\frac{x,d}{K(x)}\right)$ over $K(x)$. In order to prove $B_0 \notin \mathrm{Br}(K)$, it suffices to show $B_0' \notin \mathrm{Br}(K)$ by the commutative diagram (2.2). By the bottom exact sequence in (2.2), it follows from that the point $P := 0 \in \mathbb{P}^1_K = \mathbb{A}^1_K \cup \{\infty\}$ satisfies

$$\begin{aligned} \mathrm{res}_P(B_0') &= (-1)^{\mathrm{ord}_P(x)\mathrm{ord}_P(d)} x^{\mathrm{ord}_P(d)} d^{-\mathrm{ord}_P(x)} \bmod K(P)^{\times 2} \\ &= d^{-1} \bmod K^{\times 2} \neq 1 \end{aligned}$$

because of the assumption $d \notin K^{\times 2}$.

In the case (iii), $\mathrm{Br}(X)/\mathrm{Br}(K)$ is generated by $\bar{B}_0$ and $\bar{B}_1$. Indeed, in this case we can see that $\bar{B}_i = B_i \bmod \mathrm{Br}(K)$ defines a nonzero element of

$\mathrm{Br}(X)/\mathrm{Br}(K)$ for each $i \in \{0,1,2\}$ by the same argument as above, and therefore $\bar{B}_0 \neq \bar{B}_1$ since

$$B_0 + B_1 = \{f(x), d\}_F - B_2 = \{y^2 - dz^2, d\}_F - B_2 \equiv B_2 \not\equiv 0 \pmod{\mathrm{Br}(K)}.$$

This completes the proof of Theorem 2.1.

3. Brauer–Manin pairing

Let X be a cubic Châtelet surface over a p-adic field K defined as in Proposition 2.4 with $c = 1$ and $n = 3$. In this section, we will give a vanishing condition for the values of the map

$$\mathrm{Br}(X)/\mathrm{Br}(K) \to \mathrm{A}_0(X)^\vee$$

induced by the Brauer–Manin pairing (1.1) in the case (iii) (see Lemma 3.1). We will use it in Section 4.

Theorem 1.1 is proven by computing the image of the composite map

$$\chi : X(K) \to \mathrm{H}^1(K, \mathrm{Hom}(\mathrm{Pic}(\bar{X}), \bar{K}^\times))$$

of the surjection

$$X(K) \twoheadrightarrow \mathrm{A}_0(X); P \mapsto [P] - [P_0] \tag{3.1}$$

and the injection

$$\mathrm{A}_0(X) \hookrightarrow \mathrm{H}^1(K, \mathrm{Hom}(\mathrm{Pic}(\bar{X}), \bar{K}^\times)) \tag{3.2}$$

defined in [1], [5]. Here $X(K)$ denotes the set of K-valued points on X, and P_0 is a fixed point in $X(K)$. The surjectivity of (3.1) is proven by Colliot-Thélène and Coray [3]. We remark $X(K) \neq \emptyset$, since X is birational to a smooth projective model of the affine surface defined by

$$y^2 - dz^2 = x \cdot x^3 f(x^{-1}).$$

By Proposition 2.3 and the local duality, we have

$$(\mathrm{Br}\,(X)/\mathrm{Br}\,(K))^{\vee} \cong \mathrm{H}^1(K, \mathrm{Pic}\,(\bar{X}))^{\vee} \cong \mathrm{H}^1(K, \mathrm{Hom}\,(\mathrm{Pic}\,(\bar{X}), \bar{K}^{\times})). \quad (3.3)$$

Identify the both sides of (3.3). Then, the map (3.2) coincides with the map induced by the pairing (1.1), since the diagram

$$\begin{array}{ccccc}
\mathrm{A}_0(X) & \times & \mathrm{Br}\,(X)/\mathrm{Br}\,(K) & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Q}/\mathbb{Z} \\
{\scriptstyle (3.2)}\downarrow & & \downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{inv}}\uparrow{\scriptstyle \cong} \\
\mathrm{H}^1(K, \mathrm{Hom}\,(\mathrm{Pic}\,(\bar{X}), \bar{K}^{\times})) & \times & \mathrm{H}^1(K, \mathrm{Pic}\,(\bar{X})) & \xrightarrow{\text{cup product}} & \mathrm{H}^2(K, \bar{K}^{\times}) = \mathrm{Br}\,(K)
\end{array}$$

commutes (see [1, Proposition A.1]).

