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Isometric realization of cross caps as formal power series and its applications

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Abstract. Two cross caps in Euclidean 3-space are said to be *infinitesimally isometric* if their Taylor expansions of the first fundamental forms coincide by taking a local coordinate system. For a given C^∞ cross cap f, we give a method to find all cross caps which are infinitesimally isomeric to f. More generally, we show that for a given C^∞ metric with singularity having certain properties like as induced metrics of cross caps (called a *Whitney metric*), there exists locally a C^∞ cross cap infinitesimally isometric to the given one. Moreover, the Taylor expansion of such a realization is uniquely determined by a given C^∞ function with a certain property (called *characteristic function*). As an application, we give a countable family of intrinsic invariants of cross caps which recognizes infinitesimal isometry classes completely.

Key words: cross cap, Whitney umbrella, positive semi-definite metric, isometric deformation, intrinsic invariant.

Introduction

Singular points of a positive semi-definite metric $d\sigma^2$ are the points where the metric is not positive definite. In the authors' previous work [4] with Hasegawa and Saji, a class of positive semi-definite metrics on 2-manifolds called 'Whitney metrics' was given. Singularities of Whitney metrics are isolated and the pull-back metrics of cross caps in Euclidean 3-space $\boldsymbol{R}^3$ are typical examples of Whitney metrics.

In [5] with Hasegawa, the authors gave three intrinsic invariants $\alpha_{2,0}$, $\alpha_{1,1}$ and $\alpha_{0,2}$ for cross caps. After that they were generalized in [4] as invari-

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ants of Whitney metrics. In this paper, we construct a series of invariants $\{\alpha_{i,j}\}_{i+j\geq 2}$ as an extension of $\alpha_{2,0}, \alpha_{1,1}$ and $\alpha_{0,2}$. This series of invariants can distinguish isometric classes of real analytic Whitney metric completely (see Section 5), and are related to the following problem:

Problem *Can each singular point of a Whitney metric locally be isometrically realized as a cross cap in $\boldsymbol{R}^3$?*

The authors expect the answer will be affirmative, under the assumption that the metric is real analytic. In fact, for real analytic cuspidal edges and swallowtails, the corresponding problems are solved affirmatively (see [4] and [6]). Moreover, the moduli of isometric deformations of a given generic real analytic germ of cuspidal edge and swallowtail singularity was completely determined in [6] and [8]. In this paper, we construct all isometric realizations of a given Whitney metric germ at its singular point as formal power series solutions of the problem. The above family of invariants $\{\alpha_{i,j}\}_{i+j\geq 2}$ corresponds to the coefficients of the Taylor expansion of a certain realization (called a 'normal cross cap') of the Whitney metric associated to a given cross cap singular point. So we can give an explicit algorithm to compute the invariants (cf. Section 5). Although it seems difficult to show the convergence of the power series, we can approximate it by C^∞ maps by applying Borel's theorem (cf. [3, Lemma 2.5 in Chapter IV]), and get our main result (cf. Theorem 1.11).

1. Preliminaries and main results

1.1. Characteristic functions of cross caps

We recall fundamental properties of cross caps (cf. [9], [2], [5], [4]). Let $f : U \to \boldsymbol{R}^3$ be a C^∞ map, where U is a domain in $\boldsymbol{R}^2$. A point p $(\in U)$ is called a *singular point* if f is not an immersion at p. Consider such a map given by

$$f_0(u, v) = (u, uv, v^2), \tag{1.1}$$

which has an isolated singular point at the origin $(0, 0)$, and is called the *standard cross cap*. A singular point p of the map $f : U \to \boldsymbol{R}^3$ is called a *cross cap* or a *Whitney umbrella* if there exist a local diffeomorphism φ on $\boldsymbol{R}^2$ and a local diffeomorphism Φ on $\boldsymbol{R}^3$ satisfying $\Phi \circ f = f_0 \circ \varphi$ such that $\varphi(p) = (0, 0)$ and $\Phi(f(p)) = (0, 0, 0)$.

Let $f : (U; u, v) \to \boldsymbol{R}^3$ be a C^∞ map such that $(u, v) = (0, 0)$ is a cross cap singular point and $f_v(0, 0) = (\partial f/\partial v)(0, 0) = 0$. Since cross cap singularities are of co-rank one, $f_u(0, 0) \neq 0$. We call the line

$$\{f(0,0) + tf_u(0,0); t \in \boldsymbol{R}\}$$

the *tangential line* at the cross cap. The plane passing through $f(0, 0)$ spanned by $f_u(0, 0)$ and $f_{vv}(0, 0)$ is called the *principal plane*. The principal plane is determined independently of the choice of the local coordinate system (u, v) satisfying $f_v(0, 0) = 0$. By definition, the principal plane contains the tangential line.

On the other hand, the plane passing through $f(0, 0)$ perpendicular to the tangential line is called the *normal plane*. The unit normal vector $\nu(u, v)$ near the cross cap at $(u, v) = (0, 0)$ can be extended as a C^∞ function of (r, θ) by setting $u = r \cos\theta$ and $v = r \sin\theta$, and the limiting normal vector

$$\nu(\theta) := \lim_{r \to 0} \nu(r\cos\theta, r\sin\theta) \in T_{f(0,0)}\boldsymbol{R}^3$$

lies in the normal plane.

We have the following normal form of f at a cross cap singularity:

Fact 1.1 (West [9]) *Let $f : (U; u, v) \to \boldsymbol{R}^3$ be a germ of cross cap at $(u, v) = (0, 0)$. Then there exist an orientation preserving isometry T and a local diffeomorphism $(x, y) \mapsto (u(x, y), v(x, y))$ such that $f(x, y) := f(u(x, y), v(x, y))$ satisfies*

$$T \circ f(x, y) = (x,\ xy + b(y),\ z(x, y)), \tag{1.2}$$

where $b(y)$ and $z(x, y)$ are smooth functions satisfying

$$\begin{gathered} b(0) = b'(0) = b''(0) = 0, \quad z(0,0) = z_x(0,0) = z_y(0,0) = 0, \\ z_{yy}(0,0) > 0. \end{gathered} \tag{1.3}$$

Moreover, if we assume

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} > 0 \tag{1.4}$$

at $(u,v) = (0,0)$, then the function germs $x = x(u,v), y = y(u,v)$, $z = z(x,y)$ and $b = b(y)$ are uniquely determined.

