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Applications of Campanato spaces with variable growth condition to the Navier-Stokes equation

Dedicated to Professor Yoshikazu Giga on the occasion of his sixtieth birthday

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Abstract. We give new viewpoints of Campanato spaces with variable growth condition for applications to the Navier-Stokes equation. Namely, we formulate a blowup criteria along maximum points of the 3D-Navier-Stokes flow in terms of stationary Euler flows and show that the properties of Campanato spaces with variable growth condition are very useful for this formulation, since variable growth condition can control the continuity and integrability of functions on the neighborhood at each point. Our criterion is different from the Beale-Kato-Majda type and Constantin-Fefferman type criterion. If geometric behavior of the velocity vector field near the maximum point has a kind of stationary Euler flow configuration up to a possible blowup time, then the solution can be extended to be the strong solution beyond the possible blowup time. As another application we also mention the Cauchy problem for the Navier-Stokes equation.

Key words: Campanato spaces with variable growth condition, blowup criterion, 3D Navier-Stokes equation, stationary 3D Euler flow, Cauchy problem.

1. Introduction

In this paper we consider the properties of Campanato spaces with variable growth condition and give their applications to the Navier-Stokes equation. More precisely, we construct a blowup criteria along maximum points of the 3D-Navier-Stokes flow in terms of stationary 3D Euler flows. As another application we also mention existence of a time local solution to the Cauchy problem for the Navier-Stokes equation.

Campanato spaces was introduced and studied in [3], [4], [28], [29], etc, and their variant with variable growth condition was introduced in [25] to characterize pointwise multipliers on BMO. Recently, it turned out that Campanato spaces with variable growth condition were the dual spaces of Hardy spaces $H^{p(\cdot)}$ with variable exponent by [24]. In this paper we recall properties of Campanato spaces with variable growth condition investigated

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in [18], [20], [21], [23] and give new viewpoints of them for applications to the Navier-Stokes equation.

The Navier-Stokes equation is expressed as

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla p = 0 & \text{in } \mathbb{R}^n \times [0,T), \\ \nabla\cdot v = 0 & \text{in } \mathbb{R}^n \times [0,T), \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $v = (v_1, \dots, v_n)$ is a vector field representing velocity of the fluid, p is the pressure, and

$$\nabla\cdot v = \sum_{j=1}^n \partial_j v_j, \quad v\cdot\nabla = \sum_{j=1}^n v_j\partial_j, \quad \Delta = \sum_{j=1}^n \partial_j^2.$$

It is known that the pair of solutions (v,p) satisfies the relation

$$p = \sum_{i,j=1}^n R_iR_j(v_iv_j),$$

where R_j $(j = 1, \dots, n)$ are the Riesz transforms (see [13], [15], [26] for example). Therefore, to estimate the solutions in some function space we need the properties of the Riesz transforms and pointwise multipliers (pointwise product operators) on the function space. Namely, we investigate the following norm boundedness:

$$\|fg\|_{\mathcal{L}^\natural_{q,\psi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}\|g\|_{\mathcal{L}^\natural_{p,\phi}}, \tag{1.2}$$

$$\|R_jf\|_{\mathcal{L}^\natural_{q,\psi}} \le C\|f\|_{\mathcal{L}^\natural_{q,\psi}}, \tag{1.3}$$

for Campanato spaces $\mathcal{L}^\natural_{p,\phi}$ and $\mathcal{L}^\natural_{q,\psi}$ with variable growth condition. Moreover, to consider blowup (or non-blowup) criterion we will use the following estimate on the value of functions at a certain point x_0:

$$|f(x_0)| \le C\|f\|_{\mathcal{L}^\natural_{q,\psi}}. \tag{1.4}$$

For this estimate Campanato spaces with variable growth condition are very useful, since variable growth condition can control the continuity and integrability of functions on the neighborhood at each point.

In the next section we define Campanato spaces with variable growth condition and state their several properties. We give the boundedness of the Riesz transforms, operators of convolution type, and pointwise multipliers on Campanato spaces with variable growth condition in Sections 3, 4 and 5, respectively. Then we formulate a blowup criteria for the 3D Navier-Stokes flow in Section 6, which is proved in Section 7. The most significant blowup criterion must be the Beale-Kato-Majda criterion [1]. On the other hand, Constantin and Fefferman [7] (see also [8]) took into account geometric structure of the vortex stretching term in the vorticity equations to get another kind of blowup condition. These two separate forms of criteria controlling the blow-up by magnitude and the direction of the vorticity respectively are interpolated by Chae [5]. In this paper, we give a different type of blowup criterion from them. We focus on a geometric behavior of the velocity vector field near the each maximum points. Further, we give in Section 8 the specific function spaces of Campanato spaces with variable growth condition as suitable examples for our blowup criteria. Finally, as another application of Campanato spaces with variable growth condition, we give an existence theorem on the Cauchy problem for the Navier-Stokes equation in Section 9.

2. Campanato spaces with variable growth condition

In this section we define Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}$ with variable growth condition. We state basic properties of the function spaces $\mathcal{L}^{\natural}_{p,\phi}$. To do this we also define Morrey spaces and Hölder spaces with variable growth condition.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^n$ and of radius $r > 0$, that is,

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ and χ_G the Lebesgue measure of G and the characteristic function of G, respectively.

We consider variable growth functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r)$, write $\phi(B)$ in place of $\phi(x,r)$. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

and for a ball B, let

$$f_B = |B|^{-1} \int_B f(x)\, dx.$$

Then we define Campanato spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$, Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$, and Hölder spaces $\Lambda_\phi(\mathbb{R}^n)$ and $\Lambda^{\natural}_\phi(\mathbb{R}^n)$ with variable growth functions ϕ as the following:

Definition 2.1 For $1 \le p < \infty$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, function spaces $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$, $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$, $L_{p,\phi}(\mathbb{R}^n)$, $\Lambda_\phi(\mathbb{R}^n)$, $\Lambda^{\natural}_\phi(\mathbb{R}^n)$ are the sets of all functions f such that

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p\, dx \right)^{1/p} < \infty,$$

$$\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} = \|f\|_{\mathcal{L}_{p,\phi}} + |f_{B(0,1)}| < \infty,$$

$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x)|^p\, dx \right)^{1/p} < \infty,$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y\in\mathbb{R}^n,\ x\neq y} \frac{2|f(x)-f(y)|}{\phi(x,|x-y|)+\phi(y,|y-x|)} < \infty,$$

$$\|f\|_{\Lambda^{\natural}_\phi} = \|f\|_{\Lambda_\phi} + |f(0)| < \infty,$$

respectively.

We regard $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and $L_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions, $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ as spaces of functions modulo null-functions and constant functions, $\Lambda^{\natural}_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$, and $\Lambda_\phi(\mathbb{R}^n)$ as a space of functions defined at all $x \in \mathbb{R}^n$ modulo constant functions. Then these five functionals are norms and thereby these spaces are all Banach spaces.

In order to apply the Campanato spaces $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ to the blowup criterion (more precisely, in order to find specific function spaces satisfying (1.2), (1.3) and (1.4)), we state several properties of $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and the relation between ϕ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$.

For two variable growth functions ϕ_1 and ϕ_2, we write $\phi_1 \sim \phi_2$ if there exists a positive constant C such that

$$C^{-1}\phi_1(B) \le \phi_2(B) \le C\phi_1(B) \quad \text{for all balls } B.$$

In this case, two spaces defined by ϕ_1 and by ϕ_2 coincide with equivalent norms. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is the usual $\mathrm{BMO}(\mathbb{R}^n)$. For $\phi(x,r) = r^\alpha$, $0 < \alpha \le 1$, we denote $\Lambda_{r^\alpha}(\mathbb{R}^n)$ and $\Lambda^\natural_{r^\alpha}(\mathbb{R}^n)$ by $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ and $\mathrm{Lip}^\natural_\alpha(\mathbb{R}^n)$, respectively. In this case,

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup_{x,y\in\mathbb{R}^n,\ x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \quad \text{and} \quad \|f\|_{\mathrm{Lip}^\natural_\alpha} = \|f\|_{\mathrm{Lip}_\alpha} + |f(0)|.$$

If $\phi(x,r) = \min(r^\alpha, 1)$, $0 < \alpha \le 1$, then

$$\|f\|_{\Lambda^\natural_\phi} \sim \|f\|_{\mathrm{Lip}_\alpha} + \|f\|_{L^\infty}.$$

From the definition it follows that

$$\|f\|_{\mathcal{L}_{p,\phi}} \le 2\|f\|_{L_{p,\phi}}, \quad \|f\|_{\mathcal{L}^\natural_{p,\phi}} \le (2+\phi(0,1))\|f\|_{L_{p,\phi}}.$$

If $\phi(B) = |B|^{-1/p}$ for all balls B, then

$$\|f\|_{L_{p,\phi}} = \|f\|_{L^p}.$$

We consider the following conditions on variable growth function ϕ:

$$\frac{1}{A_1} \le \frac{\phi(x,s)}{\phi(x,r)} \le A_1, \qquad \frac{1}{2} \le \frac{s}{r} \le 2, \tag{2.1}$$

$$\frac{1}{A_2} \le \frac{\phi(x,r)}{\phi(y,r)} \le A_2, \qquad d(x,y) \le r, \tag{2.2}$$

$$\phi(x,r) \le A_3\phi(x,s), \quad 0 < r < s < \infty, \tag{2.3}$$

where A_i, $i = 1,2,3$, are positive constants independent of $x, y \in \mathbb{R}^n$ and $r, s > 0$. Note that (2.2) and (2.3) imply that there exists a positive constant C such that

$$\phi(x,r) \le C\phi(y,s) \quad \text{for} \quad B(x,r) \subset B(y,s),$$

where the constant C is independent of balls $B(x,r)$ and $B(y,s)$.

The following three theorems are known:

Theorem 2.1 ([22]) *If ϕ satisfies (2.1), (2.2) and (2.3), then, for every $1 \le p < \infty$, $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}_{1,\phi}(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathcal{L}^{\natural}_{1,\phi}(\mathbb{R}^n)$ with equivalent norms, respectively.*

Theorem 2.2 ([21]) *If ϕ satisfies (2.1), (2.2), (2.3), and there exists a positive constant C such that*

$$\int_0^r \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r > 0, \tag{2.4}$$

then, for every $1 \le p < \infty$, each element in $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ can be regarded as a continuous function, (that is, each element is equivalent to a continuous function modulo null-functions) and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n)$ and $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \Lambda^{\natural}_\phi(\mathbb{R}^n)$ with equivalent norms, respectively. In particular, if $\phi(x,r) = r^\alpha$, $0 < \alpha \le 1$, then, for every $1 \le p < \infty$, $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}^{\natural}_\alpha(\mathbb{R}^n)$ and $\mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n)$ with equivalent norms, respectively.

Theorem 2.3 ([21]) *Let $1 \le p < \infty$. If ϕ satisfies (2.1), (2.2), and there exists a positive constant C such that*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C\phi(x,r), \quad x \in \mathbb{R}^n,\ r > 0, \tag{2.5}$$

then, for $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, the limit $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$ exists and

$$\|f\|_{\mathcal{L}_{p,\phi}} \sim \|f - \sigma(f)\|_{L_{p,\phi}}.$$

That is, the mapping $f \mapsto f - \sigma(f)$ is bijective and bicontinuous from $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ (modulo constants) to $L_{p,\phi}(\mathbb{R}^n)$.

