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# Lipschitz continuity of $\alpha$-harmonic functions

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**Abstract.** The aim of this paper is to discuss the Lipschitz continuity of $\alpha$-harmonic functions.

*Key words*: $\alpha$-harmonic function, $\alpha$-harmonic equation, Lipschitz continuity, majorant.

## 1. Introduction and statement of the main results

Let $\mathbb{C}$ denote the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a,r) = \{z : |z-a| < r\}$ $(r > 0)$ and $\mathbb{D}(0,r) = \mathbb{D}_r$. Also, we use the notations $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial\mathbb{D}$, the boundary of $\mathbb{D}$.

### 1.1. $\alpha$-harmonic functions

For $\alpha > -1$, a complex-valued function $f$ is said to be *$\alpha$-harmonic* if $f$ is twice continuously differentiable in $\mathbb{D}$ and satisfies the $\alpha$-harmonic equation:

$$\Delta_\alpha(f(z)) = \partial z(1 - |z|^2)^{-\alpha}\partial\overline{z} f(z) = 0 \tag{1.1}$$

in $\mathbb{D}$ (see [21, Proposition 1.5] for the reason of this restriction on $\alpha$), where

$$\partial z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \partial\overline{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Obviously, $\alpha$-harmonicity coincides with harmonicity when $\alpha = 0$. See [4] and the references therein for the properties of harmonic mappings.

Denote the associated *Dirichlet boundary value problem* of functions $f$ satisfying Equation (1.1) by

$$\begin{cases} \Delta_\alpha(f) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \mathbb{T}. \end{cases} \tag{1.2}$$

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Here the boundary data $f^*$ is a distribution on $\mathbb{T}$, i.e. $f^* \in \mathcal{D}^1(\mathbb{T})$, and the boundary condition in (1.2) is understood as $f_r \to f^* \in \mathcal{D}^1(\mathbb{T})$ as $r \to 1^-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

for $e^{i\theta} \in \mathbb{T}$ and $r \in [0, 1)$.

In [21], Olofsson and Wittsten showed that if an $\alpha$-harmonic function $f$ satisfies

$$\lim_{r \to 1^-} f_r = f^* \in \mathcal{D}^1(\mathbb{T}) \ (\alpha > -1),$$

then it has the form of a *Poisson type integral*

$$f(z) = \mathcal{P}_\alpha[f^*](z) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta \tag{1.3}$$

in $\mathbb{D}$, where

$$\mathcal{P}_\alpha(z) = \frac{(1 - |z|^2)^{\alpha+1}}{(1 - z)(1 - \overline{z})^{\alpha+1}}.$$

See [20] for related discussions in this line.

In the following, we always assume that every $\alpha$-harmonic function has such a representation which plays a key role in the discussions of this paper.

## 1.2. Main results

### 1.2.1 Lipschitz continuity

Let $D$ and $\Omega$ be domains in $\mathbb{C}$, and let $L$ be a constant. Then a mapping $f : D \to \Omega$ is said to be *L-Lipschitz* if $|f(z) - f(w)| \leq L|z - w|$ for all $z$, $w \in D$.

In [22], Pavlović proved that the quasiconformality of harmonic homeomorphisms between $\mathbb{D}$ can be characterized in terms of their bi-Lipschitz continuity ([22, Theorem 1.2]). See [2], [3], [8], [10], [11], [12], [13], [14], [17], [19] and references therein for the discussions in this topic in $\mathbb{C}$. For the discussions of this line in space, in [1], Arsenović, Kojić and Mateljević showed that the Lipschitz continuity of $\phi : \mathbb{S}^{n-1} \to \mathbb{R}^n$ implies the Lipschitz continuity of its harmonic extension $P[\phi] : \mathbb{B}^n \to \mathbb{R}^n$ provided that $P[\phi]$ is a $K$-quasiregular mapping ([1, Theorem 1]), where $\mathbb{B}^n$ (resp. $\mathbb{S}^{n-1}$) denotes the unit ball (resp. the boundary of $\mathbb{B}^n$) in $\mathbb{R}^n$ and $P$ stands for the usual Poisson kernel with respect to $\Delta$, the standard Laplacian in $\mathbb{R}^n$. As [1, Example 1] shows that the assumption "$P[\phi]$ being $K$-quasiregular" in [1, Theorem 1] is necessary. Meanwhile, by assuming that $P[\phi] : \mathbb{B}^n \to \mathbb{B}^n$ is a $K$-quasiconformal harmonic mapping with $P[\phi](0) = 0$ and $\phi \in C^{1,\alpha}$, Kalaj [9] also proved the Lipschitz continuity of $P[\phi]$ ([9, Theorem 2.1]).