This special local coordinate system (x,y) is called the *canonical coordinate system* or the *normal form* of f at the cross cap singularity. In particular, the function $b(y)$ is called the *characteristic function* associated to the cross cap f. Historically, West [9] initially introduced this normal form of cross caps (see also [1]). An argorithmic approach to determine the coefficients of the Taylor expansions of $b(y)$ and $z(x,y)$ can be found in Fukui-Hasegawa [2, Proposition 2.1], which we will apply at Section 5. For the sake of the later discussions, we give here a proof of the last assertion of Fact 1.1 as follows:

Proof of the uniqueness of the normal form. Without loss of generality, we may assume that $f(0,0) = (0,0,0)$. Suppose that there exists another such normal form

$$\tilde{T} \circ \tilde{f}(\tilde{x},\tilde{y}) = \bigl(\tilde{x}, \tilde{x}\tilde{y} + \tilde{b}(\tilde{y}), \tilde{z}(\tilde{x},\tilde{y})\bigr), \tag{1.5}$$

where $\tilde{f}(\tilde{x},\tilde{y}) := f(u(\tilde{x},\tilde{y}), v(\tilde{x},\tilde{y}))$. Since $f(0,0) = (0,0,0)$, two isometries T and $\tilde{T}$ can be considered as matrices in SO(3). By (1.2) and (1.5), it holds that $T(f_x(0,0)) = \tilde{T}(\tilde{f}_{\tilde{x}}(0,0)) = \boldsymbol{e}_1$, where $\boldsymbol{e}_1 := (1,0,0)$. Since the tangential lines of f and $\tilde{f}$ coincide, we have

$$\tilde{T} \circ T^{-1}(\boldsymbol{e}_1) = \tilde{T}(f_x(0,0)) = \tilde{T}\bigl(\tilde{f}_{\tilde{x}}(0,0)\bigr) = \boldsymbol{e}_1.$$

Hence, $\boldsymbol{e}_1$ is an eigenvector of the matrix $S := \tilde{T} \circ T^{-1}$. On the other hand, by (1.2) and (1.5) again, both of $T(f_{yy}(0,0))$ and $\tilde{T}(\tilde{f}_{\tilde{y}\tilde{y}}(0,0))$ must be proportional to $\boldsymbol{e}_3 := (0,0,1)$. Since the principal planes of f and $\tilde{f}$ coincide, we have

$$\begin{aligned}\tilde{T} \circ T^{-1}(\boldsymbol{R}\boldsymbol{e}_1 + \boldsymbol{R}\boldsymbol{e}_3) &= \tilde{T}(\boldsymbol{R}f_x(0,0) + \boldsymbol{R}f_{yy}(0,0)) \\ &= \tilde{T}\bigl(\boldsymbol{R}\tilde{f}_{\tilde{x}}(0,0) + \boldsymbol{R}\tilde{f}_{\tilde{y}\tilde{y}}(0,0)\bigr) = \boldsymbol{R}\boldsymbol{e}_1 + \boldsymbol{R}\boldsymbol{e}_3.\end{aligned}$$

Since we know that $\boldsymbol{e}_1$ is an eigenvector of S, we can conclude that $\boldsymbol{e}_3$ is also an eigenvector of S. Thus $\boldsymbol{e}_2 = (0,1,0)$ is also an eigenvector of S, and we can write

$$S = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 \end{pmatrix} \qquad (\varepsilon_i = \pm 1,\ i = 1, 2).$$

Then we get the expression

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 \end{pmatrix} \begin{pmatrix} x \\ xy + b(y) \\ z(x, y) \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{x}\tilde{y} + \tilde{b}(\tilde{y}) \\ \tilde{z}(\tilde{x}, \tilde{y}) \end{pmatrix}.$$

Comparing the first components, we have

$$\varepsilon_1 x = \tilde{x}. \tag{1.6}$$

Next, comparing the second components, we have

$$\varepsilon_2(xy + b(y)) = \varepsilon_1 x\tilde{y} + \tilde{b}(\tilde{y}). \tag{1.7}$$

Substituting $x = 0$, we get $\varepsilon_2 b(y) = \tilde{b}(\tilde{y})$, and therefore $\varepsilon_2 xy = \varepsilon_1 x\tilde{y}$. So we can conclude that $\tilde{y} = \varepsilon_1\varepsilon_2 y$. By (1.4), we have $\varepsilon_2 = 1$. By comparing the third components, $\varepsilon_1 z(x, y) = \tilde{z}(\varepsilon_1 x, \varepsilon_1 y)$ holds. Hence we have

$$\varepsilon_1 z_{yy}(x, y) = \tilde{z}_{yy}(\varepsilon_1 x, \varepsilon_1 y) = \tilde{z}_{\tilde{y}\tilde{y}}(\tilde{x}, \tilde{y}) > 0.$$

Since $z_{yy}(x, y) > 0$, we can conclude that $\varepsilon_1 = 1$. In particular, we have $x = \tilde{x}$, $y = \tilde{y}$, and $z(x, y)$ coincides with $\tilde{z}(\tilde{x}, \tilde{y})$. Then (1.7) reduces to $b(y) = \tilde{b}(y)$, proving the assertion. □

In the statement of Fact 1.1, two functions b and z can be taken as real analytic if f is real analytic. The following assertion was proved in [5]:

Fact 1.2 *Under the assumption that f is real analytic, the characteristic function $b(y)$ vanishes identically if and only if the set of self-intersections of f lies in the intersection of the principal plane and the normal plane.*

Definition 1.3 Cross caps whose characteristic functions vanish identically are called *normal cross caps* (cf. [5]).

Let $C^\infty_o(\boldsymbol{R}^2)$ (resp. $C^\infty_o(\boldsymbol{R})$) be the set of C^∞ function germs at the origin o of the (u, v)-plane $\boldsymbol{R}^2$ (resp. the line $\boldsymbol{R}$). Two functions $h_1(u, v)$, $h_2(u, v) \in C^\infty_o(\boldsymbol{R}^2)$ (resp. $h_1(t), h_2(t) \in C^\infty_o(\boldsymbol{R})$) are called *jet-equivalent* (denoted by $h_1 \sim h_2$) if the Taylor series of h_1 coincides with that of h_2 at the origin. By the well-known Borel theorem (cf. [3, Lemma 2.5 in Chapter IV]), the quotient space $C^\infty_o(\boldsymbol{R}^2)/\sim$ (resp. $C^\infty_o(\boldsymbol{R})/\sim$) can be identified with the space $\boldsymbol{R}[[u,v]]$ (resp. $\boldsymbol{R}[[t]]$) of formal power series in the variables u, v (resp. t) at the origin o, that is, the formal power series

$$[h] := \sum_{k,l=0}^{\infty} \frac{\partial^{k+l} h(0,0)}{\partial u^k \partial v^l} \frac{u^k v^l}{k!l!} \qquad \left(\text{resp. } [h] := \sum_{j=0}^{\infty} \frac{d^j h(0)}{dt^j} \frac{t^j}{j!}\right) \tag{1.8}$$

represents the jet-equivalent class containing h in $C^\infty_o(\boldsymbol{R}^2)/\sim$ (resp. in $C^\infty_o(\boldsymbol{R})/\sim$). The following assertion is an immediate consequence of our main result (Theorem 1.11):

Proposition 1.4 *Let $f_j : (U; u, v) \to \boldsymbol{R}^3$ $(j = 1, 2)$ be two real analytic cross cap singularities such that the first fundamental form (i.e. the pull back of the canonical metric of $\boldsymbol{R}^3$) of f_1 coincides with that of f_2. Then, f_1 coincides with f_2 up to orientation-preserving isometries in $\boldsymbol{R}^3$ if and only if the Taylor series of their characteristic functions coincide.*

Proposition 1.4 tells us that an analytic isometric deformation of cross caps can be controlled by the corresponding deformation of characteristic functions. Examples of isometric deformations of cross caps are constructed in [5] (cf. Figure 1). By the definition of normal cross caps (cf. Definition 1.3), we get the following corollary:

Corollary 1.5 (The rigidity of normal cross caps) *Two germs of real analytic normal cross caps are congruent if and only if they have the same first fundamental form.*

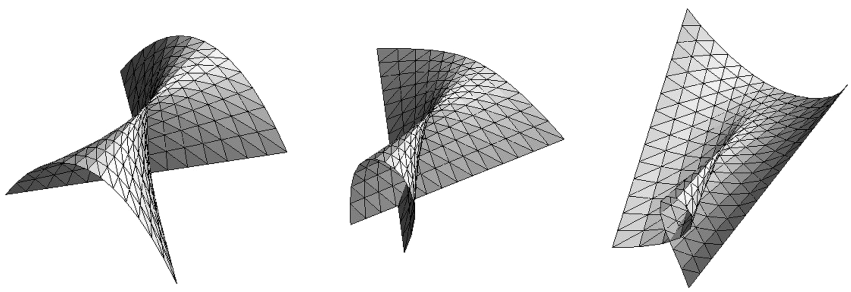

Figure 1. An isometric deformation of the standard cross cap.