Remark 2.1 The following inequality is important to prove Theorems 2.2 and 2.3. It is proven by an elementaly calculation, see [21, (3.6) on page 7].

$$|f_{B(x,r_1)} - f_{B(x,r_2)}| \le C \int_{r_1}^{2r_2} \frac{\phi(x,t)}{t}\,dt\,\|f\|_{\mathcal{L}_{p,\phi}} \quad \text{for } x \in \mathbb{R}^n,\ r_1 < r_2. \tag{2.6}$$

Remark 2.2 If $\int_1^\infty \phi(0,t)/t\,dt < \infty$, then, for every $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$, there exists a constant $\sigma(f)$ such that $\lim_{r\to\infty} f_{B(x,r)} = \sigma(f)$ for all $x \in \mathbb{R}^n$, see [21, Lemma 3.2].

Remark 2.3 If $\int_1^\infty \phi(0,t)/t\,dt < \infty$, then $\phi(0,r) \to 0$ as $r \to \infty$. Hence, for $f \in L_{p,\phi}(\mathbb{R}^n)$, we have

$$|\sigma(f)| = \lim_{r\to\infty} |f_{B(0,r)}| \le \lim_{r\to\infty} \phi(0,r)\|f\|_{L_{p,\phi}} \to 0 \quad \text{as} \quad r \to \infty.$$

That is, $\sigma(f) = 0$.

For a ball $B_* \subset \mathbb{R}^n$ and $0 < \alpha \le 1$, let

$$\|f\|_{\mathrm{Lip}_\alpha(B_*)} = \sup_{x,y\in B_*,\, x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We also conclude the following:

Proposition 2.4 *Let $1 \le p < \infty$ and $0 < \alpha \le 1$. Assume that, for a ball B_*,*

$$\phi(x,r) = r^\alpha \quad \textit{for all balls } B(x,r) \subset B_*. \tag{2.7}$$

Then each element f in $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ can be regarded as a continuous function on the ball B_*, and, there exists a positive constant C such that

$$\|f\|_{\mathrm{Lip}_\alpha(B_*)} \le C\|f\|_{\mathcal{L}_{p,\phi}},$$

where C is dependent only on n and α. In particular, if (2.7) *holds for $B_* = B(0,1)$, then each $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ is α-Lipschitz continuous near the origin and*

$$\|f\|_{\mathcal{L}^\natural_{p,\phi}} \sim \|f\|_{\mathcal{L}_{p,\phi}} + |f(0)|.$$

Proof. By (2.6) we have that, if $B(x,r)$, $B(y,r) \subset B_*$, then

$$|f_{B(x,r)} - f_{B(y,r)}| \le C\int_r^{2r+|x-y|} \frac{t^\alpha}{t}\,dt\ \|f\|_{\mathcal{L}_{p,\phi}} \le C_*(2r+|x-y|)^\alpha\ \|f\|_{\mathcal{L}_{p,\phi}},$$

since $B(x,r)$, $B(y,r) \subset B((x+y)/2, r+|x-y|/2)$, where C_* is dependent only on n and α. Letting $r \to 0$, we have

$$|f(x) - f(y)| \leq C_* |x-y|^\alpha \ \|f\|_{\mathcal{L}_{p,\phi}},$$

for almost every $x, y \in B_*$. In this case we can regard that f is a continuous function modulo null-functions and we have

$$\|f\|_{\mathrm{Lip}_\alpha(B_*)} \leq C_* \|f\|_{\mathcal{L}_{p,\phi}}.$$

If $B_* = B(0,1)$, then

$$|f_{B(0,r)} - f_{B(0,1)}| \leq C \int_r^2 \frac{t^\alpha}{t}\, dt \ \|f\|_{\mathcal{L}_{p,\phi}} \leq C \|f\|_{\mathcal{L}_{p,\phi}}.$$

Letting $r \to 0$, we have

$$|f(0) - f_{B(0,1)}| \leq C \|f\|_{\mathcal{L}_{p,\phi}}.$$

This shows that $\|f\|_{\mathcal{L}_{p,\phi}} + |f_{B(0,1)}| \sim \|f\|_{\mathcal{L}_{p,\phi}} + |f(0)|$. □

Proposition 2.5 *Let $1 \leq p < \infty$. Assume that there exists a positive constant A such that*

$$\phi(B) \leq A|B|^{-1/p} \quad \textit{for all balls } B.$$

Then there exists a positive constant C such that, if $f \in \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then $f \in L^p(\mathbb{R}^n)$ and

$$\|f\|_{L^p} \leq C \|f\|_{\mathcal{L}_{p,\phi}}.$$

Proof. Let $\tilde{\phi}(B) = |B|^{-1/p}$. Then $\tilde{\phi}$ satisfies (2.5). Hence, by Theorem 2.3 we have

$$\|f\|_{L_{p,\tilde{\phi}}} = \|f - \sigma(f)\|_{L_{p,\tilde{\phi}}} \sim \|f\|_{\mathcal{L}_{p,\tilde{\phi}}}.$$

Since $\|f\|_{L_{p,\tilde{\phi}}} = \|f\|_{L^p}$ and $\|f\|_{\mathcal{L}_{p,\tilde{\phi}}} \leq A\|f\|_{\mathcal{L}_{p,\phi}}$, we have the conclusion. □

3. Singular integral operators

In this section we consider the singular integral theory to show the boundedness of Riesz transforms in Campanato spaces with variable growth condition. We denote by $L^p_c(\mathbb{R}^n)$ the set of all $f \in L^p(\mathbb{R}^n)$ with compact support. Let $0 < \kappa \le 1$. We shall consider a singular integral operator T with measurable kernel K on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following properties:

$$|K(x,y)| \le \frac{C}{|x-y|^n} \quad \text{for} \quad x \ne y, \tag{3.1}$$

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le \frac{C}{|x-y|^n} \left(\frac{|x-z|}{|x-y|} \right)^{\kappa} \quad \text{for} \quad |x-y| \ge 2|x-z|, \tag{3.2}$$

$$\int_{r \le |x-y| < R} K(x,y)\,dy = \int_{r \le |x-y| < R} K(y,x)\,dy = 0 \quad \text{for } 0 < r < R < \infty \text{ and } x \in \mathbb{R}^n, \tag{3.3}$$

where C is a positive constant independent of $x, y, z \in \mathbb{R}^n$. For $\eta > 0$, let

$$T_\eta f(x) = \int_{|x-y| \ge \eta} K(x,y) f(y)\,dy.$$

Then $T_\eta f(x)$ is well defined for $f \in L^p_c(\mathbb{R}^n)$, $1 < p < \infty$. We assume that, for all $1 < p < \infty$, there exists positive constant C_p independently $\eta > 0$ such that,

$$\|T_\eta f\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{for} \quad f \in L^p_c(\mathbb{R}^n),$$

and $T_\eta f$ converges to Tf in $L^p(\mathbb{R}^n)$ as $\eta \to 0$. By this assumption, the operator T can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. We shall say the operator T satisfying the above conditions is a singular integral operator of type κ. For example, Riesz transforms are singular integral operators of type 1.

Now, to define T for functions $f \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$, we first define the modified version of T_η by

$$\tilde{T}_\eta f(x) = \int_{|x-y|\geq\eta} f(y)\big[K(x,y) - K(0,y)(1-\chi_{B(0,1)}(y))\big]\,dy. \tag{3.4}$$

Then we can show that the integral in the definition above converges absolutely for each x and that $\tilde{T}_\eta f$ converges in $L^p(B)$ as $\eta \to 0$ for each ball B. We denote the limit by $\tilde{T}f$. If both $\tilde{T}f$ and Tf are well defined, then the difference is a constant.

We can show the following results. Theorem 3.1 is an extension of [23, Theorem 4.1] and Theorem 3.3 is an extension of [19, Theorem 2]. The proofs are almost the same.

Theorem 3.1 *Let $0 < \kappa \leq 1$ and $1 < p < \infty$. Assume that ϕ and ψ satisfy* (2.1) *and that there exists a positive constant A such that, for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$r^\kappa \int_r^\infty \frac{\phi(x,t)}{t^{1+\kappa}}\,dt \leq A\psi(x,r). \tag{3.5}$$

If T is a singular integral operator of type κ, then $\tilde{T}$ is bounded from $\mathcal{L}_{p,\phi}(\mathbb{R}^n)$ to $\mathcal{L}_{p,\psi}(\mathbb{R}^n)$ and from $\mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ to $\mathcal{L}^\natural_{p,\psi}(\mathbb{R}^n)$, that is, there exists a positive constants C such that

$$\|\tilde{T}f\|_{\mathcal{L}_{p,\psi}} \leq C\|f\|_{\mathcal{L}_{p,\phi}}, \quad \|\tilde{T}f\|_{\mathcal{L}^\natural_{p,\psi}} \leq C\|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

Moreover, if ϕ and ψ satisfy (2.2) *and* (2.3) *also, then $\tilde{T}$ is bounded from $\mathcal{L}^\natural_{1,\phi}(\mathbb{R}^n)$ to $\mathcal{L}^\natural_{1,\psi}(\mathbb{R}^n)$.*

Corollary 3.2 *Under the assumption in Theorem* 3.1*, if ϕ and ψ satisfies* (2.2)*,* (2.3) *and* (2.4)*, then $\tilde{T}$ is bounded from $\Lambda_\phi(\mathbb{R}^n)$ to $\Lambda_\psi(\mathbb{R}^n)$ and from $\Lambda^\natural_\phi(\mathbb{R}^n)$ to $\Lambda^\natural_\psi(\mathbb{R}^n)$.*

For Morrey spaces $L_{p,\phi}(\mathbb{R}^n)$, we have the following.

Theorem 3.3 *Let $0 < \kappa \leq 1$ and $1 < p < \infty$. Assume that ϕ and ψ satisfy* (2.1) *and that there exists a positive constant A such that, for all $x \in \mathbb{R}^n$ and $r > 0$,*

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \leq A\psi(x,r).$$

If T is a singular integral operator of type κ, then T is bounded from $L_{p,\phi}(\mathbb{R}^n)$ to $L_{p,\psi}(\mathbb{R}^n)$.

Now we state the boundedness of Riesz transforms. For f in Schwartz class, the Riesz transforms of f are defined by

$$R_j f(x) = c_n \lim_{\varepsilon \to 0} R_{j,\varepsilon} f(x), \quad j = 1, \dots, n,$$

where

$$R_{j,\varepsilon} f(x) = \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad c_n = \Gamma\left(\frac{n+1}{2}\right) \pi^{-(n+1)/2}.$$

Then it is known that there exists a positive constant C_p independently $\varepsilon > 0$ such that,

$$\|R_{j,\varepsilon} f\|_{L^p} \le C_p \|f\|_{L^p} \quad \text{for} \quad f \in L^p_c(\mathbb{R}^n),$$

and $R_{j,\varepsilon} f$ converges to $R_j f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$. That is, the operator R_j can be extended as a continuous linear operator on $L^p(\mathbb{R}^n)$. Hence, we can define modified Riesz transforms of f as

$$\tilde{R}_j f(x) = c_n \lim_{\varepsilon \to 0} \tilde{R}_{j.\varepsilon} f(x), \quad j = 1, \dots, n,$$

and

$$\tilde{R}_{j.\varepsilon} f(x) = \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} \right) f(y)\, dy.$$

We note that, if both $R_j f$ and $\tilde{R}_j f$ are well defined on $\mathbb{R}^n$, then $R_j f - \tilde{R}_j f$ is a constant function. More precisely,

$$R_j f(x) - \tilde{R}_j f(x) = c_n \int_{\mathbb{R}^n} \frac{(-y_j)(1 - \chi_{B(0,1)}(y))}{|y|^{n+1}} f(y)\, dy.$$

Remark 3.1 If f is a constant function, then $\tilde{R}_j f = 0$. Actually, for $f \equiv 1$,

$$\begin{aligned}\tilde{R}_{j.\varepsilon}1(x) &= \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{(x_j-y_j)\chi_{B(x,1)}}{|x-y|^{n+1}}\,dy \\ &\quad + \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \left(\frac{(x_j-y_j)(1-\chi_{B(x,1)})}{|x-y|^{n+1}} - \frac{(-y_j)(1-\chi_{B(0,1)}(y))}{|y|^{n+1}}\right) dy \\ &= \int_{B(x,\varepsilon)} \frac{(-y_j)(1-\chi_{B(0,1)}(y))}{|y|^{n+1}}\,dy \to 0 \quad \text{as } \varepsilon\to 0,\end{aligned}$$

since

$$\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{(x_j-y_j)\chi_{B(x,1)}}{|x-y|^{n+1}}\,dy = \int_{B(0,1)\setminus B(0,\varepsilon)} \frac{y_j}{|y|^{n+1}}\,dy = 0$$

and

$$\int_{\mathbb{R}^n} \left(\frac{(x_j-y_j)(1-\chi_{B(x,1)})}{|x-y|^{n+1}} - \frac{(-y_j)(1-\chi_{B(0,1)}(y))}{|y|^{n+1}}\right) dy = 0.$$

Hence $\tilde{R}_j 1(x) = 0$ for all $x \in \mathbb{R}^n$.