The purpose of this paper is to consider the results of the above type for $\alpha$-harmonic functions. Our result is as follows.

**Theorem 1.1** *Suppose that $f$ is an $\alpha$-harmonic function in $\mathbb{D}$ with $\alpha \in (0, \infty)$ and that $f^*$ satisfies the Lipschitz condition*:

$$|f^*(e^{i\theta}) - f^*(e^{i\varphi})| \le L|e^{i\theta} - e^{i\varphi}|,$$

*where $L$ is a constant. Then for $z_1, z_2 \in \mathbb{D}$,*

$$|f(z_1) - f(z_2)| \le \mu_1 |z_1 - z_2|,$$

*where*

$$\mu_1 = L\tau_1, \quad \tau_1 = \frac{\alpha+2}{3} 2^{\alpha+1} \left(1 + 6\frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2}\right)$$

*and $\Gamma$ is the Gamma function.*

When $\alpha = 0$, in [1], the authors proved that there is a Lipschitz continuous function $f^* : \mathbb{T} \to \mathbb{R}$ such that $f = \mathcal{P}_\alpha[f^*]$ is not Lipschitz continuous. Further, we will construct an example (Example 3.1 below) to show that for any $\alpha \in (-1, 0)$, there is a Lipschitz continuous function $f^* : \mathbb{T} \to \mathbb{C}$ such that $f = \mathcal{P}_\alpha[f^*]$ is not Lipschitz continuous in $\mathbb{D}$. This demonstrates that the assumption $\alpha \in (0, \infty)$ in Theorem 1.1 is necessary. We remark that Lemma 1.6 in [21] also indicates the invalidity of Theorem 1.1 when $\alpha \in (-1, 0)$.

In fact, we can establish a more general result on Lipschitz continuity of $\alpha$-harmonic functions, which we shall discuss next.

#### 1.2.2 $\omega$-Lipschitz continuity

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. For some $\delta_0 > 0$ and

$0 < \delta < \delta_0$, a majorant $\omega$ is called *fast* if

$$\int_0^\delta \frac{\omega(t)}{t} dt \leq C\omega(\delta),$$

where $C$ is a positive constant.

Given a subset $\Omega$ of $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant $C$ such that

$$|f(z) - f(w)| \leq C\omega(|z - w|) \tag{1.4}$$

for all $z, w \in \Omega$ (cf. [5], [6]).

Let $\Omega$ be a proper subdomain of $\mathbb{C}$. We say that a function $f : \Omega \to \mathbb{C}$ belongs to the *local Lipschitz space* $loc\Lambda_\omega(\Omega)$ if (1.4) holds, whenever $z \in \Omega$ and $|z - w| < (1/2)d(z, \partial\Omega)$, where $C$ is a positive constant and $d(z, \partial\Omega)$ denotes the Euclidean distance from $z$ to the boundary $\partial\Omega$ of $\Omega$. Further, $\Omega$ is said to be a $\Lambda_\omega$*-extension domain* if $\Lambda_\omega(\Omega) = loc\Lambda_\omega(\Omega)$. The geometric characterization of $\Lambda_\omega$-extension domains was first given by Gehring and Martio [7]. Then Lappalainen [15] extended it to the general case and proved that $\Omega$ is a $\Lambda_\omega$-extension domain if and only if each pair of points $z_1$, $z_2 \in \Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ satisfying

$$\int_\gamma \frac{\omega(d(z, \partial\Omega))}{d(z, \partial\Omega)} ds(z) \leq \tau_2\omega(|z_1 - z_2|) \tag{1.5}$$

with some fixed positive constant $\tau_2 = \tau_2(\Omega, \omega)$, where $ds$ stands for the arc length measure on $\gamma$. Furthermore, from [15, Theorem 4.12], we know that $\Lambda_\omega$-extension domains exist for fast majorants $\omega$ only. It is known that $\mathbb{D}$ is a $\Lambda_\omega$-extension domain for any fast $\omega$ (cf. [6, Section 1]).

The following result establishes the $\omega$-Lipschitz continuity of $\alpha$-harmonic functions with $\omega$ being a fast majorant.