Corollary 1.5 suggests us the following:

Question *Can a given cross cap germ in $\boldsymbol{R}^3$ be isometrically deformed into a normal cross cap?*

If the answer to the problem in the introduction is affirmative, so it is for the above question. Since the standard cross cap (cf. (1.1)) is normal, the deformation of the standard cross cap in Figure 1 can be re-interpreted as a normalization of the rightmost cross cap to the normal cross cap (i.e. the leftmost cross cap). We give here another example:

Example 1.6 We consider a cross cap germ

$$f_1(u,v) = \left(u, uv + \frac{v^3}{6}, \frac{u^2}{2} + \frac{v^2}{2}\right).$$

Here, (u, v) gives the canonical coordinate system at $(0, 0)$ (see Figure 2, left). Since $b \neq 0$, this cross cap is not normal. We suppose that there exists a real analytic germ f_2 of a normal cross cap which is isometric to f_1. By Corollary 1.5, we know the uniqueness of f_2. Moreover, for a given positive integer n, we can determine the coefficients of its Taylor expansion of order at most n using our algorithm as in the proof of Theorem 1.11. Figure 2, right is an approximation of f_2 by setting $n = 10$. The main difference between the figures of f_1 and f_2 appears on the set of self-intersection. The set of self-intersection of the figure of f_2 consists of a straight line perpendicular to the tangential direction of the surface at $(0, 0)$.

1.2. Whitney metrics

We fix a 2-manifold M^2, and a positive semi-definite metric $d\sigma^2$ on M^2. A point $p \in M^2$ is called a *singular point* of the metric $d\sigma^2$ if the metric is not positive definite at p.

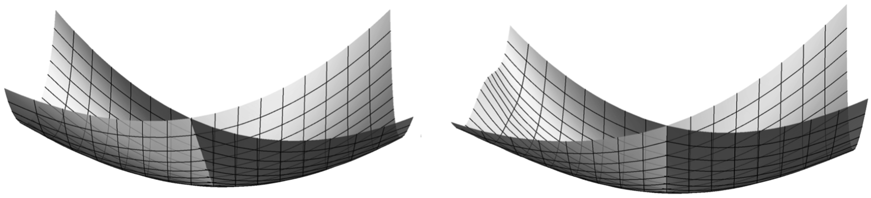

Figure 2. Example 1.6: The cross cap f_1 (left) and an approximation of its corresponding normal cross cap f_2 (right).

Let p be a singular point of $d\sigma^2$, and (u,v) a local coordinate system centered at p, and assume that the null space of $d\sigma^2$ at p is one-dimensional. We set

$$d\sigma^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{1.9}$$

The local coordinate system (u,v) is called *admissible* if $\partial/\partial v$ is a null direction of the metric $d\sigma^2$ at the origin, that is, it holds that $F(0,0) = G(0,0) = 0$. Since the nullity of $d\sigma^2$ is 1 at the origin, $E(0,0) > 0$ holds.

Definition 1.7 A singular point p of the metric $d\sigma^2$ is called *admissible*[1] if there exists an admissible local coordinate system (u,v) centered at p satisfying $E > 0$ and

$$E_v = 2F_u, \qquad G_u = G_v = 0$$

at the origin. If each singular point of $d\sigma^2$ is admissible, then $d\sigma^2$ is called *admissible.*

Definition 1.8 ([4]) Let p be a singular point of an admissible (positive semi-definite) metric $d\sigma^2$ on M^2 in the sense of Definition 1.7. Let (u,v) be an admissible local coordinate system centered at p and set

$$\delta := EG - F^2,$$

where E, F, G are functions satisfying (1.9). If the Hessian

$$\mathrm{Hess}_{u,v}(\delta) := \det \begin{pmatrix} \delta_{uu} & \delta_{uv} \\ \delta_{uv} & \delta_{vv} \end{pmatrix}$$

does not vanish at p, then p is called an *intrinsic cross cap* of $d\sigma^2$. Moreover, if $d\sigma^2$ admits only intrinsic cross cap singularities on M^2, then it is called a *Whitney metric* on M^2.

The definition of intrinsic cross caps is independent of the choice of admissible coordinate systems. A Gauss-Bonnet type formula for Whitney metrics is given in [4]. The following fact is important:

[1] Admissibility was originally introduced by Kossowski [7]. He called it $d(\langle,\rangle)$-*flatness.* Our definition of admissibility is equivalent to the original one, see [4, Proposition 2.7].

Fact 1.9 ([4]) *Let $f : (U; u, v) \to \boldsymbol{R}^3$ be a C^∞ map. Then $p \in U$ is a cross cap singular point of f if and only if p is an intrinsic cross cap of the first fundamental form of f.*

Definition 1.10 Two metrics $d\sigma_j^2$ $(j = 1, 2)$ defined on a neighborhood of $p \in M^2$ are called *jet-equivalent* at p if, for each local coordinate system (u, v) of M^2 centered at p,

$$[E_1] = [E_2], \quad [F_1] = [F_2], \quad [G_1] = [G_2]$$

hold at $(0, 0)$ (see (1.8) for the definition of the bracket $[\ \ ]$), where

$$d\sigma_j^2 = E_j\, du^2 + 2F_j\, du\, dv + G_j\, dv^2 \qquad (j = 1, 2).$$

We write $d\sigma_1^2 \approx d\sigma_2^2$ if two metrics are jet-equivalent.

The following is the main result of this paper:

Theorem 1.11 *Let p be a singular point of a Whitney metric $d\sigma^2$. For any choice of C^∞ function germ $b \in C_o^\infty(\boldsymbol{R})$ satisfying $b(0) = b'(0) = b''(0) = 0$, there exist a local coordinates (x, y) centered at $(0, 0)$ and a C^∞ map germ $f(x, y)$ into $\boldsymbol{R}^3$ having a cross cap singularity at p satisfying the following two properties*:

(1) *$f(x, y)$ is a normal form of cross cap,*
(2) *the first fundamental form of f (i.e. the pull-back of the canonical metric of $\boldsymbol{R}^3$ by f) is jet-equivalent to $d\sigma^2$ at p,*
(3) *the characteristic function of f is jet-equivalent to b, that is, it has the same Taylor expansion at 0 as b.*

Moreover, such an f is uniquely determined up to addition of flat functions[2] *at p. In other words, the Taylor expansion of f gives a unique formal power series solution for the realization problem of the Whitney metric $d\sigma^2$ as a cross cap.*

If the problem in the introduction is affirmative, then the set of analytic cross cap germs which have the same first fundamental form can be identified with the set of convergent power series in one variable.

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[2] A C^∞ function $h(u, v)$ is called *flat* (at p) if $\partial^{k+l} h(p)/\partial u^k \partial v^l$ vanishes at p for all non-negative integers k, l.