From now on, we consider the case (iii). In this case, we will use the explicit expression of χ in Section 4. From (3.3), Theorem 2.1 and the local class field theory, we obtain

$$\mathrm{H}^1(K, \mathrm{Hom}\,(\mathrm{Pic}\,(\bar{X}), \bar{K}^{\times})) \cong (K^{\times}/N_{L/K}L^{\times})^2 \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

We identify the target of χ with $(K^{\times}/N_{L/K}L^{\times})^2$ and $(\mathbb{Z}/2\mathbb{Z})^2$ by these isomorphisms. Take a point $(0, (0 : 0 : 1)) \in X_2$ as P_0. For each point $P \in X(K)$, we denote by x_P the x_1-coordinate of P if $P \in X_1$, and define $x_P = \infty$ if $P \notin X_1$. Colliot-Thélène and Sansuc [4] proved that χ is given by

$$\chi(P) = \begin{cases} (\overline{x_P}, \overline{x_P - e_1}) & \text{if } x_P \neq 0,\ e_1,\ \infty, \\ (\overline{e_1 e_2}, \overline{-e_1}) & \text{if } x_P = 0, \\ (\overline{e_1}, \overline{e_1(e_1 - e_2)}) & \text{if } x_P = e_1, \\ (0, 0) & \text{if } x_P = \infty, \end{cases} \quad (3.4)$$

where the bar denotes the natural projection $K^{\times} \to K^{\times}/N_{L/K}L^{\times}$. In the proof of Theorem 1.2, to evaluate generators of $\mathrm{Br}\,(\mathscr{X})$, we will use the following fact:

Lemma 3.1 *Let X be the cubic Châtelet surface over a p-adic field K defined as above. Suppose that we are in the case* (iii). *Then, for each $i \in \{0, 1\}$, the following two conditions are equivalent*:

(A) $\langle \xi, \bar{B}_i \rangle = 0$ *for any* $\xi \in \mathrm{A}_0(X)$.
(B) $\mathrm{pr}_{i+1}(\chi(P)) = 0$ *for any* $P \in X(K)$.

Here pr_j *denotes the projection from* $(\mathbb{Z}/2\mathbb{Z})^2$ *to the* j*-th component.*

Proof. Fix $i \in \{0,1\}$. Put $e_0 = 0$. From the equation (1.3) and a basic property of quaternion algebras, we have

$$B_i = \left\{ \frac{y^2 - dz^2}{\prod_{j\neq i}(x - e_j)}, d \right\}_F = \left\{ \prod_{j\neq i}(x - e_j), d \right\}_F = \left\{ \prod_{j\neq i}(1 - e_j x^{-1}), d \right\}_F .$$

By definition, this implies

$$\mathrm{inv}\,({P_0}^*(B_i)) = \mathrm{inv}\,\{1, d\}_K = 0$$

and therefore

$$\langle P - P_0, B_i \rangle = \mathrm{inv}\,(P^*(B_i)) = \begin{cases} \mathrm{inv}\,\{x_P - e_i, d\}_K & \text{if } x_P \neq e_i,\ \infty, \\ \mathrm{inv}\,\left\{\prod_{j\neq i}(e_i - e_j), d\right\}_K & \text{if } x_P = e_i, \\ 0 & \text{if } x_P = \infty \end{cases}$$

for given $P \in X(K)$, where P^* denotes the pull-back along $P \to X$. Here $\{a, b\}_K$ denotes the Brauer equivalence class of the quaternion algebra $\left(\frac{a,b}{K}\right)$ over K, for any a, $b \in K^\times$. Since the invariant map inv : $\mathrm{Br}\,(K) \to \mathbb{Q}/\mathbb{Z}$ is an isomorphism, we have

$$\mathrm{inv}\,\{a, d\}_K = 0 \iff \{a, d\}_K = 0 \iff a \in N_{L/K}L^\times .$$

These imply that $\langle P - P_0, B_i \rangle = 0$ holds if and only if the $(i+1)$-st component of the right hand side of (3.4) is zero. (This reproduces (3.4).) From the surjectivity of (3.1), the conditions (A) and (B) are equivalent. □

4. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Let X be a cubic Châtelet surface over a p-adic field K defined as in Proposition 2.4 with $c = 1$ and $n = 3$. We use the same notation as in Sections 2 and 3. Since $X(K) \neq \emptyset$ as seen in Section 3, the natural map $\mathrm{Br}\,(K) \to \mathrm{Br}\,(X)$ is injective. We regard $\mathrm{Br}\,(K)$ as a subgroup of $\mathrm{Br}\,(X)$ by this injection. Take a point $P_0 \in X(K)$. From the properness of $\mathscr{X}$, there exists a morphism $\mathrm{Spec}\,(\mathfrak{O}) \to \mathscr{X}$ such that the diagram

$$\begin{array}{ccc} \mathscr{X} & \longleftarrow & \operatorname{Spec}(\mathfrak{O}) \\ \uparrow & & \uparrow \\ X & \xleftarrow{P_0} & \operatorname{Spec}(K) \end{array}$$

commutes. Since we obtain the commutative diagram

$$\begin{array}{ccccccc} \operatorname{Br}(\mathscr{X}) \cap \operatorname{Br}(K) & \longrightarrow & \operatorname{Br}(\mathscr{X}) & \longrightarrow & \operatorname{Br}(\mathfrak{O}) & = & \{0\} \\ \downarrow & & \downarrow & & \downarrow & & \\ \operatorname{Br}(K) & \longrightarrow & \operatorname{Br}(X) & \xrightarrow{{P_0}^*} & \operatorname{Br}(K) & & \end{array}$$

(see [13, Proposition 1.6, Chapter IV]), and the composite of the bottom maps coincides with the identity map on $\operatorname{Br}(K)$, we can find

$$\operatorname{Br}(\mathscr{X}) \cap \operatorname{Br}(K) = \{0\}. \tag{4.1}$$

Thus we have

$$(\operatorname{Br}(\mathscr{X}) + \operatorname{Br}(K))/\operatorname{Br}(K) \cong \operatorname{Br}(\mathscr{X})/(\operatorname{Br}(\mathscr{X}) \cap \operatorname{Br}(K)) \cong \operatorname{Br}(\mathscr{X}),$$

by the isomorphism theorem, and obtain the exact sequence

$$\{0\} \to \operatorname{Br}(\mathscr{X}) \to \operatorname{Br}(X)/\operatorname{Br}(K) \to \operatorname{Br}(X)/(\operatorname{Br}(\mathscr{X}) + \operatorname{Br}(K)) \to \{0\}.$$

By combining with the isomorphism (1.2) due to Saito and Sato [16], we obtain the exact sequence

$$\begin{array}{ccccccc} \{0\} \to \operatorname{Br}(\mathscr{X}) \to & \operatorname{Br}(X)/\operatorname{Br}(K) & \to & \mathrm{A}_0(X)^\vee & \to \{0\} \\ & \bar{B} & \mapsto & (\xi \mapsto \langle \xi, \bar{B} \rangle). & \end{array} \tag{4.2}$$

Case (i). In this case, we have $\operatorname{Br}(X)/\operatorname{Br}(K) = \{0\}$ by Theorem 2.1. This implies $\operatorname{Br}(\mathscr{X}) = \{0\}$ by (4.1).

Case (ii). In this case, we have $\operatorname{Br}(X)/\operatorname{Br}(K) = \bar{B}_0 \cdot \mathbb{Z}/2\mathbb{Z}$ by Theorem 2.1. This implies that we have $\operatorname{Br}(\mathscr{X}) = B_0 \cdot \mathbb{Z}/2\mathbb{Z}$ if $\mathrm{A}_0(X) \cong \{0\}$, and $\operatorname{Br}(\mathscr{X}) = \{0\}$ if $\mathrm{A}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$, by (4.2).