**Theorem 1.2** *Suppose that $f$ is an $\alpha$-harmonic function in $\mathbb{D}$ with $\alpha \in (0, \infty)$, that $\omega$ is a fast majorant and that $f^*$ satisfies the Lipschitz condition*

$$|f^*(e^{i\theta}) - f^*(e^{i\varphi})| \leq \omega(|e^{i\theta} - e^{i\varphi}|).$$

*Then for $z_1, z_2 \in \mathbb{D}$,*

$$|f(z_1) - f(z_2)| \leq \mu_2 \omega(|z_1 - z_2|),$$

*where* $\mu_2 = \tau_1 \tau_2$ *and* $\tau_2 = \tau_2(\mathbb{D}, \omega)$ *is from* (1.5).

We will prove Theorems 1.1 and 1.2 in Section 2, and in Section 3, an example will be constructed, which implies that Theorems 1.1 and 1.2 are no longer valid for any $\alpha \in (-1, 0)$.

## 2. Lipschitz continuity of $\alpha$-harmonic functions

The aim of this section is to prove Theorems 1.1 and 1.2. First, we need some notations.

Suppose $f = u + iv$ has both partial derivatives at $z = x + iy$ in $\Omega$, where $\Omega$ is a domain in $\mathbb{C}$, $u$ and $v$ are real functions. The Jacobian matrix of $f$ at $z$ is denoted by

$$Df(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Then

$$|Df(z)| = \sup\{|Df(z)\varsigma| : |\varsigma| = 1\} = |f_z(z)| + |f_{\overline{z}}(z)| \tag{2.1}$$

and

$$l(Df(z)) = \inf\{|Df(z)\varsigma| : |\varsigma| = 1\} = \big||f_z(z)| - |f_{\overline{z}}(z)|\big|. \tag{2.2}$$

We first prove Theorem 1.2. For this, a result from [18] is needed.

**Lemma A** ([18, Lemma 2.1]) *If* $\alpha > -1$ *and* $f^* \in C(\mathbb{T})$, *then*

$$\frac{\partial}{\partial z} \int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial z} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta$$

*and*

$$\frac{\partial}{\partial \overline{z}} \int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial \overline{z}} \mathcal{P}_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) d\theta.$$

Now, we are ready to show Theorem 1.2.

*Proof of Theorem 1.2.* From the proof of [21, Theorem 2.5], we can easily know that for $z \in \mathbb{D}$,

$$\frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta})d\theta = 1. \tag{2.3}$$

For any $z \in \mathbb{D}$ and $\varphi \in [0, 2\pi]$, it follows from (1.3) and (2.3) that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_\alpha(ze^{-i\theta})\big(f^*(e^{i\theta}) - f^*(e^{i\varphi})\big)d\theta + f^*(e^{i\varphi}),$$

and then by Lemma A

$$f_z(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{P}_\alpha(ze^{-i\theta})}{\partial z}\big(f^*(e^{i\theta}) - f^*(e^{i\varphi})\big)d\theta$$

and

$$f_{\overline{z}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \mathcal{P}_\alpha(ze^{-i\theta})}{\partial \overline{z}}\big(f^*(e^{i\theta}) - f^*(e^{i\varphi})\big)d\theta.$$

To finish the proof, we let $z = re^{i\varphi}$, where $r \in [0, 1)$. Since

$$\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{(1-|z|^2)^\alpha\left[e^{-i\theta}(1-|z|^2) - (\alpha+1)\overline{z}(1-ze^{-i\theta})\right]}{(1-ze^{-i\theta})^2(1-\overline{z}e^{i\theta})^{\alpha+1}}$$

and

$$\frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta}) = \frac{(\alpha+1)(1-|z|^2)^\alpha e^{i\theta}}{(1-\overline{z}e^{i\theta})^{\alpha+2}}, \tag{2.4}$$

it follows from the obvious fact:

$$1-|z|^2 \le |1-ze^{-i\theta}|(1+|z|)$$

that

$$\left|\frac{\partial}{\partial z}\mathcal{P}_\alpha(ze^{-i\theta})\right| \le \frac{(1-|z|^2)^\alpha(1+(\alpha+2)|z|)}{|1-\overline{z}e^{i\theta}|^{\alpha+2}}$$

and

$$\left|\frac{\partial}{\partial \overline{z}}\mathcal{P}_\alpha(ze^{-i\theta})\right| \leq \frac{(\alpha+1)(1-|z|^2)^\alpha}{|1-\overline{z}e^{i\theta}|^{\alpha+2}}.$$