Proof of Proposition 1.4. The uniqueness of f modulo flat functions and the second assertion of Theorem 1.11 immediately imply Proposition 1.4, by setting $d\sigma^2 = f_1^* ds^2_{\boldsymbol{R}^3}$, where $ds^2_{\boldsymbol{R}^3}$ is the canonical metric of the Euclidean 3-space $\boldsymbol{R}^3$. □

1.3. The strategy of the proof of Theorem 1.11

From now on, we fix a Whitney metric

$$d\sigma^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

defined on a neighborhood U of the origin $o = (0,0)$ in the (u,v)-plane $\boldsymbol{R}^2$. We suppose that o is a singular point of $d\sigma^2$. We set

$$\mathcal{E} := [E], \qquad \mathcal{F} := [F], \qquad \mathcal{G} := [G],$$

that is, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the formal power series in $\boldsymbol{R}[[u,v]]$ associated to the coefficients of the metric $d\sigma^2$.

Definition 1.12 A formal power series

$$P := \sum_{k,l=0}^{\infty} \frac{P(k,l)}{k!l!} u^k v^l \qquad (P(k,l) \in \boldsymbol{R}) \tag{1.10}$$

in $\boldsymbol{R}[[u,v]]$ is said to be of *order at least* m if

$$P(k,l) = 0 \qquad (k + l < m).$$

We denote by $\mathcal{O}_m$ the ideal of $\boldsymbol{R}[[u,v]]$ consisting of series of order at least m. By definition, $\mathcal{O}_0 = \boldsymbol{R}[[u,v]]$. In [4], the following assertion was given:

Fact 1.13 ([4, Theorem 4.11]) *One can choose a local coordinate system (u,v) centered at the singular point of $d\sigma^2$ so that*

$$\mathcal{E}\ (= [E]) = 1 + a_{2,0}^2 u^2 + 2a_{2,0}a_{1,1}uv + (1 + a_{1,1}^2)v^2 + O_3(u,v), \tag{1.11}$$

$$\begin{aligned}\mathcal{F}\ (= [F]) = {} & a_{2,0}a_{1,1}u^2 + (a_{2,0}a_{0,2} + a_{1,1}^2 + 1)uv \\ & + a_{1,1}a_{0,2}v^2 + O_3(u,v),\end{aligned} \tag{1.12}$$

$$\mathcal{G}\ (=[G]) = (1+a_{1,1}^2)u^2 + 2a_{1,1}a_{0,2}uv + a_{0,2}^2v^2 + O_3(u,v), \tag{1.13}$$

where $a_{0,2}$ (>0) *and* $a_{2,0}$, $a_{1,1}$ *are real numbers*[3], *and* $O_3(u,v)$ *is a certain element of* $\mathcal{O}_3$.

So we can assume that our local coordinate system (u,v) satisfies (1.11), (1.12) and (1.13). We write these formal power series as follows:

$$\mathcal{E} = \sum_{k,l=0}^{\infty} \frac{\mathcal{E}(k,l)}{k!l!}u^kv^l \qquad (\mathcal{E}(k,l) \in \boldsymbol{R}),$$

$$\mathcal{F} = \sum_{k,l=0}^{\infty} \frac{\mathcal{F}(k,l)}{k!l!}u^kv^l \qquad (\mathcal{F}(k,l) \in \boldsymbol{R}),$$

$$\mathcal{G} = \sum_{k,l=0}^{\infty} \frac{\mathcal{G}(k,l)}{k!l!}u^kv^l \qquad (\mathcal{G}(k,l) \in \boldsymbol{R}).$$

For example, by (1.11), $\mathcal{E}(2,0) = 2a_{2,0}^2$ holds. We now fix a C^∞ function germ $b(t)$ satisfying $b(0) = b'(0) = b''(0) = 0$.

Lemma 1.14 *Let* $f = (x,\ xy+b(y),\ z(x,y))$ *be a* C^∞ *map germ that gives a normal form of cross cap at* $(0,0)$ *satisfying* (1.4) *whose first fundamental form coincides with the Whitney metric* $d\sigma^2$. *Suppose that* (u,v) *satisfies* (1.11), (1.12) *and* (1.13). *Then,* f *can be rewritten as*

$$f(u,v) = \big(x(u,v),\ x(u,v)y(u,v) + b(y(u,v)),\ z(u,v)\big), \tag{1.14}$$

where $z(u,v) := z(x(u,v), y(u,v))$ *and*

$$\begin{aligned} x(0,0) = y(0,0) = 0, \quad z(0,0) = z_u(0,0) = z_v(0,0) = 0,& \\ z_{vv}(0,0) > 0,& \end{aligned} \tag{1.15}$$

$$x_u(0,0) = \pm 1, \quad x_v(0,0) = 0, \quad x_u(0,0)y_v(0,0) > 0 \tag{1.16}$$

hold.

Proof. (1.15) follows from the fact that f is a normal from. Then, we have

[3]As shown in [4], $a_{2,0}$, $a_{1,1}$ and $a_{0,2}$ are invariants of the Whitney metric $d\sigma^2$.

$f_u(0,0) = (x_u(0,0), 0, 0)$. In particular, (the dot denotes the inner product of $\boldsymbol{R}^3$)

$$1 = \mathcal{E}(0,0) = f_u(0,0) \cdot f_u(0,0) = x_u(0,0)^2$$

holds and $x_u(0,0) = \pm 1$. On the other hand, we have

$$0 = \mathcal{F}(0,0) = f_u(0,0) \cdot f_v(0,0) = x_u(0,0) x_v(0,0),$$

and we get $x_v(0,0) = 0$. By (1.4),

$$0 < \left. \frac{\partial(x,y)}{\partial(u,v)} \right|_{(u,v)=(0,0)} = x_u(0,0) y_v(0,0)$$

holds, proving the assertion. □

Replacing (u,v) by $(-u,-v)$ if necessary, we may assume that

$$x_u(0,0) = 1, \qquad y_v(0,0) > 0. \tag{1.17}$$

The map f as in Lemma 1.14 satisfies

$$\begin{aligned} f_u \cdot f_u &= (1+y^2) x_u^2 + 2\bigl(x + b'(y)\bigr) y x_u y_u \\ &\quad + \bigl(x^2 + 2x b'(y) + b'(y)^2\bigr) y_u^2 + z_u^2, \end{aligned} \tag{1.18}$$

$$\begin{aligned} f_u \cdot f_v &= (1+y^2) x_u x_v + \bigl(x + b'(y)\bigr) y (x_u y_v + x_v y_u) \\ &\quad + \bigl(x^2 + 2x b'(y) + b'(y)^2\bigr) y_u y_v + z_u z_v, \end{aligned} \tag{1.19}$$

$$\begin{aligned} f_v \cdot f_v &= (1+y^2) x_v^2 + 2\bigl(x + b'(y)\bigr) y x_v y_v \\ &\quad + \bigl(x^2 + 2x b'(y) + b'(y)^2\bigr) y_v^2 + z_v^2, \end{aligned} \tag{1.20}$$

where

$$b'(y) := \frac{db(y)}{dy}. \tag{1.21}$$

Definition 1.15 We call the functions

$$x_u^2, \quad y^2 x_u^2, \quad xyx_uy_u, \quad b'(y)yx_uy_u, \quad x^2y_u^2,$$
$$xb'(y)y_u^2, \quad b'(y)^2y_u^2, \quad z_u^2, \tag{1.22}$$

obtained by expanding the right-hand side of (1.18), the *terms of* $f_u \cdot f_u$. Similarly, the *terms of* $f_u \cdot f_v$ (resp. the *terms of* $f_v \cdot f_v$) are also defined.