Case (iii). In this case, we have $\mathrm{Br}\,(X)/\mathrm{Br}\,(K) = \bar{B}_0 \cdot \mathbb{Z}/2\mathbb{Z} \oplus \bar{B}_1 \cdot \mathbb{Z}/2\mathbb{Z}$ by Theorem 2.1. This implies that we have $\mathrm{Br}\,(\mathscr{X}) = B_0 \cdot \mathbb{Z}/2\mathbb{Z} \oplus B_1 \cdot \mathbb{Z}/2\mathbb{Z}$ if $\mathrm{A}_0(X) \cong \{0\}$, and $\mathrm{Br}\,(\mathscr{X}) = \{0\}$ if $\mathrm{A}_0(X) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, by (4.2). In the remaining case, it is proven that $\mathrm{A}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$ in [7, Proposition 4.7], [8, Proposition 2], [9, Proposition 3] by showing

$$\chi(X(K)) = \{0\} \times \mathbb{Z}/2\mathbb{Z}. \tag{4.3}$$

For the reader's convenience, we reproduce the proof for the remaining case of (iii-a) due to Colliot-Thélène [7]. That is, we will show (4.3) in the following case:

(iii-a)' L/K is unramified, $r = v(e_1) = v(e_2) \in 2\mathbb{Z}$ and $r' := v(e_1 - e_2) > r$.

In this case, the values of χ can be identified with the pairs of their valuations modulo 2, since the valuation $v : K^\times \to \mathbb{Z}$ induces the isomorphism

$$K^\times / N_{L/K} L^\times \cong \mathbb{Z}/2\mathbb{Z}; \bar{a} \mapsto v(a) \bmod 2. \tag{4.4}$$

We will show $\mathrm{pr}_1(\chi(P)) = 0$ for given $P \in X(K)$. We can assume $x_P \neq \infty$.

- Case: $x_P = 0$. In this case, we have $\mathrm{pr}_1(\chi(P)) = v(e_1 e_2) = 2r \bmod 2 = 0$.
- Case: $x_P \neq 0$ and $v(x_P) = r$. In this case, we have $\mathrm{pr}_1(\chi(P)) = v(x_P) \bmod 2 = r \bmod 2 = 0$.
- Case: $x_P \neq 0$ and $v(x_P) \neq r$. Put $P = (x_P, (y_P : z_P : 1))$ in X_1. Then we have

  $$v({y_P}^2 - d{z_P}^2) = v(x_P(x_P - e_1)(x_P - e_2)) = \begin{cases} v(x_P) + 2r & \text{if } v(x_P) > r, \\ 3v(x_P) & \text{if } v(x_P) < r. \end{cases}$$

  Since this value is even by (4.4), we have $\mathrm{pr}_1(\chi(P)) = v(x_P) \bmod 2 = 0$.

The existence of a point $P \in X(K)$ satisfying $\mathrm{pr}_2(\chi(P)) = 1$ is shown as follows:

- Case: $r' \notin 2\mathbb{Z}$. In this case, the point $P := (e_1, (0 : 0 : 1)) \in X_1$ satisfies

$$\mathrm{pr}_2(\chi(P)) = v(e_1(e_1 - e_2)) \bmod 2 = r + r' \bmod 2 = 1.$$

- Case: $r' \in 2\mathbb{Z}$. Take a uniformizer π of K. Put $a = e_1 + \pi^{r+1}$. Then we have

  $$v(a) = r, \quad v(a-e_1) = r+1, \quad v(a-e_2) = v((a-e_1)+(e_1-e_2)) = r+1$$

  since $r < r+1 < r'$, and therefore

  $$v(a(a-e_1)(a-e_2)) = r + 2(r+1).$$

  Since this value is even, we have $a(a-e_1)(a-e_2) \in N_{L/K}L^\times$ by (4.4). This implies that a is the x-coordinate of a K-valued point P on X_1, since $L = K(\sqrt{d})$. This point P satisfies $\mathrm{pr}_2(\chi(P)) = 1$.

This is why we have (4.3) in the case (iii-a)'.

In each remaining case of (iii-a) or (iii-b), the equation (4.3) implies $\langle \xi, \bar{B}_0 \rangle = 0$ for any $\xi \in \mathrm{A}_0(X)$ by Lemma 3.1. We conclude that $\mathrm{Br}(\mathscr{X}) = B_0 \cdot \mathbb{Z}/2\mathbb{Z}$ if $\mathrm{A}_0(X) \cong \mathbb{Z}/2\mathbb{Z}$ by (4.2) and Theorem 2.1.

This completes the proof of Theorem 1.2.

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