Theorem 3.4 *Let $1 < p < \infty$, and let ϕ satisfy (2.1) and*

$$r\int_r^\infty \frac{\phi(x,t)}{t^2}\,dt \le A\phi(x,r), \tag{3.6}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Assume that there exists a growth function $\tilde{\phi}$ such that $\phi \le \tilde{\phi}$ and that $\tilde{\phi}$ satisfies (2.1), (2.2) and (2.5). If $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then $R_j f$, $j = 1,2,\dots,n$, are well defined, $\sigma(R_j f) = \lim_{r\to\infty} (R_j f)_{B(0,r)} = 0$, and

$$\|R_j f\|_{\mathcal{L}^\natural_{p,\phi}} \le C\|f\|_{\mathcal{L}^\natural_{p,\phi}}, \quad j = 1,2,\dots,n,$$

where C is a positive constant independent of f.

Proof. Let $f \in \mathcal{L}^\natural_{p,\phi}(\mathbb{R}^n)$ and $\sigma(f) = 0$. Then, by Theorem 2.3,

$$\|f\|_{L_{p,\tilde{\phi}}} = \|f-\sigma(f)\|_{L_{p,\tilde{\phi}}} \sim \|f\|_{\mathcal{L}_{p,\tilde{\phi}}} \le \|f\|_{\mathcal{L}_{p,\phi}} \le \|f\|_{\mathcal{L}^\natural_{p,\phi}}.$$

By Theorem 3.3 $R_j f$ is well defined and

$$\|R_j f\|_{L_{p,\tilde{\phi}}} \le C\|f\|_{L_{p,\tilde{\phi}}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

This shows that $\sigma(R_j f) = 0$ by Remark 2.3 and

$$\begin{aligned} |(R_j f)_{B(0,1)}| &\le \left(\frac{1}{|B(0,1)|} \int_{B(0,1)} |R_j f(x)|^p \, dx \right)^{1/p} \\ &\le \tilde{\phi}(0,1)\|R_j f\|_{L_{p,\tilde{\phi}}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}. \end{aligned}$$

Since $R_j f - \tilde{R}_j f$ is a constant, by Theorem 3.1, we have

$$\|R_j f\|_{\mathcal{L}_{p,\phi}} = \|\tilde{R}_j f\|_{\mathcal{L}_{p,\phi}} \le C\|f\|_{\mathcal{L}_{p,\phi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Therefore, we have $\|R_j f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}$. □

Remark 3.2 Under the assumption in Theorem 3.4, for $f \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ with $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, $R_i R_j f$ is well defined and

$$\|R_i R_j f\|_{\mathcal{L}^{\natural}_{p,\phi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}, \quad i, j = 1, \ldots, n.$$

4. Convolution

In this section we prove the boundedness of operators of convolution type with nice functions like the heat kernel. For a function g and $s > 0$, let $g_s(x) = g(x/s)/s^n$.

Theorem 4.1 *Let $1 \le p_1, p_2, p_3 < \infty$ and $1+1/p_1 = 1/p_2+1/p_3$. Assume that ϕ satisfies* (2.1) *and* (2.5)*. Let $g \in L^{p_3}(\mathbb{R}^n)$ and there exists a positive constant C_0 such that*

$$|g(x)| \le \frac{C_0}{|x|^n} \quad \text{for} \quad x \ne 0. \tag{4.1}$$

Then there exists a positive constant C such that, for all $s \in (0,\infty)$ and $f \in L_{p_2,\phi}(\mathbb{R}^n)$,

$$\|g_s * f\|_{L_{p_1,\theta}} \le C(1 + s^{-(1/p_2 - 1/p_1)n})\|f\|_{L_{p_2,\phi}}.$$

where $\theta(x,r) = (1 + r^{(1/p_2 - 1/p_1)n})\phi(x,r)$.

Theorem 4.2 *Let* $1 \le p_1, p_2, p_3 < \infty$ *and* $1+1/p_1 = 1/p_2 + 1/p_3$. *Assume that* ϕ *satisfies* (2.1) *and* (3.6). *Let* $g \in L^{p_3}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ *and there exists a positive constant* C_0 *such that*

$$|g(x)| \le C_0 \min\left(\frac{1}{|x|^n}, \frac{1}{|x|^{n+1}}\right), \quad |\nabla g(x)| \le \frac{C_0}{|x|^{n+1}} \quad \text{for} \quad x \ne 0. \tag{4.2}$$

Then there exists a positive constant C *such that, for all* $s \in (0,\infty)$ *and* $f \in \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$,

$$\|g_s * f\|_{\mathcal{L}_{p_1,\theta}} \le C(1 + s^{-(1/p_2 - 1/p_1)n})\|f\|_{\mathcal{L}_{p_2,\phi}},$$

where $\theta(x,r) = (1+r^{(1/p_2 - 1/p_1)n})\phi(x,r)$. *Moreover, assume that there exists a positive constant* C_ϕ *such that, for all* $x \in \mathbb{R}^n$, $\int_1^\infty (\phi(x,t)/t)dt \le C_\phi$. *Then there exists a positive constant* C *such that, for all* $s \in (0,\infty)$ *and* $f \in \mathcal{L}^\natural_{p_2,\phi}(\mathbb{R}^n)$,

$$\|g_s * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le C(1 + s^{-(1/p_2 - 1/p_1)n})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}.$$

Further, if $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \int_r^\infty (\phi(x,t)/t)dt = 0$, *then* $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$ *implies* $\sigma(g_s * f) = \lim_{r\to\infty} (g_s * f)_{B(0,r)} = 0$.

Theorem 4.3 *Let* $1 \le p_1, p_2, p_3 < \infty$ *and* $1+1/p_1 = 1/p_2 + 1/p_3$. *Assume that* ψ *satisfies* (2.1) *and* (3.6) *and that there exists a positive constant* C_ψ *such that, for all* $x \in \mathbb{R}^n$, $\int_0^\infty (\psi(x,t)/t)dt \le C_\psi$. *Assume also that* $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \int_r^\infty (\psi(x,t)/t)dt = 0$. *Let*

$$\phi(x,r) = \begin{cases} \psi(x,r) & r < 1, \\ \psi(x,r)^{p_2/p_1} & r \ge 1. \end{cases}$$

Let $g \in L^{p_3}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ *and* g *satisfy* (4.2). *Then there exists a positive constant* C *such that, for all* $s \in (0,\infty)$, *if* $f \in \mathcal{L}^\natural_{p_2,\psi}(\mathbb{R}^n)$ *and* $\sigma(f) =$

$\lim_{r\to\infty} f_{B(0,r)} = 0$, *then* $\sigma(g_s * f) = \lim_{r\to\infty}(g_s * f)_{B(0,r)} = 0$, *and*

$$\|g_s * f\|_{\mathcal{L}^{\natural}_{p_1,\phi}} \le C(1 + s^{-(1/p_2-1/p_1)n})\|f\|_{\mathcal{L}^{\natural}_{p_2,\psi}}.$$

Next we apply above theorems to the heat kernel. Then we have the following corollaries.

Corollary 4.4 *Let* $1 \le p_2 \le p_1 < \infty$. *Assume that* ϕ *satisfies* (2.1) *and* (3.6). *Let*

$$h_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \quad \textit{for} \quad x \in \mathbb{R}^n,\ t \in (0,\infty). \tag{4.3}$$

Then there exists a positive constant C *such that, for all* $t \in (0,\infty)$ *and* $f \in \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$,

$$\|h_t * f\|_{\mathcal{L}_{p_1,\theta}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}_{p_1,\theta}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}_{p_2,\phi}},$$

where $\theta(x,r) = (1+r^{(1/p_2-1/p_1)n})\phi(x,r)$. *Moreover, assume that there exists a positive constant* C_ϕ *such that, for all* $x \in \mathbb{R}^n$, $\int_1^\infty (\phi(x,t)/t)dt \le C_\phi$. *Then there exists a positive constant* C *such that, for all* $t \in (0,\infty)$ *and* $f \in \mathcal{L}^{\natural}_{p_2,\phi}(\mathbb{R}^n)$,

$$\|h_t * f\|_{\mathcal{L}^{\natural}_{p_1,\theta}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^{\natural}_{p_2,\phi}},$$
$$\|(\nabla h_t) * f\|_{\mathcal{L}^{\natural}_{p_1,\theta}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^{\natural}_{p_2,\phi}}.$$

Further, if $\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \int_r^\infty (\phi(x,t)/t)dt = 0$, *then* $\sigma(f) = 0$ *implies* $\sigma(h_t * f) = \sigma((\nabla h_t) * f) = 0$.

Corollary 4.5 *Let* $1 \le p_2 \le p_1 < \infty$. *Assume that* ϕ *and* ψ *satisfy the same conditions in Theorem 4.3. Let* h_t *be the function defined by* (4.3). *Then there exists a positive constant* C *such that, for all* $t \in (0,\infty)$, *if* $f \in \mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n)$ *and* $\sigma(f) = 0$, *then* $\sigma(h_t * f) = \sigma((\nabla h_t) * f) = 0$ *and*

$$\|h_t * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le C(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}},$$

$$\|(\nabla h_t) * f\|_{\mathcal{L}^\natural_{p_1,\phi}} \le Ct^{-1/2}(1 + t^{-(1/p_2-1/p_1)n/2})\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}.$$

To prove Theorems 4.1, 4.2 and 4.3 we state the following lemma which is proven by the same way as [23, Lemmas 6.5 and 6.6].

Lemma 4.6 *If ϕ satisfies* (2.1), *then there exists a positive constant C such that, for all balls $B(x,r)$,*

$$\int_{\mathbb{R}^n\setminus B(x,r)} \frac{|f(y)|}{|x-y|^n}\,dy \le C\int_r^\infty \frac{\phi(x,t)}{t}\|f\|_{L_{1,\phi}},$$

and

$$\int_{\mathbb{R}^n\setminus B(x,r)} \frac{|f(y)-f_{B(x,r)}|}{|x-y|^{n+1}}\,dy \le C\int_r^\infty \frac{\phi(x,t)}{t^2}\|f\|_{\mathcal{L}_{1,\phi}}.$$

Proof of Theorem 4.1. First note that, from (4.1) it follows that $|g_s(x)| \le C_0/|x|^n$ for $x \ne 0$.