Let $\pi_m : \boldsymbol{R}[[u,v]] \to \boldsymbol{R}[u,v]$ be a homomorphism defined by

$$\pi_m(P) := \sum_{i+j \le m} \frac{P(i,j)}{i!j!} u^i v^j, \tag{1.23}$$

where $\boldsymbol{R}[u,v]$ is the set of the real polynomial ring in two variables. We set

$$X := [x] = \sum_{k,l=0}^{\infty} \frac{X(k,l)}{k!l!} u^k v^l, \tag{1.24}$$

$$Y := [y] = \sum_{k,l=0}^{\infty} \frac{Y(k,l)}{k!l!} u^k v^l, \tag{1.25}$$

$$Z := [z] = \sum_{k,l=0}^{\infty} \frac{Z(k,l)}{k!l!} u^k v^l. \tag{1.26}$$

Proposition 1.16 *Let $f(u,v)$ be a germ of cross cap satisfying the properties of Lemma* 1.14 *and* (1.17)*. Then, for each $m \ge 2$, the m-th order coefficients*

$$\mathcal{E}(i,j), \quad \mathcal{F}(i,j), \quad \mathcal{G}(i,j) \qquad (i+j=m)$$

of $\mathcal{E}, \mathcal{F}, \mathcal{G}$ (cf. (1.11), (1.12), (1.13)*) can be expressed as polynomials in the variables*

$$X(k_1,l_1), \quad Y(k_2,l_2), \quad Z(k_3,l_3),$$

where

$$k_1 + l_1 \le m+1, \quad k_2 + l_2 \le m-1, \quad k_3 + l_3 \le m.$$

Proof. Since, the first fundamental form of f is $d\sigma^2$, we have

$$\mathcal{E} = [f_u \cdot f_u], \quad \mathcal{F} = [f_u \cdot f_v], \quad \mathcal{G} = [f_v \cdot f_v].$$

By (1.15), (1.18), (1.19) and (1.20), the assertion can be proved using the following lemma (i.e. Lemma 1.17). □

Lemma 1.17 *Each m-th order coefficient of the power series associated to $b'(y(u,v))$ can be expressed as a polynomial in the variables $Y(k,l)$ $(k+l \le m-1)$.*

Proof. We can write (cf. (1.21))

$$\mathcal{B}'(t) := [b'(t)] = \sum_{r=2}^{\infty} \frac{b_{r+1}}{r!} t^r, \tag{1.27}$$

where

$$(\mathcal{B}(t) :=)[b(t)] = \sum_{r=3}^{\infty} \frac{b_r}{r!} t^r. \tag{1.28}$$

Since $Y \in \mathcal{O}_1$ and the index satisfies $r \ge 2$, each coefficient of $\pi_m(Y^r)$ is a polynomial in the variables $Y(k,l)$ $(k+l \le m-1)$. □

Regarding the statement of Proposition1.16, we consider the following three polynomials in u, v:

$$\mathcal{X}_{m+1} := u + \sum_{2 \le k+l \le m+1} \frac{\hat{X}(k,l)}{k!l!} u^k v^l, \tag{1.29}$$

$$\mathcal{Y}_{m-1} := \sum_{1 \le k+l \le m-1} \frac{\hat{Y}(k,l)}{k!l!} u^k v^l, \tag{1.30}$$

$$\mathcal{Z}_m := \sum_{2 \le k+l \le m} \frac{\hat{Z}(k,l)}{k!l!} u^k v^l. \tag{1.31}$$

We set

$$f^m := \big(\mathcal{X}_{m+1}, \mathcal{X}_{m+1}\mathcal{Y}_{m-1} + \mathcal{B}_{m+1}(\mathcal{Y}_{m-1}), \mathcal{Z}_m\big),$$

$$\mathcal{B}_{m+1}(t) := \sum_{j=2}^{m+1} \frac{b_{j+1}}{j!} t^j, \qquad (1.32)$$

where $\{b_j\}_{j=3}^{\infty}$ is a sequence determined by (1.28). By definition, each coefficient of f^m is a polynomial in the variables u and v. To describe the key assertion (cf. Proposition 1.20), we prepare a terminology as follows:

Definition 1.18 A triple of polynomials $(\mathcal{X}_{m+1}, \mathcal{Y}_{m-1}, \mathcal{Z}_m)$ as in (1.29), (1.30) and (1.31) are called the *m-th formal solution* with respect to $d\sigma^2$ and $b \in C^\infty_o(\boldsymbol{R})$ if they satisfy (cf. (1.15) and (1.17))

$$\hat{X}(0,0) = \hat{Y}(0,0) = 0, \quad \hat{X}(1,0) = 1, \quad \hat{Y}(0,1) > 0,$$

$$\hat{Z}(0,0) = \hat{Z}(1,0) = \hat{Z}(0,1) = 0, \quad \hat{Z}(0,2) > 0$$

and

$$\pi_m(\mathcal{E}) \sim f^m_u \cdot f^m_u, \quad \pi_m(\mathcal{F}) \sim f^m_u \cdot f^m_v, \quad \pi_m(\mathcal{G}) \sim f^m_v \cdot f^m_v. \qquad (1.33)$$

As a consequence of Proposition 1.16, we get the following:

Corollary 1.19 *Let $f(u,v)$ be a germ of cross cap satisfying the properties of Lemma* 1.14 *and* (1.17). *Then, for each $m \geq 2$, the m-th formal solution f^m satisfies $\pi_m([f]) = \pi_m(f^m)$, that is,*

$$\pi_{m+1}([x]) = \mathcal{X}_{m+1}, \quad \pi_{m-1}([y]) = \mathcal{Y}_{m-1}, \quad \pi_m([z]) = \mathcal{Z}_m$$

hold, where $\mathcal{X}_{m+1}, \mathcal{Y}_{m-1}, \mathcal{Z}_m$ are power series associated with the m-th formal solution f^m (cf. (1.32)). In particular, we get the following relations

$$X(k,l) = \hat{X}(k,l), \quad (k+l \leq m+1),$$

$$Y(k,l) = \hat{Y}(k,l), \quad (k+l \leq m-1),$$

$$Z(k,l) = \hat{Z}(k,l), \quad (k+l \leq m).$$

The key assertion, which we would like to prove in Section 4, is stated as follows:

Proposition 1.20 *Let (u,v) be a local coordinate system satisfying* (1.11), (1.12) *and* (1.13). *Then for each $m \geq 2$, there exists a unique m-th formal solution. Moreover, it satisfies $\pi_{m-1}(f^m) = \pi_{m-1}(f^{m-1})$.*

We prove here the case $m = 2$ of the proposition:

Lemma 1.21 *Let (u,v) be a local coordinate system satisfying* (1.11), (1.12) *and* (1.13). *Then there exists a unique second formal solution. More precisely it has the following expressions*:

$$\mathcal{X}_3 = u, \qquad \mathcal{Y}_1 = v, \qquad \mathcal{Z}_2 = \frac{1}{2}(a_{2,0}u^2 + 2a_{1,1}uv + a_{0,2}v^2).$$

In particular,

$$x = u + O_4(u,v), \tag{1.34}$$

$$y = v + O_2(u,v), \tag{1.35}$$

$$z = \frac{1}{2}(a_{2,0}u^2 + 2a_{1,1}uv + a_{0,2}v^2) + O_3(u,v) \tag{1.36}$$

hold, where x, y, z are the functions as in (1.14), *and $O_j(u,v)$ ($j = 2,3,4$) are functions belonging to the set $\mathcal{O}_j$.*