Hence (2.1) leads to

$$\begin{aligned}|Df(z)| &\leq 2(\alpha+2)(1-r^2)^\alpha \frac{1}{2\pi}\int_0^{2\pi} \frac{|f^*(e^{i\theta})-f^*(e^{i\varphi})|}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta \\ &\leq 2(\alpha+2)(1-r^2)^\alpha I_1,\end{aligned}$$

where

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\omega(|e^{i\theta}-e^{i\varphi}|)}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta.$$

In order to estimate $I_1$, we split $[0,2\pi]$ into two subsets:

$$E_1 = \{\theta \in [0,2\pi] : |e^{i\theta}-e^{i\varphi}| \leq 1-r\}$$

and

$$E_2 = \{\theta \in [0,2\pi] : |e^{i\theta}-e^{i\varphi}| > 1-r\}.$$

Then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} \frac{\omega(|e^{i\theta}-e^{i\varphi}|)}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta \\ &\quad= \frac{1}{2\pi}\int_{E_1} \frac{\omega(|e^{i\theta}-e^{i\varphi}|)}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta + \frac{1}{2\pi}\int_{E_2} \frac{\omega(|e^{i\theta}-e^{i\varphi}|)}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta.\end{aligned}$$

First, we know that

$$\frac{1}{2\pi}\int_{E_1} \frac{\omega(|e^{i\theta}-e^{i\varphi}|)}{|1-ze^{-i\theta}|^{\alpha+2}}d\theta \leq \frac{2\arcsin\frac{1-r}{2}}{\pi(1-r)^{\alpha+1}}\cdot\frac{\omega(1-r)}{1-r}.$$

Second, since $|e^{i\theta}-z| \geq 1-r$ and since

$$|e^{i\theta} - e^{i\varphi}| \le |e^{i\theta} - z| + |e^{i\varphi} - z| = |e^{i\theta} - z| + 1 - r,$$

we have that

$$|e^{i\theta} - z| \ge \frac{1}{2}|e^{i\theta} - e^{i\varphi}|,$$

which, together with the assumption "$\omega(t)/t$ being non-increasing for $t > 0$", implies

$$\frac{1}{2\pi}\int_{E_2} \frac{\omega(|e^{i\theta} - e^{i\varphi}|)}{|1 - ze^{-i\theta}|^{\alpha+2}} d\theta \le \frac{2\omega(1-r)}{1-r} \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{\alpha+1}}.$$

Further, for any given $\alpha > 0$, by [21, Lemma 2.3], we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{\alpha+1}} \le \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2} \cdot \frac{1}{(1-r^2)^\alpha},$$

where $\Gamma$ is the Gamma function, and so

$$\frac{1}{2\pi}\int_{E_2} \frac{\omega(|e^{i\theta} - e^{i\varphi}|)}{|1 - ze^{-i\theta}|^{\alpha+2}} d\theta \le \frac{2\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2(1-r^2)^\alpha} \cdot \frac{\omega(1-r)}{1-r}.$$

Hence

$$\begin{aligned} |Df(z)| &\le 4(\alpha+2)(1+r)^\alpha \left( \frac{\arcsin\frac{1-r}{2}}{\pi(1-r)} + \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})^2(1+r)^\alpha} \right) \frac{\omega(1-r)}{1-r} \\ &\le \tau_1 \frac{\omega(1-r)}{1-r}, \end{aligned} \tag{2.5}$$

where $\tau_1 = ((\alpha+2)/3)2^{\alpha+1}\left(1 + 6(\Gamma(\alpha)/\Gamma((\alpha+1)/2)^2)\right)$.

Finally, given any points $z_1, z_2 \in \mathbb{D}$, let $\gamma$ denote a curve in $\mathbb{D}$ joining $z_1$ and $z_2$ and satisfying (1.5). Integrating (2.5) along $\gamma$, we obtain

$$|f(z_2) - f(z_1)| = \int_\gamma |Df(z)| ds(z) \le \tau_1 \int_\gamma \frac{\omega(1-|z|)}{1-|z|} ds(z) \le \mu_2 \omega(|z_1 - z_2|),$$

where $\mu_2 = \tau_1\tau_2$ and $\tau_2 = \tau_2(\mathbb{D}, \omega)$ is from (1.5). Hence the proof of this theorem is complete. $\square$

In order to prove Theorem 1.1, we need an auxiliary lemma, which is as follows.