Let $f \in L_{p_2,\phi}(\mathbb{R}^n)$. For any ball $B = B(z,r)$, let $f_1 = f\chi_{2B}$ and $f_2 = f - f_1$. Then we have

$$\begin{aligned}\left(\int_B |g_s * f_1(x)|^{p_1}\,dx\right)^{1/p_1} &\le \|g_s * f_1(x)\|_{L^{p_1}} \le \|g_s\|_{L^{p_3}}\|f_1\|_{L^{p_2}}\\ &= s^{-(1/p_2-1/p_1)n}\|g\|_{L^{p_3}}\left(\int_{2B}|f(x)|^{p_2}\,dx\right)^{1/p_2}\\ &\le s^{-(1/p_2-1/p_1)n}\|g\|_{L^{p_3}}|2B|^{1/p_2}\phi(2B)\|f\|_{L_{p_2,\phi}}.\end{aligned}$$

That is,

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |g_s * f_1(x)|^{p_1}\,dx\right)^{1/p_1}\\ &\quad \le Cs^{-(1/p_2-1/p_1)n}|B|^{1/p_2-1/p_1}\phi(B)\|f\|_{L_{p_2,\phi}}.\end{aligned} \tag{4.4}$$

Next, for $x \in B$, using Lemma 4.6 and (2.5), we have

$$|g_s * f_2(x)| = \left|\int_{\mathbb{R}^n} g_s(x-y)f_2(y)\,dy\right| \le C_0 \int_{\mathbb{R}^n\setminus 2B} \frac{|f(y)|}{|x-y|^n}\,dy$$
$$\le C\int_{2r}^{\infty} \frac{\phi(z,t)}{t}\|f\|_{L_{1,\phi}} \le C\phi(z,r)\|f\|_{L_{p_2,\phi}}.$$

This shows that $g_s * f$ is well defined and that

$$\left(\frac{1}{|B|}\int_B |g_s * f_2(x)|^{p_1}\,dx\right)^{1/p_1} \le C\phi(B)\|f\|_{L_{p_2,\phi}}. \tag{4.5}$$

By (4.4) and (4.5) we have the conclusion. □

Proof of Theorem 4.2. First note that, from (4.2) it follows that

$$|g_s(x)| \le C_0 \min\left(\frac{1}{|x|^n}, \frac{s}{|x|^{n+1}}\right), \quad \text{for} \quad x \ne 0,$$

and that

$$|g_s(x-y) - g_s(z-y)| \le 2^{n+1}C_0 \frac{|x-z|}{|x-y|^{n+1}} \quad \text{for} \quad |x-y| \ge 2|x-z|.$$

Let $f \in \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$. We first show that $g_s * f$ is well defined. For any $r > 0$, let $f_1 = f\chi_{B(0,2r)}$ and $f_2 = f - f_1$. Then $g_s * f_1$ is well defined, since $g_s \in L^{p_3}$ and $f_1 \in L^{p_2}(\mathbb{R}^n)$. On the other hand, for $x \in B(0,r)$,

$$|g_s * f_2(x)| \le \int_{\mathbb{R}^n} |g_s(x-y)f_2(y)|\,dy$$
$$\le C_0 s \int_{\mathbb{R}^n\setminus B(0,2r)} \frac{|f(y) - f_{B(0,2r)}| + |f_{B(0,2r)}|}{|x-y|^{n+1}}\,dy.$$

Using Lemma 4.6 and (3.6), we have

$$|g_s * f_2(x)| \le Cs\left(\int_{2r}^{\infty} \frac{\phi(0,t)}{t^2}\|f\|_{\mathcal{L}_{1,\phi}} + \frac{|f_{B(0,2r)}|}{r}\right)$$
$$\le Cs\left(\frac{\phi(0,r)}{r}\|f\|_{\mathcal{L}_{p_2,\phi}} + \frac{|f_{B(0,2r)}|}{r}\right).$$

This shows that $g_s * f$ is well defined. It is also clear that $g_t * 1$ is a constant function.

Next we estimate the norm of $g_s * f$. For any ball $B = B(z,r)$, let $\tilde{f}_1 = (f - f_{(2B)})\chi_{2B}$ and $\tilde{f}_2 = (f - f_{(2B)})\chi_{\mathbb{R}^n \setminus 2B}$. Then

$$\begin{aligned} &g_s * f(x) - g_s * f(z) + g_s * \tilde{f}_1(z) \\ &\quad = g_s * (f - f_{(2B)})(x) - g_s * (f - f_{(2B)})(z) + g_s * \tilde{f}_1(z) \\ &\quad = g_s * \tilde{f}_1(x) + g_s * \tilde{f}_2(x) - g_s * \tilde{f}_2(z). \end{aligned}$$

For the term $g_s * \tilde{f}_1$, we have

$$\begin{aligned} &\left(\int_B |g_s * \tilde{f}_1(x)|^{p_1} \, dx \right)^{1/p_1} \\ &\quad \le \|g_s * \tilde{f}_1\|_{L^{p_1}} \le \|g_s\|_{L^{p_3}} \|\tilde{f}_1\|_{L^{p_2}} \\ &\quad = s^{-(1/p_2 - 1/p_1)n} \|g\|_{L^{p_3}} \left(\int_{2B} |f(x) - f_{(2B)}|^{p_2} \, dx \right)^{1/p_2} \\ &\quad \le s^{-(1/p_2 - 1/p_1)n} \|g\|_{L^{p_3}} |2B|^{1/p_2} \phi(2B) \|f\|_{\mathcal{L}_{p_2,\phi}}. \end{aligned}$$

That is,

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B |g_s * \tilde{f}_1(x)|^{p_1} \, dx \right)^{1/p_1} \\ &\quad \le C s^{-(1/p_2 - 1/p_1)n} |B|^{1/p_2 - 1/p_1} \phi(B) \|f\|_{\mathcal{L}_{p_2,\phi}}. \end{aligned} \tag{4.6}$$

For $x \in B$, using Lemma 4.6 and (3.6), we have

$$\begin{aligned} |g_s * \tilde{f}_2(x) - g_s * \tilde{f}_2(z)| &= \left| \int_{\mathbb{R}^n} (g_s(x-y) - g_s(z-y)) \tilde{f}_2(y) \, dy \right| \\ &\le \int_{\mathbb{R}^n \setminus 2B} |g_s(x-y) - g_s(z-y)| |f(y) - f_{2B}| \, dy \\ &\le 2^{n+1} C_0 \int_{\mathbb{R}^n \setminus 2B} \frac{|x-z|}{|z-y|^{n+1}} |f(y) - f_{2B}| \, dy \end{aligned}$$

$$\leq Cr \int_r^\infty \frac{\phi(z,t)}{t^2} \|f\|_{\mathcal{L}_{1,\phi}}$$
$$\leq C\phi(z,r)\|f\|_{\mathcal{L}_{p_2,\phi}}.$$

Then,

$$\left(\frac{1}{|B|}\int_B |g_s * \tilde{f}_2(x) - g_s * \tilde{f}_2(z)|^{p_1}\,dx\right)^{1/p_1} \leq C\phi(B)\|f\|_{\mathcal{L}_{p_2,\phi}}. \tag{4.7}$$

By (4.6) and (4.7) we have

$$\left(\frac{1}{|B|}\int_B |g_s * f(x) - g_s * f(z) + g_s * \tilde{f}_1(z)|^{p_1}\,dx\right)^{1/p_1}$$
$$\leq C(1 + s^{-(1/p_2 - 1/p_1)n})(1 + |B|^{1/p_2 - 1/p_1})\phi(B)\|f\|_{\mathcal{L}_{p_2,\phi}}.$$

That is,

$$\|g_s * f\|_{\mathcal{L}_{p_1,\theta}} \leq C(1 + s^{-(1/p_2 - 1/p_1)n})\|f\|_{\mathcal{L}_{p_2,\phi}}.$$

Next we show that

$$|(g_s * f)_{B(0,1)}| \leq C\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}, \tag{4.8}$$

under the assumption that $\int_1^\infty (\phi(x,t)/t)dt \leq C_\phi$. By Remark 2.1 and an elementary calculation we have

$$\begin{aligned}|f_{B(y,1)} - f_{B(0,1)}| &= |f_{B(y,1)} - f_{B(y,1+|y|)}| + |f_{B(y,1+|y|)} - f_{B(0,1+2|y|)}| \\ &\quad + |f_{B(0,1+2|y|)} - f_{B(0,1)}| \\ &\leq \left(C\int_1^\infty \frac{\phi(y,t)+\phi(0,t)}{t}\,dt + 2^n\phi(0,1+2|y|)\right)\|f\|_{\mathcal{L}_{p_2,\phi}} \\ &\leq CC_\phi\|f\|_{\mathcal{L}_{p_2,\phi}}.\end{aligned}$$

Then

$$|f_{B(y,1)}| \leq CC_\phi\|f\|_{\mathcal{L}_{p_2,\phi}} + |f_{B(0,1)}| \leq CC_\phi\|f\|_{\mathcal{L}^\natural_{p_2,\phi}} \quad \text{for all } y \in \mathbb{R}^n,$$

and

$$\left|\frac{1}{|B(0,1)|}\int_{B(0,1)} g_s * f(x)\,dx\right| = \left|\frac{1}{|B(0,1)|}\int_{B(0,1)}\int_{\mathbb{R}^n} g_s(y)f(x-y)\,dy\,dx\right|$$
$$= \left|\int_{\mathbb{R}^n} g_s(y) f_{B(-y,1)}\,dy\right|$$
$$\le CC_\phi \|g_s\|_{L^1}\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}.$$

That is, we have (4.8) and

$$\|g_s * f\|_{\mathcal{L}^\natural_{p_1,\theta}} \le C(1+s^{-(1/p_2-1/p_1)n})\|f\|_{\mathcal{L}^\natural_{p_2,\phi}}.$$

If $\sup_{x\in\mathbb{R}^n}\int_r^\infty (\phi(x,t)/t)dt \to 0$ as $r\to\infty$ and $\sigma(f)=0$, then we have by Remark 2.1

$$|f_{B(y,r)}| = |f_{B(y,r)} - \sigma(f)| \le C\int_r^\infty \frac{\phi(y,t)}{t}\,dt\|f\|_{\mathcal{L}_{p_2,\phi}},$$

and

$$\left|\frac{1}{|B(0,r)|}\int_{B(0,r)} g_s * f(x)\,dx\right| = \left|\int_{\mathbb{R}^n} g_s(y) f_{B(-y,r)}\,dy\right|$$
$$\le \|g_s\|_{L^1}\sup_{y\in\mathbb{R}^n}\int_r^\infty \frac{\phi(y,t)}{t}\,dt\|f\|_{\mathcal{L}_{p_2,\phi}} \to 0 \quad \text{as } r\to\infty.$$