Proof. By a straightforward calculation using $b(0) = b'(0) = b''(0) = 0$, we have

$$[f_u \cdot f_u] = 1 + 2uX(2,0) + 2vX(1,1) + O_2(u,v).$$

Since $1 = \mathcal{E} = [f_u \cdot f_u] + O_2(u,v)$, we can conclude that $X(2,0) = X(1,1) = 0$. Similarly, using $z_u(0,0) = z_v(0,0) = 0$, we have

$$0 = \mathcal{F} = [f_u \cdot f_v] = vX(0,2) + O_2(u,v).$$

In particular, $X(0,2) = 0$. Using the fact $X(j,k) = 0$ ($j + k = 2$), we have

$$\begin{aligned}[f_u \cdot f_u] = 1 &+ u^2 \left(X(3,0) + 4Y(1,0)^2 + Z(2,0)^2\right) \\ &+ 2uv\left(X(2,1) + 2Y(0,1)Y(1,0) + Z(1,1)Z(2,0)\right) \\ &+ v^2\left(X(1,2) + Y(0,1)^2 + Z(1,1)^2\right) + O_3(u,v).\end{aligned}$$

By the first equation of (1.33), we have

$$X(3,0) + 4Y(1,0)^2 + Z(2,0)^2 = a_{2,0}^2, \tag{1.37}$$

$$X(2,1) + 2Y(0,1)Y(1,0) + Z(1,1)Z(2,0) = a_{2,0}a_{1,1}, \tag{1.38}$$

$$X(1,2) + Y(0,1)^2 + Z(1,1)^2 = 1 + a_{1,1}^2. \tag{1.39}$$

On the other hand, by the second equation of (1.33), we have

$$\frac{1}{2}X(2,1) + 2Y(0,1)Y(1,0) + Z(1,1)Z(2,0) = a_{2,0}a_{1,1}, \tag{1.40}$$

$$X(1,2) + Y(0,1)^2 + Z(1,1)^2 + Z(0,2)Z(2,0) = 1 + a_{1,1}^2 + a_{2,0}a_{0,2}, \tag{1.41}$$

$$X(0,3) + 2Z(0,2)Z(1,1) = 2a_{1,1}a_{0,2}. \tag{1.42}$$

Similarly, the third equation of (1.33) yields

$$Y(0,1)^2 + Z(1,1)^2 = 1 + a_{1,1}^2, \tag{1.43}$$

$$Z(0,2)Z(1,1) = a_{1,1}a_{0,2}, \tag{1.44}$$

$$Z(0,2)^2 = a_{0,2}^2. \tag{1.45}$$

Since $Z(0,2)$ and $a_{0,2}$ are positive (cf. Fact 1.13 and (1.15)), (1.45) reduces to $Z(0,2) = a_{0,2}$. Then (1.44) yields that $Z(1,1) = a_{1,1}$. Moreover, (1.43) reduces to $Y(0,1) = 1$ because of $Y(0,1) > 0$ (cf. (1.17)). On the other hand, (1.42) implies $X(0,3) = 0$. Also $X(1,2) = 0$ follows from (1.39). Then (1.41) yields $Z(2,0) = a_{2,0}$. Finally, (1.38) and (1.40) reduce to

$$X(2,1) + 2Y(1,0) = 0, \quad X(2,1) + 4Y(1,0) = 0.$$

So we have $X(2,1) = Y(1,0) = 0$. Moreover, (1.37) yields $X(3,0) = 0$. By Corollary 1.19, we have

$$X(i,j) = \hat{X}(i,j) \qquad (i+j \le 3),$$

$$Y(i,j) = \hat{Y}(i,j) \qquad (i+j \le 1),$$

$$Z(i,j) = \hat{Z}(i,j) \qquad (i+j \le 2),$$

and get the assertion. □

An outline of the proof of Proposition 1.20. By Lemma 1.21, the second formal solution f^2 is found. So we prove the case for $m(\geq 3)$ by induction. Suppose that f^{m-1} has been uniquely determined from the equations obtained by at most $(m-1)$-th order terms of (1.33). We then try to find the m-th order solution f^m of (1.33). If it exists, (1.33) induce a $3m$-family of equations which can be considered as a system of linear equations with unknown $3m$-variables

$$X_{m+1} := \{X(j,k)\}_{j+k=m+1}, \tag{1.46}$$

$$Y_{m-1} := \{Y(j,k)\}_{j+k=m-1}, \tag{1.47}$$

$$Z_m := \{Z(j,k)\}_{j+k=m} \tag{1.48}$$

that can be rewritten in the form

$$\Omega_m \zeta_m = \eta_m, \tag{1.49}$$

where $\zeta_m \in \boldsymbol{R}^{3m}$ is a column matrix given by

$$\zeta_m = \begin{pmatrix} X_{m+1} \\ Y_{m-1} \\ Z_m \end{pmatrix},$$

and Ω_m and η_m are a $(3m)\times(3m)$-matrix and a $3m$-dimensional column vector, respectively, which are both written in terms of

$$X(j,k) \qquad (j+k<m+1),$$

$$Y(j,k) \qquad (j+k<m-1),$$

$$Z(j,k) \qquad (j+k<m).$$

We then show that Ω_m is a non-singular matrix and then X_{m+1}, Y_{m-1}, Z_m are determined uniquely. As a consequence, the existence of f^m follows. The precise proof is given in Section 4. □

Now we can prove Theorem 1.11 under the assumption that Proposition 1.20 is proved:

Proof of Theorem 1.11. We fix a germ of the Whitney metric $d\sigma^2$ and $b \in C_0^\infty(\boldsymbol{R})$. Suppose that there exists a desired normal form $f(x, y)$ of a cross cap germ at $(x, y) = (0, 0)$ satisfying the properties of Theorem 1.11. Take a local coordinate system (u, v) satisfying (1.18), (1.19) and (1.20). Then x, y can be considered as functions of u, v, and $f = f(u, v)$ can be expressed as (1.14). We may assume that f satisfies (1.15) and (1.17) by Lemma 1.14. To prove the uniqueness of the desired f, it is sufficient to show that the Taylor expansions of $x(u, v), y(u, v)$ and $z(u, v)$ are uniquely determined by the first fundamental form $d\sigma^2$ and the characteristic function b. In fact, as shown in the outline of the proof of Proposition 1.20, $\pi_m(f)$ coincides with $\pi_m(f^m)$, where f^m is the m-th formal solution with respect to $d\sigma^2$ and $b \in C_o^\infty(\boldsymbol{R})$. Thus the uniqueness of f^m shown in Proposition 1.20 implies the uniqueness of desired f.