**Lemma 2.1** *Let $f$ be a function which is continuously differentiable in $\mathbb{D}$. Then $f$ is $L$-Lipschitz continuous with $L > 0$ if and only if*

$$|Df(z)| \leq L$$

*in $\mathbb{D}$.*

*Proof.* Assume that $|Df(z)| \leq L$ in $\mathbb{D}$. Then for $z_1$, $z_2 \in \mathbb{D}$,

$$|f(z_2) - f(z_1)| = \left| \int_{[z_1,z_2]} f_z(z)dz + f_{\overline{z}}(z)d\overline{z} \right| \leq L|z_1 - z_2|,$$

where $[z_1, z_2]$ stands for the segment in $\mathbb{D}$ with the endpoint $z_1$ and $z_2$. Hence the sufficiency is true. To prove the necessity, we assume that $f$ is $L$-Lipschitz continuous. Let $\partial_\alpha f$ denote the directional derivative of $f$. Then

$$\left|\partial_\alpha f\right| = \left| \lim_{r\to 0} \frac{f(z + re^{i\alpha}) - f(z)}{r} \right| = \lim_{r\to 0} \frac{\left|f(z + re^{i\alpha}) - f(z)\right|}{r} \leq L.$$

We know from [16] that $|Df(z)| = \max_\alpha \left|\partial_\alpha f(z)\right|$. Hence $|Df(z)| \leq L$, as required. □

Now, we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* By taking $\omega(t) = Lt$ in Theorem 1.2, we easily see from (2.5) that

$$|Df(z)| \leq L\tau_1,$$

and then Lemma 2.1 leads to

$$|f(z_1) - f(z_2)| \leq \mu_1|z_1 - z_2|,$$

where $\mu_1 = L\tau_1$. Hence the proof of this theorem is complete. □

## 3. An example

In this section, we will construct an example to show that when $\alpha \in (-1, 0)$, Theorems 1.1 and 1.2 are no longer valid in $\mathbb{D}$.

**Example 3.1** For $b \in \mathbb{C}$ with $0 < |b| < 1$, let $f^*(e^{i\theta}) = 1/(e^{i\theta} - b)$, where $\theta \in [0, 2\pi]$. Then

(1) $f^*(e^{i\theta})$ is Lipschitz continuous in $\mathbb{T}$;
(2) $f(z) = \mathcal{P}_\alpha[f^*](z)$ is an $\alpha$-harmonic function in $\mathbb{D}$ for any $\alpha$ in $(-1, 0)$;
(3) $f(z) = \mathcal{P}_\alpha[f^*](z)$ is not Lipschitz continuous in $\mathbb{D}$ for any $\alpha$ in $(-1, 0)$.

*Proof.* Obviously, $f^*(z) = 1/(z - b)$ $(z \in \mathbb{T})$. By letting $z = e^{i\theta}$, we have

$$|Df^*(z)| = \left|\frac{df^*(e^{i\theta})}{d\theta}\right| = \left|\frac{-ie^{i\theta}}{(e^{i\theta} - b)^2}\right| \leq \frac{1}{(1 - |b|)^2}.$$

Hence we know from Lemma 2.1 that $f^*$ is Lipschitz continuous in $\mathbb{T}$.

To prove the rest two statements in the example, by (2.1) and Lemma 2.1, it is sufficient to show the unboundedness of $f_{\overline{z}}(z)$. The proof is as follows.

First, by Lemma A and (2.4), we get the following expression of $f_{\overline{z}}(z)$:

$$f_{\overline{z}}(z) = (\alpha + 1)(1 - |z|^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - be^{-i\theta})(1 - \overline{z}e^{i\theta})^{\alpha+2}}.$$

Since for any $\beta > 0$,

$$\frac{1}{(1 - z)^\beta} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \beta)}{n!\Gamma(\beta)} z^n$$

for $z \in \mathbb{D}$ (cf. [21, (1.13)]), it follows from Parseval's theorem that

$$f_{\overline{z}}(z) = (\alpha + 1)(1 - |z|^2)^\alpha \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} b^n e^{-in\theta}\right) \times \left(\sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha + 2)}{n!\Gamma(\alpha + 2)} \overline{z}^n e^{in\theta}\right) d\theta$$

$$= (\alpha+1)(1-|z|^2)^\alpha \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)} b^n \overline{z}^n$$
$$= (\alpha+1)\frac{(1-|z|^2)^\alpha}{(1-b\overline{z})^{\alpha+2}}.$$

Obviously, $f$ is an $\alpha$-harmonic function in $\mathbb{D}$, and aslo

$$|f_{\overline{z}}(z)| \to \infty$$

as $|z| \to 1$. Hence the example is proved. □

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