The proof is complete. □

Proof of Theorem 4.3. Let $f\in\mathcal{L}^\natural_{p_2,\psi}(\mathbb{R}^n)$. By the same way in the proof of Theorem 4.2 we see that $g_s * f$ is well defined. Next we estimate the norm of $g_s * f$. As in the proof of Theorem 4.2, for any ball $B=B(z,r)$, let $\tilde{f}_1 = (f - f_{(2B)})\chi_{2B}$ and $\tilde{f}_2 = (f - f_{(2B)})\chi_{\mathbb{R}^n\setminus 2B}$. Then, by the same way as (4.6) and (4.7) we have

$$\left(\frac{1}{|B|}\int_B |g_s * \tilde{f}_1(x)|^{p_1}\,dx\right)^{1/p_1} \le Cs^{-(1/p_2-1/p_1)n}|B|^{1/p_2-1/p_1}\psi(B)\|f\|_{\mathcal{L}_{p_2,\psi}} \tag{4.9}$$

and

$$\left(\frac{1}{|B|}\int_B |g_s * \tilde{f}_2(x) - g_s * \tilde{f}_2(z)|^{p_1}\,dx\right)^{1/p_1} \le C\psi(B)\|f\|_{\mathcal{L}_{p_2,\psi}}, \qquad (4.10)$$

respectively. On the other hand, by the assumption that $\int_0^\infty (\psi(x,t)/t)dt \le C_\psi$ we have that $f \in L^\infty(\mathbb{R}^n)$ and $\|f\|_{L^\infty} \le C\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}$, see Remark 5.1 below. Hence, we have also

$$\begin{aligned}\left(\int_B |g_s * \tilde{f}_1(x)|^{p_1}\,dx\right)^{1/p_1} &\le \|g_s * \tilde{f}_1\|_{L^{p_1}} \le \|g_s\|_{L^1}\|\tilde{f}_1\|_{L^{p_1}} \\ &= \|g\|_{L^1}\left(\int_{2B} |f(x) - f_{(2B)}|^{p_1}\,dx\right)^{1/p_1} \\ &\le C\|f\|_{L^\infty}^{1-p_2/p_1}\left(\int_{2B} |f(x) - f_{(2B)}|^{p_2}\,dx\right)^{1/p_1} \\ &\le C\|f\|_{L^\infty}^{1-p_2/p_1}|2B|^{1/p_1}\psi(2B)^{p_2/p_1}\|f\|_{\mathcal{L}_{p_2,\psi}}^{p_2/p_1}.\end{aligned}$$

That is,

$$\left(\frac{1}{|B|}\int_B |g_s * \tilde{f}_1(x)|^{p_1}\,dx\right)^{1/p_1} \le C\psi(B)^{p_2/p_1}\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}. \qquad (4.11)$$

Here we note that $\psi(z,r) \le C\int_r^{2r}(\psi(z,t)/t)dt \le CC_0$. Then $\psi(B) \le C\psi(B)^{p_2/p_1}$ and

$$\min(|B|^{1/p_2-1/p_1}\psi(B), \psi(B)^{p_2/p_1}) + \psi(B) \le C\phi(B).$$

Combining (4.9), (4.10) and (4.11), we have

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |g_s * f(x) - g_s * f(z) + g_s * \tilde{f}_1(z)|^{p_1}\,dx\right)^{1/p_1} \\ &\quad \le C(1 + s^{-(1/p_2-1/p_1)n})\phi(B)\|f\|_{\mathcal{L}^\natural_{p_2,\psi}}.\end{aligned}$$

That is,

$$\|g_s * f\|_{\mathcal{L}_{p_1,\phi}} \le C(1 + s^{-(1/p_2-1/p_1)n})\|f\|_{\mathcal{L}^{\natural}_{p_2,\psi}}.$$

Moreover, as in the proof of Theorem 4.2, we also have that

$$\|g_s * f\|_{\mathcal{L}^{\natural}_{p_1,\phi}} \le C(1 + s^{-(1/p_2-1/p_1)n})\|f\|_{\mathcal{L}^{\natural}_{p_2,\psi}}$$

and that $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$ implies $\sigma(g_s * f) = \lim_{r\to\infty} (g_s * f)_{B(0,r)} = 0$. □

5. Pointwise multiplication

Let $L^0(\mathbb{R}^n)$ be the set of all measurable functions on $\mathbb{R}^n$. Let X_1 and X_2 be subspaces of $L^0(\mathbb{R}^n)$ and $g \in L^0(\mathbb{R}^n)$. We say that g is a pointwise multiplier from X_1 to X_2 if $fg \in X_2$ for all $f \in X_1$. We denote by $\mathrm{PWM}(X_1, X_2)$ the set of all pointwise multipliers from X_1 to X_2.

For $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$, we define

$$\Phi^*(x,r) = \int_1^{\max(2,|x|,r)} \frac{\phi(0,t)}{t}\,dt, \tag{5.1}$$

$$\Phi^{**}(x,r) = \int_r^{\max(2,|x|,r)} \frac{\phi(x,t)}{t}\,dt. \tag{5.2}$$

Let $1 \le p < \infty$ and ϕ satisfy the doubling condition (2.1). Then, for $f \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and ball $B = B(x,r)$,

$$|f_B| \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}(\Phi^*(x,r) + \Phi^{**}(x,r)),$$

see [18, Lemma 3.2] or [20, Lemma 3.2]. Using Φ^* and Φ^{**}, we can characterize pointwise multipliers on $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$.

Proposition 5.1 ([20, Proposition 4.4]) *Suppose that ϕ_1 and ϕ_2 satisfy the doubling condition* (2.1)*. For ϕ_1, define Φ_1^* and Φ_1^{**} by* (5.1) *and* (5.2)*, respectively. Let $\phi_3 = \phi_2/(\Phi_1^* + \Phi_1^{**})$. If $1 \le p_2 < p_1 < \infty$ and $p_4 \ge p_1p_2/(p_1 - p_2)$, then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n)) \supset \mathcal{L}^{\natural}_{p_2,\phi_3}(\mathbb{R}^n) \cap L_{p_4,\phi_2/\phi_1}(\mathbb{R}^n),$$

$$\|g\|_{\mathrm{Op}} \le C(\|g\|_{\mathcal{L}_{p_2,\phi_3}} + \|g\|_{L_{p_4,\phi_2/\phi_1}}),$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi_1}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi_2}(\mathbb{R}^n))$.

Lemma 5.2 ([20, Lemma 3.5]) *Let* $1 \le p < \infty$. *Suppose that* ϕ *satisfies the doubling condition* (2.1). *Then*

$$\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) \subset L_{p,\Phi^*+\Phi^{**}}(\mathbb{R}^n) \quad \textit{and} \quad \|f\|_{L_{p,\Phi^*+\Phi^{**}}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Corollary 5.3 *Suppose that* ϕ *satisfies the doubling condition* (2.1). *Let* $\psi = \phi(\Phi^* + \Phi^{**})$. *If* $1 \le p_2 < p_1 < \infty$ *and* $p_4 \ge p_1p_2/(p_1 - p_2)$, *then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n)) \supset \mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n),$$
$$\|g\|_{\mathrm{Op}} \le C\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}},$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n))$. *This implies that*

$$\|fg\|_{\mathcal{L}^{\natural}_{p_2,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p_1,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}.$$

For example, we can take $p_1 = p_4 = 4$ and $p_2 = 2$.

Proof. By Lemma 5.2 we have the inclusion

$$\mathcal{L}^{\natural}_{p_2,\phi}(\mathbb{R}^n) \cap L_{p_4,\Phi^*+\Phi^{**}}(\mathbb{R}^n) \supset \mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n),$$
$$\|g\|_{\mathcal{L}^{\natural}_{p_2,\phi}} + \|g\|_{L_{p_4,\Phi^*+\Phi^{**}}} \le C\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}.$$

Then, using Proposition 5.1, we have the conclusion. □

Corollary 5.4 *Suppose that* ϕ *satisfies the doubling condition* (2.1) *and that there exists a positive constant* C_ϕ *such that, for all* $x \in \mathbb{R}^n$, $\int_0^\infty (\phi(x,t)/t)dt \le C_\phi$. *If* $1 \le p_2 < p_1 < \infty$ *and* $p_4 \ge p_1p_2/(p_1 - p_2)$, *then*

$$\mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi}(\mathbb{R}^n)) \supset \mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n),$$
$$\|g\|_{\mathrm{Op}} \le C\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}},$$

where $\|g\|_{\mathrm{Op}}$ *is the operator norm of* $g \in \mathrm{PWM}(\mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n), \mathcal{L}^{\natural}_{p_2,\phi}(\mathbb{R}^n))$*. This implies that*

$$\|fg\|_{\mathcal{L}^{\natural}_{p_2,\phi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p_1,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}. \tag{5.3}$$

Proof. From the assumption it follows that $\Phi^*+\Phi^{**} \sim 1$ and $\phi(\Phi^*+\Phi^{**}) \sim \phi$. By Corollary 5.3 we have the conclusion. □

Remark 5.1 If $\theta \equiv 1$, then $L_{p,\theta}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ and $\|f\|_{L_{p,\theta}} = \|f\|_{L^\infty}$. Hence, if $\Phi^* + \Phi^{**} \sim 1$, then, by Lemma 5.2 we have $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n) \subset L_{p,\Phi^*+\Phi^{**}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ and $\|f\|_{L^\infty} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}$. Therfore, in (5.3), if $\sigma(f)=0$ or $\sigma(g)=0$, then $\sigma(gf)=0$, where $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$.

Corollary 5.5 *Let* $1 \le p_2 < p_1 < \infty$ *and* $1/p_4 = 1/p_2 - 1/p_1$*. Suppose that* ϕ *satisfies the doubling condition* (2.1) *and that there exists a positive constant* C_ϕ *such that*

$$\int_0^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi \qquad \text{for all } x \in \mathbb{R}^n, \tag{5.4}$$

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \le C_\phi\,\phi(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \ge 1. \tag{5.5}$$

Let

$$\psi(x,r) = \begin{cases} \phi(x,r) & r<1, \\ \phi(x,r)^2 & r \ge 1. \end{cases}$$

If $f \in \mathcal{L}^{\natural}_{p_1,\phi}(\mathbb{R}^n)$, $g \in \mathcal{L}^{\natural}_{p_4,\phi}(\mathbb{R}^n)$ *and* $\sigma(f)=\sigma(g)=0$*, then* $fg \in \mathcal{L}^{\natural}_{p_2,\psi}(\mathbb{R}^n)$*,* $\sigma(fg)=0$ *and*

$$\|fg\|_{\mathcal{L}^{\natural}_{p_2,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p_1,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p_4,\phi}}. \tag{5.6}$$

Remark 5.2 In Corollary 5.5, by (2.1) there exists a positive constant C'_ϕ such that

$$\phi(x,r) \le C'_\phi \int_r^{2r} \frac{\phi(x,t)}{t}\,dt \le C'_\phi C_\phi.$$

This shows that $\phi(x,r)^2 \le C'_\phi C_\phi\, \phi(x,r)$. Then $\|fg\|_{\mathcal{L}^\natural_{p_2,\phi}} \le \|fg\|_{\mathcal{L}^\natural_{p_2,\psi}}$ and $\mathcal{L}_{p_2,\psi}(\mathbb{R}^n) \subset \mathcal{L}_{p_2,\phi}(\mathbb{R}^n)$.