Thus, it is sufficient to show the existence. Applying Proposition 1.20 and letting $m \to \infty$, we get formal power series X, Y, $Z \in \boldsymbol{R}[[u, v]]$ such that the vector-valued formal power series $\Phi := (X, XY + b(Y), Z)$ satisfies

$$\mathcal{E} = [\Phi_u \cdot \Phi_u], \quad \mathcal{F} = [\Phi_u \cdot \Phi_v], \quad \mathcal{G} = [\Phi_v \cdot \Phi_v]. \tag{1.50}$$

Then by Borel's theorem, there exist C^∞ functions x, y, z whose Taylor series are X, Y, Z, respectively. If we set

$$f(u, v) := \bigl(x(u, v), x(u, v)y(u, v) + b\bigl(y(u, v)\bigr), z(u, v)\bigr),$$

then the first fundamental form of f is jet-equivalent to $d\sigma^2$. By Fact 1.9, $(0, 0)$ is a cross cap singularity of f. By Lemma 1.21, the map $(u, v) \mapsto (x(u, v), y(u, v))$ is a local diffeomorphism at the origin. Taking (x, y) as a new local coordinate system, we can write $u = u(x, y)$ and $v = v(x, y)$. So (x, y) gives the canonical coordinate system of the cross cap germ f. Thus f satisfies (2) and (3) of Theorem 1.11. □

2. Properties of power series

In this section, we prepare several properties of power series to prove the case $m \geq 3$ of Proposition 1.20. As in Section 1, we denote by $\boldsymbol{R}[[u, v]]$ the ring of formal power series with two variables u, v in real coefficients. Each element of $\boldsymbol{R}[[u, v]]$ can be written as in (1.10). Each $P(k, l)$ $(k, l \geq 0)$ is called the (k, l)-*coefficient* of the power series P. Moreover, the sum $k + l$ is called the *order* of the coefficient $P(k,l)$. In particular, $P(k,l)$ $(k+l=m)$ consist of all coefficients of order m. The formal partial derivatives of P denoted by

$$P_u := \partial P/\partial u, \qquad P_v := \partial P/\partial v$$

are defined in the usual manner.

Lemma 2.1 *The (k,l)-coefficient of the (formal) partial derivatives P_u and P_v of P are given by*

$$P_u(k,l) = P(k+1,l), \qquad P_v(k,l) = P(k,l+1).$$

Linear operations on power series also have a simple description as follows:

Lemma 2.2 *Let P, Q be two power series in $\boldsymbol{R}[[u,v]]$, and let α, $\beta \in \boldsymbol{R}$. Then*

$$(\alpha P + \beta Q)(k,l) = \alpha P(k,l) + \beta Q(k,l).$$

The coefficient formula for products is as follows:

Lemma 2.3 *Let $P_1,\dots,P_N$ be power series in $\boldsymbol{R}[[u,v]]$. Then*

$$(P_1 \cdots P_N)(k,l) = k!l! \sum_{\substack{s_1+\cdots+s_N=k,\\ t_1+\cdots+t_N=l}} \frac{P_1(s_1,t_1)\cdots P_N(s_N,t_N)}{s_1!t_1!\cdots s_N!t_N!}. \tag{2.1}$$

If $N=2$, and $P_1=P$ and $P_2=Q$, then the formula (2.1) reduces to the following:

$$(PQ)(k,l) := k!l! \sum_{s=0}^{k}\sum_{t=0}^{l} \frac{P(s,t)Q(k-s,l-t)}{s!t!(k-s)!(l-t)!}. \tag{2.2}$$

Moreover, setting Q to be the monomial u or v, we get the following:

Corollary 2.4

$$(uP)(k,l) = kP(k-1,l), \quad (vP)(k,l) = lP(k,l-1)$$

hold, where coefficients with negative induces $P(-k,l)$, $P(m,-n)$ $(k,n>0, l,m\in \boldsymbol{Z})$ are considered as 0.

In Definition 1.12, we defined the ideal $\mathcal{O}_m$ of $\boldsymbol{R}[[u,v]]$ consisting of formal power series of order at least m. The following assertion is obvious:

Lemma 2.5 *If $P\in\mathcal{O}_n$ and $Q\in\mathcal{O}_m$, then $PQ\in\mathcal{O}_{n+m}$ and $P+Q\in\mathcal{O}_r$, where $r=\min\{n,m\}$.*

Let P_k $(k=1,\dots,N)$ be a power series in $\mathcal{O}_{n_k}$. For a given integer m satisfying $m\geq n_1+\cdots+n_N$, we set

$$\langle P_j|P_1,\dots,P_N\rangle_m := m-\sum_{k\neq j} n_k. \tag{2.3}$$

Roughly speaking, this number is an upper bound of the degree of the terms of P_j to compute the m-th order term of the product $P_1\cdots P_N$, as follows.

Proposition 2.6 *Let P_j be a power series in $\mathcal{O}_{n_j}$ $(j=1,\dots,N)$. Then the product $P_1\cdots P_N$ belongs to the class $\mathcal{O}_{n_1+\cdots+n_N}$. Moreover, each (k,l)-coefficient $(P_1\cdots P_N)(k,l)$ $(k+l=m)$ of order m can be written as a linear combination of monomials of degree N of the following form*

$$\prod_{i=1}^{N} P_i(s_i,t_i) \qquad \bigl(n_i\leq s_i+t_i\leq \langle P_i|P_1,\dots,P_N\rangle_m\bigr).$$

Proof. Take a non-zero term in (2.1) in the following form:

$$\frac{k!l!}{s_1!t_1!\cdots s_N!t_N!}P_1(s_1,t_1)\cdots P_N(s_N,t_N).$$

Here, $s_k+t_k\geq n_k$ holds for each k because $P_k\in\mathcal{O}_{n_k}$ and $P_k(s_k,t_k)\neq 0$. Hence for each j, we have

$$\begin{aligned} n_j\leq s_j+t_j &= (s_1+\cdots+s_N)+(t_1+\cdots+t_N)-\sum_{k\neq j}(s_k+t_k)\\ &=(k+l)-\sum_{k\neq j}(s_k+t_k)\leq m-\sum_{k\neq j}n_k=\langle P_j|P_1,\dots,P_N\rangle_m. \end{aligned}$$

Hence we have the conclusion. $\square$

As a consequence of Proposition 2.6, we get the following:

Corollary 2.7 *Let* $\pi_m : \boldsymbol{R}[[u,v]] \to \boldsymbol{R}[[u,v]]$ *be the canonical homomorphism defined by* (1.23). *The* m*-th order term of the product* $P_1 \cdots P_N$ *coincides with the* m*-th order term of the product of*

$$\pi_{m_1}(P_1), \dots, \pi_{m_N}(P_N),$$

where

$$m_i := \langle P_i | P_1, \dots, P_N \rangle_m \qquad (i = 1, \dots, N).$$

In other words, the m*-th order terms of* $P_1 \cdots P_N$ *depend only on the terms of* P_i *of degree at most* m_i $(i = 1, \dots, N)$.

Example 2.8 We set $P \in \mathcal{O}_2$ and $Q \in \mathcal{O}_1$ as

$$P := c_1 uv + c_2 v^2 + c_3 u^3 v + O_5(u,v), \quad Q := d_1 u + d_2 v + d_3 u^3 + O_4(u,v),$$

respectively, where $O_j(u,v)$ $(j = 4, 5)$ are terms in $\mathcal{O}_j$. Then it holds that

$$\langle P | P, Q \rangle_3 = 3 - 1 = 2, \qquad \langle Q | P, Q \rangle_3 = 3 - 2 = 1.$$

To compute PQ modulo $\mathcal{O}_4$, we need the information of $\pi_2(P)$ and $\pi_1(Q)$. So, we have that

$$\begin{aligned} PQ &= (c_1 uv + c_2 v^2)(d_1 u + d_2 v) + O_4(u,v) \\ &= (c_1 d_2 + c_2 d_1) uv^2 + c_1 d_1 u^2 v + c_2 d_2 v^3 + O_4(u,v), \end{aligned}$$

and so $PQ \in \mathcal{O}_3$, where $O_4(u,v)$ is an element of $\mathcal{O}_4$. The coefficients of the terms of order 3 are

$$c_1 d_2 + c_2 d_1, \qquad c_1 d_1, \qquad c_2 d_2.$$

They are homogeneous polynomials of degree 2 in the variables c_1, c_2, d_1, d_2.