Proof of Corollary 5.5. Let $f \in \mathcal{L}^\natural_{p_1,\phi}(\mathbb{R}^n)$, $g \in \mathcal{L}^\natural_{p_4,\phi}(\mathbb{R}^n)$ and $\sigma(f) = \sigma(g) = 0$. Then from Corollary 5.4 and Remark 5.1 it follows that $fg \in \mathcal{L}^\natural_{p_2,\phi}(\mathbb{R}^n)$, $\sigma(fg) = 0$ and (5.3). Hence, it is enough to prove that

$$\sup_{B=B(z,r),\ r\ge 1} \frac{1}{\phi(B)^2} \left(\frac{1}{|B|} \int_B |(fg)(x) - (fg)_B|^{p_2}\, dx \right)^{1/p_2} \le C\|f\|_{\mathcal{L}^\natural_{p_1,\phi}} \|g\|_{\mathcal{L}^\natural_{p_4,\phi}}.$$

From Remarks 2.1 and 2.2 it follows that

$$|f_{B(x,r)} - \sigma(f)| \le C \int_r^\infty \frac{\phi(x,t)}{t}\, dt\, \|f\|_{\mathcal{L}_{p_1,\phi}}.$$

Combining this inequality and the assumption (5.5), we have that, if $B = B(z,r)$ and $r \ge 1$, then $|f_B| = |f_B - \sigma(f)| \le C\phi(B)\|f\|_{\mathcal{L}_{p_1,\phi}}$ and

$$\left(\frac{1}{|B|} \int_B |f(x)|^{p_1}\, dx \right)^{1/p_1} \le \left(\frac{1}{|B|} \int_B |f(x) - f_B|^{p_1}\, dx \right)^{1/p_1} + |f_B| \le C\phi(B)\|f\|_{\mathcal{L}_{p_1,\phi}}.$$

By the same way we have $|g_B| \le C\phi(B)\|g\|_{\mathcal{L}_{p_4,\phi}}$ and

$$\left(\frac{1}{|B|} \int_B |g(x)|^{p_4}\, dx \right)^{1/p_4} \le C\phi(B)\|g\|_{\mathcal{L}_{p_4,\phi}}.$$

Using Hölder's inequality we have

$$\left(\frac{1}{|B|} \int_B |(fg)(x)|^{p_2}\, dx \right)^{1/p_2} \le C\phi(B)^2 \|f\|_{\mathcal{L}_{p_1,\phi}} \|g\|_{\mathcal{L}_{p_4,\phi}}.$$

Then, for $B = B(z,r)$ with $r \ge 1$,

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |(fg)(x)-(fg)_B|^{p_2}\,dx\right)^{1/p_2} \\
&\le 2\left(\frac{1}{|B|}\int_B |(fg)(x)-f_Bg_B|^{p_2}\,dx\right)^{1/p_2} \\
&\le 2\left(\frac{1}{|B|}\int_B |(fg)(x)|^{p_2}\,dx\right)^{1/p_2} + 2|f_Bg_B| \\
&\le C\phi(B)^2\|f\|_{\mathcal{L}_{p_1,\phi}}\|g\|_{\mathcal{L}_{p_4,\phi}}.
\end{aligned}$$

This shows the conclusion. □

6. A blowup criteria for the 3D Navier-Stokes flow

In this section we construct a blowup criteria along maximum points of the 3D-Navier-Stokes flow in terms of stationary 3D Euler flows and function spaces with variable growth condition. The most significant blowup criterion must be the Beale-Kato-Majda criterion [1]. The Beale-Kato-Majda criterion is as follows:

Theorem 6.1 *Let $s > 1/2$, and let v_0 be in the Sobolev space $H^s(\mathbb{R}^3)$ with div $v_0 = 0$ in distribution sense. Suppose that v is a strong solution of the Navier-Stokes equation* (1.1) *with $n = 3$. If*

$$\int_0^T \|\,curl\ v(t)\,\|_{L^\infty}\,dt < \infty, \tag{6.1}$$

then v can be extended to the strong solution up to some T' with $T' > T$.

This blowup criterion was further improved by Giga [12], Kozono and Taniuchi [16], the authors [27], etc. On the other hand, Constantin and Fefferman [7] (see also [8]) took into account geometric structure of the vortex stretching term in the vorticity equations to get another kind of blowup condition. They imposed vortex direction condition to the high vorticity part. This criterion was also further improved by, for example, Deng, Hou and Yu [9]. These two separate forms of criteria controlling the blow-up by magnitude and the direction of the vorticity respectively are interpolated by Chae [5]. For the detail of the blowup problem of the Navier-Stokes equation, see Fefferman [11] for example.

In this section, we give a different type of blowup criterion from them. We focus on a geometric behavior of the velocity vector field near the each maximum points (c.f. [14]). In order to state our blowup criterion, we need to give several definitions.

Let us denote a maximum point of $|v|$ at a time t as $x_M = x_{M(t)} \in \mathbb{R}^3$ (if there are several maximum points at a time t, then we choose one maximum point. We sometimes abbreviate the time t). We use rotation and transformation and bring a maximum point to the origin and its direction parallel to x_3-axis. Moreover, we decompose v into two parts: stationary 3D Euler flow part and its remainder. If the remainder part is small, then we can prove that the solution never blowup.

Let us explain precisely. We denote the unit tangent vector as

$$\tau(x_M) = \tau(x_{M(t)}) = (v/|v|)(x_{M(t)}, t),$$

and we choose unit normal vectors $n_1(x_M)$ and $n_2(x_M)$ as

$$\tau(x_M) \cdot n_1(x_M) = \tau(x_M) \cdot n_2(x_M) = n_1(x_M) \cdot n_2(x_M) = 0.$$

Note that n_1 and n_2 are not uniquely determined. We now construct a Cartesian coordinate system with a new y_1-axis to be the straight line which passes through the maximum point and is parallel to n_1, and a new y_2-axis to be the straight line which passes through the maximum point and is parallel to n_2. We set y_3-axis by τ in the same process. Here we fix the maximum point $x_M = x_{M(t_*)}$ at $t = t_*$ for some time. Then v can be expressed as

$$v(x,t) = \tilde{u}_1(x,t) n_1(x_{M(t_*)}) + \tilde{u}_2(x,t) n_2(x_{M(t_*)}) + \tilde{u}_3(x,t) \tau(x_{M(t_*)}), \quad (6.2)$$

with $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$, where

$$\begin{aligned} \tilde{u}_1(x,t) &= v(x,t) \cdot n_1(x_{M(t_*)}), \\ \tilde{u}_2(x,t) &= v(x,t) \cdot n_2(x_{M(t_*)}), \\ \tilde{u}_3(x,t) &= v(x,t) \cdot \tau(x_{M(t_*)}). \end{aligned}$$

Let $y = (y_1, y_2, y_3)$ be the coordinate representation of the point x in the coordinate system based at the maximum point which is specified by the orthogonal frame $\{n_1, n_2, \tau\}$. That is, the point $x \in \mathbb{R}^3$ can be realized as

$x = x_M + n_1(x_M)y_1 + n_2(x_M)y_2 + \tau(x_M)y_3$ with $x_M = x_{M(t_*)}$. Then we can rewrite $\tilde{u}(x) = \tilde{u}(x,t)$ to $u(y) = u(y,t) = u_{M(t_*)}(y,t)$ as

$$u_1(y) = u_1(y,t) = \tilde{u}_1(x_M + n_1(x_M)y_1 + n_2(x_M)y_2 + \tau(x_M)y_3, t),$$
$$u_2(y) = u_2(y,t) = \tilde{u}_2(x_M + n_1(x_M)y_1 + n_2(x_M)y_2 + \tau(x_M)y_3, t),$$
$$u_3(y) = u_3(y,t) = \tilde{u}_3(x_M + n_1(x_M)y_1 + n_2(x_M)y_2 + \tau(x_M)y_3, t).$$

In this case $u_1(0,t_*) = u_2(0,t_*) = 0$ and $u_3(0,t_*) = |v(x_{M(t_*)}, t_*)|$.

Since the Navier-Stokes equation is rotation and translation invariant, u also satisfies the Navier-Stokes equation (1.1) in y-valuable. Then ∇p, in y-valuable, can be expressed as

$$\nabla p = \sum_{i,j=1}^{3} R_i R_j \nabla(u_i u_j),$$

where R_j $(j = 1,2,3)$ are the Riesz transforms (see [13], [15], [26] for example). We decompose u into two parts; decaying stationary 3D smooth Euler flow part U and its remainder part r:

$$u = U + r, \quad p = P + p_r \quad \text{and} \quad u(0) = U(0).$$

The stationary 3D Euler flow part (U,P) can be defined as follows:

Definition 6.1 We say $U \in C^k(\mathbb{R}^3)$ (for sufficiently large $k \in \mathbb{Z}_+$) is a decaying stationary smooth Euler flow if U satisfies

$$(U \cdot \nabla)U = -\nabla P, \quad \nabla \cdot U = 0, \quad U_1(0) = U_2(0) = 0,$$

(some decaying condition with $\beta < 0$)

$$\frac{1}{|B(0,\rho)|}\int_{B(0,\rho)} |U(y)|dy = O(\rho^\beta) \quad \text{as} \quad \rho \to \infty,$$

$$\sup_{y\in\mathbb{R}^3} |\partial_3 U(y)| < C < \infty,$$

and $|U(0)| = |U_3(0)|$ attains its maximum value, with scalar function $P \in C^k(\mathbb{R}^3)$, where $B(0,\rho)$ is the ball centered at the origin and of radius ρ, and $|B(0,\rho)|$ is its Lebesgue measure.

Remark 6.1 Note that we have the following pressure formula:

$$\nabla P = \sum_{i,j=1}^{3} R_i R_j \nabla (U_i U_j),$$

and we see that $-\partial_3 P(0) = 0$, since the maximum value of $|U|$ attains at $y = 0$.

Remark 6.2 We easily have the following example of the 3D Euler flow (but this is essentially 2D-rotating flow):

$$\begin{cases} U(x_1, x_2, x_3) = f\left((x_2 - x_*)^2 + x_3^2\right)(0, x_3, -x_2 + x_*), \\ P(x_1, x_2, x_3) = F((x_2 - x_*)^2 + x_3^2) \end{cases}$$

with some smooth function $f : [0, \infty) \to \mathbb{R}$ with suitable decay condition, $\sup_{0 \le x < \infty} |f(x)| = |f(x_*)|$, F is a primitive function of f^2, namely, $F' = f^2/2$ and $f(x) = 0$ near $x = 0$. Clearly, $\mathrm{curl} U \times U \neq 0$, so, this is not the Beltrami flow (see [10] for example).

Thus we need to see the remainder part r, namely, we have the following pressure formula:

$$\partial_3 p = \partial_3 p_r = \sum_{i,j=1}^{3} R_i R_j \partial_3 \left(r_i U_j + U_i r_j + r_i r_j\right) \quad \text{at} \quad y = 0. \tag{6.3}$$

In this section, using the above formula, we construct a different type (from Beale-Kato-Majda type and Constantin-Fefferman type) of blowup criterion. In order to obtain a reasonable blowup condition from (6.3), we need two function spaces $V = (V, \|\cdot\|_V)$ and $W = (W, \|\cdot\|_W)$ on $\mathbb{R}^3$ such that

$$|f(0)| \le \|f\|_W, \tag{6.4}$$

$$\|R_i R_j f\|_W \le C\|f\|_W, \tag{6.5}$$

$$\|fg\|_W \le C\|f\|_V \|g\|_V. \tag{6.6}$$

That is, we need some smoothness condition at the origin for functions in W, the boundedness of Riesz transforms on W and the boundedness of pointwise multiplication operator as $V \times V \to W$. Moreover, it is known that there exist positive constants R and C such that

$$|v(x,t)| \le C/|x| \quad \text{for} \quad |x| > R, \tag{6.7}$$

where R and C are independent of $t \in [0,T)$. This is due to Corollary 1 in [2] (we use the partial regularity result to the decay). See also Section 1 in [6]. Thus r also satisfy

$$\frac{1}{|B(0,\rho)|}\int_{B(0,\rho)} |r(y)|dy \to 0 \quad (\rho \to \infty).$$

In these points of view, Campanato spaces with variable growth condition are very useful.

The following definition is the key in our result.

Definition 6.2 We say "v has stationary 3D Euler flow profile (near each maximum points)" with respect to the function space V, if there exist constants $C > 0$ and $\alpha < 2$ such that, for each fixed $x_{M(t_*)}$ at $t_* \in [0,T)$, $u = u_{M(t_*)}$ has the following property:

$$\begin{aligned} &\inf_{u=U+r}\left\{ \sum_{i,j} \left(\|\partial_3 r_i\|_V \|U_j\|_V + \|r_i\|_V \|\partial_3 U_j\|_V + \|r_i\|_V \|\partial_3 r_j\|_V\right)\Big|_{t=t_*}\right\} \\ &\quad \le C\frac{(T-t_*)^{-\alpha}}{u_3(0,t_*)}, \end{aligned} \tag{6.8}$$

where the infimum is taken over all decomposition $u = U + r$ with $u_3(0,t_*) = U_3(0,t_*)$ and stationary 3D Euler flow U.

Roughly saying, if $\|\partial_3 r_j\|_V$ and $\|r_j\|_V$ are sufficiently small compare to $\|\partial_3 U_j\|_V$ and $\|U_j\|_V$, then v is close to stationary Euler flows.

The following is the main theorem.

Theorem 6.2 (Blowup criteria along maximum points) *Let function spaces V and W satisfy* (6.4), (6.5) *and* (6.6). *Let v_0 be any non zero, smooth, divergence-free vector field in Schwartz class, that is,*

$$|\partial_x^\alpha v_0(x)| \le C_{\alpha,K}(1+|x|)^{-K} \quad \mathit{in} \quad \mathbb{R}^3$$

for any $\alpha \in \mathbb{Z}_+^3$ *and any* $K > 0$. *Suppose that* $v \in C^\infty([0,T) \times \mathbb{R}^3)$ *is a unique smooth solution of* (1.1) *up to* T. *If* v *has stationary 3D Euler flow profile with respect to* V, *then* v *can be extended to the strong solution up to some* T' *with* $T' > T$.

Remark 6.3 In the above blowup criteria, we do not need the well-known scaling argument to the original flow v anymore. For example, even if the original flow v is critical (even supercritical) in time, more precisely, if $|v(x_{M(t)}, t)| = |U_3(0,t)|$ and

$$|v(x_{M(t)}, t)| > C(T-t)^{-1/2}$$

for some $C > 0$, v satisfies Definition 6.2 provided that the remainder part r is identically zero near maximum points.

Remark 6.4 We need to mention that Grujić [14] proposed local sparseness of a one dimensional trace of the region of intense velocity vector field, and constructed a geometric measure-type regularity criterion on the super-level sets (near the maximum points) of solutions to (1.1). Thus it may be interesting to compare with the local sparseness and the 3D-stationary Euler flows.

7. Proof of the theorem on the blowup criteria

In this section we give a proof of Theorem 6.2. First we show a lemma.

Lemma 7.1 *Under the assumption of Theorem* 6.2, *there exist constants* $C > 0$ *and* $\alpha < 2$ *such that, for each fixed* $x_{M(t_*)}$, *the following inequalities hold*:

$$-(v \cdot \nabla p)(x_{M(t_*)}, t_*) \le C(T - t_*)^{-\alpha}, \tag{7.1}$$

$$(v \cdot \Delta v)(x_{M(t_*)}, t_*) \le 0. \tag{7.2}$$

Proof. Using the derivative ∂_3 along τ direction, we have

$$-(v \cdot \nabla p)(x_{M(t_*)}, t_*) = -(u_3 \partial_3 p)(0, t_*),$$

since $u_1(0,t_*) = u_2(0,t_*) = 0$. Then, by (6.3)–(6.6) and the definition of closeness to stationary Euler flows we get (7.1). To prove (7.2), it suffices to show

$$(u_3\Delta u_3)(0,t_*)\le 0,$$

where Δ is the Laplacian with respect to $y=(y_1,y_2,y_3)$. It directly follows from the fact that u_3 has a positive maximal value at $y=0$. □

For given time-dependent smooth vector field $v(x,t)$ in $t\in[0,T)$ with $\nabla\cdot v=0$, we define "trajectory" $\gamma:[0,T)\to\mathbb{R}^3$ starting at a time $\tilde{t}\in[0,T)$ and a point $\tilde{x}\in\mathbb{R}^3$:

$$\partial_t\gamma(\tilde{x},\tilde{t};t)=v(\gamma(\tilde{x},\tilde{t};t),t)\quad\text{with}\quad\gamma(\tilde{x},\tilde{t};\tilde{t})=\tilde{x}.$$

Then γ provides a diffeomorphism and the equation (1.1) can be rewritten as follows:

$$\partial_t\big(v(\gamma(\tilde{x},\tilde{t};t),t)\big)=(\Delta v-\nabla p)(\gamma(\tilde{x},\tilde{t};t),t)\quad(0<t<T)$$

with $\gamma(\tilde{x},\tilde{t};\tilde{t})=\tilde{x}\in\mathbb{R}^3$. Since v is bounded for fixed $t\in[0,T)$, we can define $X(t)\subset\mathbb{R}^3$ as the set of all maximum points of $|v(\cdot,t)|$ at a time $t\in[0,T)$, namely,

$$|v(x,t)|=\sup_{\xi\in\mathbb{R}^3}|v(\xi,t)|\text{ for }x\in X(t)\quad\text{and}$$
$$|v(x,t)|<\sup_{\xi\in\mathbb{R}^3}|v(\xi,t)|\text{ for }x\notin X(t).$$

By (6.7), $X(t)$ is a bounded set uniformly in t in a possible blowup scenario. For any $r>0$, we see that there is a barrier function $\beta(t)>0$ such that

$$|v(x,t)|+\beta(t)<\sup_{\xi\in\mathbb{R}^3}|v(\xi,t)|\quad\text{for}\quad x\notin\bigcup_{\xi\in X(t)}B(\xi,r).$$

Then, using Lemma 7.1 and the smoothness of the solution, we get the following:

Proposition 7.2 *Under the assumption of Theorem* 6.2*, for any* $\delta>0$ *and* $t_*\in[0,T)$*, there exists a time interval* $I_{t_*}=(t_*'',t_*')\cap[0,T)$ *and a radius* r_* *such that* $t_*\in I_{t_*}$ *and that the following two properties hold for all* $t'\in I_{t_*}$:

- $\bigcup_{\xi\in X(t_*)} B(\xi, r_*) \Subset \Omega(t')$, *where*

$$\Omega(t') := \big\{x \in \mathbb{R}^3 : (\Delta v \cdot v)(\gamma(x, t_*; t'), t') \le \delta, \\ (-\nabla p \cdot v)(\gamma(x, t_*; t'), t') \le \delta + C(T - t')^{-\alpha}\big\} \quad (7.3)$$

and C and α are the constants in Lemma 7.1,

- $|v(\gamma(x, t_*; t'), t')|^2 < \sup_{\xi\in\mathbb{R}^3} |v(\xi, t_*)|^2$ *for* $x \in \big(\bigcup_{\xi\in X(t_*)} B(\xi, r_*)\big)^{\complement}$.

Proof of Theorem 6.2. Note that the interval $[0, T)$ is covered by the collection $\{I_{t_*}\}_{t_*\in[0,T)}$ of relatively open intervals such that the interval I_{t_*} is as in Proposition 7.2 for $t_* \in [0, T)$. Since $[0, T)$ is a Lindelöf space, we can choose a sequence of the time intervals I_{t_j}, $j = 0, 1, 2, \ldots$ (finite or infinite), such that $[0, T) = \bigcup_j I_{t_j}$, and that $I_{t_j} = (t''_j, t'_j) \cap [0, T)$ and r_j satisfy the properties of Proposition 7.2 for $t_j \in [0, T)$. We may assume that

$$0 = t_0 < t'_0 < t'_1 < t'_2 < \cdots, \quad I_{t_{j-1}} \cap I_{t_j} \ne \emptyset,\ j = 0, 1, \ldots$$

For $t \in [t_0, t'_0)$ and $x \in \bigcup_{\xi\in X(t_0)} B(\xi, r_0)$, from the first property in Proposition 7.2 it follows that

$$\begin{aligned} |v(\gamma(x, t_0; t), t)|^2 &= \int_{t_0}^{t} \partial_{t'} |v(\gamma(x, t_0; t'), t')|^2 dt' + |v(x, t_0)|^2 \\ &= 2\int_{t_0}^{t} \partial_{t'} v \cdot v dt' + |v(x, t_0)|^2 \\ &= 2\int_{t_0}^{t} (\Delta v \cdot v - \nabla p \cdot v)\, dt' + |v(x, t_0)|^2 \\ &\le 2\left(2\delta(t - t_0) + C\int_{t_0}^{t} (T - t')^{-\alpha} dt'\right) + \sup_{\xi\in\mathbb{R}^3} |v(\xi, t_0)|^2. \end{aligned}$$

The case $x \in (\bigcup_{\xi\in X(t_0)} B(\xi, r_0))^{\complement}$ is straightforward by the second property in Proposition 7.2. Then we have

$$|v(z, t)|^2 \le 2\left(2\delta(t - t_0) + C\int_{t_0}^{t} (T - t')^{-\alpha} dt'\right) + \sup_{\xi\in\mathbb{R}^3} |v(\xi, t_0)|^2.$$

for all $t \in [t_0, t_0')$ and all $z \in \mathbb{R}^3$ with $z = \gamma(x, t_0; t)$, since γ gives a diffeomorphism. Repeating the above argument infinite times, and we finally have

$$|v(x,t)|^2 \le 2\left(2\delta t + C\int_0^t (T-t')^{-\alpha}dt'\right) + \sup_{\xi\in\mathbb{R}^3}|v(\xi,0)|^2$$

for all $t \in [0,T)$ and all $x \in \mathbb{R}^3$. This implies

$$\|v\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} < \infty.$$

Due to the classical regularity criterion (see [12] for example), we see that the solution v can be extended to the strong solution up to some T' with $T' > T$. □

8. Specific function spaces

We now give the specific function spaces V and W satisfying (6.4), (6.5) and (6.6).

For example, let $p > 2$, $-n/p \le \alpha_* < 0 < \alpha < 1$, $-n/p \le \beta < 0$, and

$$\begin{gathered}
\phi(x,r) = \begin{cases} r^{\alpha}, & |x| \le 2,\ 0 < r \le 2, \\ r^{\beta}, & |x| \le 2,\ r > 2, \\ r^{\alpha_*}, & |x| > 2,\ 0 < r \le 2, \\ r^{\beta}, & |x| > 2,\ r > 2, \end{cases} \\
\psi(x,r) = \begin{cases} r^{\alpha}, & |x| \le 2,\ 0 < r \le 2, \\ r^{\beta}, & |x| \le 2,\ r > 2, \\ r^{2\alpha_*}, & |x| > 2,\ 0 < r \le 2, \\ r^{\beta}, & |x| > 2,\ r > 2, \end{cases}
\end{gathered} \tag{8.1}$$

and take

$$W = \mathcal{L}^{\natural}_{p/2,\psi}(\mathbb{R}^n) \quad \text{and} \quad V = \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n),$$

then V and W satisfy (6.4), (6.5) and (6.6) when $n = 3$. We will check these properties in this section.

Firstly, we see that ϕ and ψ satisfy (2.1) and

$$\psi(x,r) = r^{\alpha} \quad \text{for all } B(x,r) \subset B(0,2).$$

Then, by Proposition 2.4, we have

$$\|f\|_{\mathrm{Lip}_{\alpha}(B(0,2))} \le C\|f\|_{\mathcal{L}_{p/2,\psi}},$$

and

$$\|f\|_{\mathcal{L}^{\natural}_{p/2,\psi}} \sim \|f\|_{\mathcal{L}_{p/2,\psi}} + |f(0)|.$$

This shows the property (6.4). Next, the properties (6.5) and (6.6) follows from Propositions 8.1 and 8.2 below, respectively. Therefore, if $f, g \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and $\sigma(fg) = \lim_{r\to\infty} (fg)_{B(0,r)} = 0$, then $\sigma(R_j R_k(fg)) = 0$ and

$$|(R_j R_k(fg))(0)| \le C\|R_j R_k(fg)\|_{\mathcal{L}^{\natural}_{p/2,\psi}} \le C\|fg\|_{\mathcal{L}^{\natural}_{p/2,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \|g\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

Further, let f be α-Lipschitz continuous on $B(0,2)$ and $|f(x)| \le C/|x|$ for $|x| \ge 2$. Then $\sigma(f) = 0$ and f is in $\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$, if p and β satisfy one of the following conditions:

$$\begin{cases} 2 < p < n & \text{and } -1 \le \beta < 0, \\ p = n & \text{and } -1 < \beta < 0, \\ n < p & \text{and } -n/p \le \beta < 0. \end{cases}$$

Moreover, if $\alpha_* = \beta/2 = -n/p$ also, then $-n/(p/2) = 2\alpha_* = \beta < 0$ and

$$\begin{aligned} &\|R_j R_k(fg)\|_{\mathrm{Lip}_{\alpha}(B(0,2))} + \|R_j R_k(fg)\|_{L^{p/2}} \\ &\quad \le C\|R_j R_k(fg)\|_{\mathcal{L}^{\natural}_{p/2,\psi}} \le C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}} \|g\|_{\mathcal{L}^{\natural}_{p,\phi}}, \end{aligned}$$

for all $f, g \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ satisfying $\sigma(fg) = 0$, see Proposition 2.5.

It is known that $\nabla u(t) \in L^{\infty}(\mathbb{R}^3)$ for fixed t, see [13]. Thus $\partial_3 r$ is bounded due to the assumption $|\partial_3 U| < C$. Hence $\sigma(\partial_3 r_i U_j) = \sigma(r_i \partial_3 U_j) = \sigma(r_i \partial_3 r_j) = 0$ for all i, j.

Proposition 8.1 *Let $p \geq 2$, $-n/p \leq \alpha_* < 0 < \alpha \leq 1$, $-n/p \leq \beta < 0$, and let ϕ and ψ be as (8.1). Then there exists a positive constant C such that, for all $f, g \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$,*

$$\|fg\|_{\mathcal{L}^{\natural}_{p/2,\psi}} \leq C\|f\|_{\mathcal{L}^{\natural}_{p,\phi}}\|g\|_{\mathcal{L}^{\natural}_{p,\phi}}. \tag{8.2}$$

Proof. For ϕ in (8.1), we have

$$\Phi^*(x,r) = \int_1^{\max(2,|x|,r)} \frac{\phi(0,t)}{t}\,dt = \int_1^2 t^{\alpha-1}\,dt + \int_2^{\max(2,|x|,r)} t^{\beta-1}\,dt \sim 1,$$

and

$$\begin{aligned}
1+\Phi^{**}(x,r) &= 1 + \int_r^{\max(2,|x|,r)} \frac{\phi(x,t)}{t}\,dt \\
&= 1 + \begin{cases} \int_r^2 t^{\alpha-1}\,dt, & |x| \leq 2,\ 0 < r \leq 2, \\ 0, & |x| \leq 2,\ r > 2, \\ \int_r^2 t^{\alpha_*-1}\,dt + \int_2^{|x|} t^{\beta-1}\,dt, & |x| > 2,\ 0 < r \leq 2, \\ \int_r^{\max(|x|,r)} t^{\beta-1}\,dt, & |x| > 2,\ r > 2, \end{cases} \\
&\sim \begin{cases} 1, & |x| \leq 2,\ 0 < r \leq 2, \\ r^{\alpha_*}, & |x| > 2,\ 0 < r \leq 2, \\ 1, & r > 2. \end{cases}
\end{aligned}$$

Hence

$$\phi(x,r)(\Phi^*(x,r) + \Phi^{**}(x,r)) \sim \psi(x,r) = \begin{cases} r^{\alpha}, & |x| \leq 2,\ 0 < r \leq 2, \\ r^{2\alpha_*}, & |x| > 2,\ 0 < r \leq 2, \\ r^{\beta}, & r > 2. \end{cases}$$

Then, using Corollary 5.3, we have the conclusion. □

Proposition 8.2 *Let $q > 1$, $-n/q \leq \delta < 0 < \alpha < 1$, $-n/q \leq \beta < 0$, and*

$$\psi(x,r) = \begin{cases} r^\alpha, & |x| \le 2,\ 0 < r \le 2, \\ r^\beta, & |x| \le 2,\ r > 2, \\ r^\delta, & |x| > 2,\ 0 < r \le 2, \\ r^\beta, & |x| > 2,\ r > 2. \end{cases}$$

Then the Riesz transforms $\tilde{R}_j$, $j = 1, 2, \dots, n$, are bounded on $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{q,\psi}(\mathbb{R}^n)$. That is, there exists a positive constant C such that, for all $f \in \mathcal{L}_{q,\psi}(\mathbb{R}^n)$,

$$\|\tilde{R}_j f\|_{\mathcal{L}_{q,\psi}} \le C\|f\|_{\mathcal{L}_{q,\psi}}, \quad \|\tilde{R}_j f\|_{\mathcal{L}^\natural_{q,\psi}} \le C\|f\|_{\mathcal{L}^\natural_{q,\psi}}, \quad j = 1, 2, \dots, n.$$

Moreover, if $f \in \mathcal{L}^\natural_{q,\psi}(\mathbb{R}^n)$ and $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)} = 0$, then the Riesz transforms $R_j f$, $j = 1, 2, \dots, n$, are well defined, $\sigma(R_j f) = \lim_{r\to\infty} (R_j f)_{B(0,r)} = 0$, and

$$\|R_j f\|_{\mathcal{L}^\natural_{q,\psi}} \le C\|f\|_{\mathcal{L}^\natural_{q,\psi}}, \quad j = 1, 2, \dots, n.$$

Proof. We see that ψ satisfies (2.1) and

$$r\int_r^\infty \frac{\psi(x,t)}{t^2}\,dt \le A\psi(x,r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Then we have the boundedness of $\tilde{R}_j$ on $\mathcal{L}_{q,\psi}(\mathbb{R}^n)$ and on $\mathcal{L}^\natural_{q,\psi}(\mathbb{R}^n)$ by Theorem 3.1. Let

$$\tilde{\psi}(x,r) = \tilde{\psi}(r) = \begin{cases} r^\delta, & 0 < r \le 2, \\ r^\beta, & r > 2. \end{cases}$$

Then $\tilde{\psi}$ satisfies (2.1), (2.2), (2.5) and $\psi \le \tilde{\psi}$. Therefore, by Theorem 3.4, we have the conclusion. □

9. Cauchy problem for the Navier-Stokes equation

Finally, we give an existence theorem on the Cauchy problem for the Navier-Stokes equation.

Theorem 9.1 *Let* $\max(2,n) < p < \infty$, $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *and*

$$\psi(x,r) = \begin{cases} \phi(x,r) & r < 1, \\ \phi(x,r)^2 & r \geq 1. \end{cases}$$

Assume that ϕ *and* ψ *satisfy the doubling condition* (2.1) *and* (3.6) *and that there exists a positive constant* C_ϕ *such that*

$$\int_0^\infty \frac{\phi(x,t)}{t}\,dt \leq C_\phi \qquad \text{for all } x \in \mathbb{R}^n,$$

$$\int_r^\infty \frac{\phi(x,t)}{t}\,dt \leq C_\phi\,\phi(x,r) \quad \text{for all } x \in \mathbb{R}^n \text{ and } r \geq 1,$$

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^n} \phi(x,r) = 0.$$

Assume also that there exists a growth function $\tilde{\psi}$ *such that* $\psi \leq \tilde{\psi}$, *that* $\tilde{\psi}$ *satisfies* (2.1), (2.2) *and* (2.5). *Then, for all* $u_0 \in (\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n$ *such that* $\nabla \cdot u_0 = 0$ *and* $\sigma(u_0) = \lim_{r\to\infty} (u_0)_{B(0,r)} = 0$, *there exist a positive constant* T (*depending only on the norm of initial data*) *and a unique solution* $u \in C([0,T); (\mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n))^n)$ *to* (1.1).

Proof. By Duhamel's principle we only solve the following equations:

$$u(t) = e^{t\Delta}u_0 + Gu(t),$$

$$Gu(t) = -\int_0^t e^{-(t-s)\Delta} P(u\cdot\nabla u)(s)\,ds = -\int_0^t \nabla e^{-(t-s)\Delta} P(u\otimes u)(s)\,ds,$$

where P is the Helmholtz projection; $P = (\delta_{jk} + R_j R_k)_{1\leq j,k\leq n}$.

Using Corollary 4.4 with $p_1 = p_2 = p$, we have

$$\|e^{t\Delta}u_0\|_{\mathcal{L}^{\natural}_{p,\phi}} \leq C\|u_0\|_{\mathcal{L}^{\natural}_{p,\phi}}.$$

From Remark 5.2 we see that $\psi \leq C\phi$. Then, combining Theorem 3.4, Remark 3.2, and Corollaries 4.5 and 5.5 with $p_1 = p_4 = p > 2$ and $p_2 = p/2$, we have that, if $u \in \mathcal{L}^{\natural}_{p,\phi}(\mathbb{R}^n)$ and $\sigma(u) = 0$, then $\sigma(\nabla e^{-(t-s)\Delta}P(u\otimes u)) = 0$ and

$$\begin{aligned}
&\|\nabla e^{-(t-s)\Delta}P(u\otimes u)\|_{\mathcal{L}^{\natural}_{p,\phi}}\\
&\quad\le C(t-s)^{-1/2}(1+(t-s)^{-n/(2p)})\|P(u\otimes u)\|_{\mathcal{L}^{\natural}_{p/2,\psi}}\\
&\quad\le C(t-s)^{-1/2}(1+(t-s)^{-n/(2p)})\|u\otimes u\|_{\mathcal{L}^{\natural}_{p/2,\psi}}\\
&\quad\le C(t-s)^{-1/2}(1+(t-s)^{-n/(2p)})\|u\|^2_{\mathcal{L}^{\natural}_{p,\phi}}.
\end{aligned}$$

We take the integral in time t, we have

$$\|Gu(t)\|_{\mathcal{L}^{\natural}_{p,\phi}}\le C(t^{1/2}+t^{1/2-n/(2p)})(\sup_{0<s<t}\|u(s)\|_{\mathcal{L}^{\natural}_{p,\phi}})^2.$$

Then we now apply the Picard contraction theorem with the above estimates, we have the desired existence theorem (see [17, Theorem 13.2] for example). □

For example, let $p>\max(2,n)$, $\alpha(\cdot):\mathbb{R}^n\to(0,1)$, $\beta(\cdot):\mathbb{R}^n\to[-n/p,0)$, and let

$$\phi(x,r)=\begin{cases}r^{\alpha(x)}, & 0<r\le 1,\\ r^{\beta(x)}, & r>1,\end{cases}\qquad \psi(x,r)=\begin{cases}r^{\alpha(x)}, & 0<r\le 1,\\ r^{2\beta(x)}, & r>1,\end{cases}$$

and $\tilde\psi(x,r)=r^{2\beta_+}$, where $\alpha(\cdot)$, $\beta(\cdot)$ and β_+ satisfy

$$\begin{aligned}
0<\inf_{x\in\mathbb{R}^n}\alpha(x)\le\sup_{x\in\mathbb{R}^n}\alpha(x)<1,\\
-n/p\le\inf_{x\in\mathbb{R}^n}\beta(x)\le\sup_{x\in\mathbb{R}^n}\beta(x)=\beta_+<0.
\end{aligned}$$

Then ϕ, ψ and $\tilde\psi$ satisfy the assumption in Theorem 9.1.

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