Let Q_1, $\dots$, Q_r be power series in $\boldsymbol{R}[[u,v]]$. We set

$$P_i = Q_{\mu_i}, \ \frac{\partial Q_{\mu_i}}{\partial u}, \ \text{ or } \ \frac{\partial Q_{\mu_i}}{\partial v} \qquad (i = 1, \ldots, r), \tag{2.4}$$

where $\mu_i \in \{1, \ldots, r\}$. Then, each coefficient of the product $P_1 \cdots P_N$ can be expressed as coefficients of $Q_1, \ldots, Q_r$. For each non-negative integer m, we set

$$\begin{aligned} &\langle Q_{\mu_i}; P_i | P_1, \ldots, P_N \rangle_m \\ &\quad = \begin{cases} \langle P_i | P_1, \ldots, P_N \rangle_m & \text{if } P_i = Q_{\mu_i}, \\ \langle P_i | P_1, \ldots, P_N \rangle_m + 1 & \text{if } P_i = (Q_{\mu_i})_u \text{ or } (Q_{\mu_i})_v. \end{cases} \end{aligned} \tag{2.5}$$

Roughly speaking,

> the number $\langle Q_{\mu_i}; P_i | P_1, \ldots, P_N \rangle_m$ is an upper bound of the degree of the terms of Q_{μ_i} which comes from the factor P_i to compute the m-th order term of the product $P_1 \cdots P_N$. (2.6)

In fact, by applying Lemma 2.6 and Lemma 2.1 to this situation, we get the following:

Proposition 2.9 *Under the convention* (2.4)*, each* (k, l)*-coefficient* $(P_1 \ldots P_N)(k, l)$ $(k + l = m)$ *of order* m *can be written as a linear combination of monomials of degree* N *of the following form*

$$\prod_{i=1}^{N} Q_{\mu_i}(s_i, t_i) \qquad \big(0 \le s_i + t_i \le \langle Q_{\mu_i}; P_i | P_1, \ldots, P_N \rangle_m\big).$$

Corollary 2.10 *The* m*-th order terms of the product* $P_1 \cdots P_N$ *coincides with that of*

$$\pi_{m'_1}(Q_{\mu_1}), \ \ldots, \ \pi_{m'_n}(Q_{\mu_N}),$$

where

$$m'_i := \langle Q_{\mu_i}; P_i | P_1, \ldots, P_N \rangle_m \qquad (i = 1, \ldots, N).$$

In other words, the m*-th order terms of* $P_1 \cdots P_N$ *depend only on the terms of* Q_{μ_i} *of degree at most* m'_i $(i = 1, \ldots, N)$*.*

Example 2.11 We set

$$Q_1 := a_1uv + a_2v^2 + a_3u^3v + O_5(u,v), \quad Q_2 := b_1u + b_2v + b_3u^3 + O_4(u,v),$$

and $P_1 := (Q_1)_u$, $P_2 := (Q_2)_u$, where $O_j(u,v)$ $(j = 4,5)$ are terms in $\mathcal{O}_j(u,v)$, respectively. Then,

$$P_1 = a_1v + 3a_3u^2v + O_4(u,v), \quad P_1P_2 = b_1 + 3b_3u^2 + O_3(u,v),$$

where $O_4(u,v) \in \mathcal{O}_4$ and $O_3(u,v) \in \mathcal{O}_3$. Since $P_1 \in \mathcal{O}_1$ and $P_2 \in \mathcal{O}_0$, we have

$$\begin{aligned} \langle Q_1; P_1|P_1, P_2\rangle_1 &= \langle P_1|P_1, P_2\rangle_1 + 1 = (1-0) + 1 = 2, \\ \langle Q_2; P_2|P_1, P_2\rangle_1 &= \langle P_2|P_1, P_2\rangle_1 + 1 = 1. \end{aligned}$$

So we have

$$\begin{aligned} Q_1Q_2 &= (\pi_2(Q_1))_u(\pi_1(Q_2))_u + O_2(u,v) \\ &= \left(a_1uv + a_2v^2\right)_u (b_1u + b_2v)_u + O_2(u,v) = a_1b_1v + O_2(u,v). \end{aligned}$$

Moreover,

$$\begin{aligned} \langle Q_1; P_1|P_1, P_2\rangle_2 &= \langle P_1|P_1, P_2\rangle_2 + 1 = (2-0) + 1 = 3, \\ \langle Q_2; P_1|P_1, P_2\rangle_2 &= \langle P_2|P_1, P_2\rangle_2 + 1 = (2-1) + 1 = 2. \end{aligned}$$

So we have

$$\begin{aligned} Q_1Q_2 &= (\pi_3(Q_1))_u\,(\pi_2(Q_2))_u + O_3(u,v) \\ &= \left(a_1uv + a_2v^2 + a_3u^3v\right)_u (b_1u + b_2v)_u + O_3(u,v) \\ &= a_1b_1v + O_3(u,v). \end{aligned}$$

In this case, the upper bound $\langle Q_2; P_2|P_1, P_2\rangle_2 = 2$ of the order of Q_2 for the contribution of the order 2 coefficients of P_1P_2 is not sharp. In fact, there are no order 2 terms for P_1P_2. Also, this does not contradict Proposition 2.9, since 0 can be considered as a homogeneous polynomial of order 2 whose coefficients are all zero.

3. The ignorable terms when determining f^m

We fix a germ of the Whitney metric $d\sigma^2$ and $b \in C_0^\infty(\boldsymbol{R})$. Take a local coordinate system (u, v) satisfying (1.18), (1.19) and (1.20). We suppose that there exists a germ of cross cap $f(u, v)$ satisfying the properties of Lemma 1.14.

3.1. The leading terms and ignorable terms

We let P be a polynomial in

$$x,\ x_u,\ x_v, \quad y,\ y_u,\ y_v, \quad z,\ z_u,\ z_v, \quad b'(y).$$

Each term of $(f_u \cdot f_u)$, $(f_u \cdot f_v)$ or $(f_v \cdot f_v)$ is a typical example of such polynomials. We denote by $P|_m$ the finite formal power series in u, v (i.e. a polynomial in u, v) that results after the substitutions

$$x := \pi_{m+1}(X), \quad y := \pi_{m-1}(Y), \quad z := \pi_{m+1}(Z), \quad b(y) := \sum_{j=3}^{m+1} \frac{b_j}{j!} y^j$$

into P, where $X := [x], Y := [y]$ and $Z := [z]$.

Definition 3.1 A term T of $(f_u \cdot f_u)$, $(f_u \cdot f_v)$ or $(f_v \cdot f_v)$ is called an *m-ignorable term* ($m \geq 3$) if each m-th order coefficient

$$(T|_m)(j, k) \qquad (j + k = m)$$

does not contain any of the top term coefficients (1.46), (1.47), (1.48) of $\pi_{m+1}(X)$, $\pi_{m-1}(Y)$, $\pi_m(Z)$ (cf. (1.29), (1.30) and (1.31)). In the proof of Proposition 1.20 at the end of this section, m-ignorable terms of $f_u \cdot f_u$, $f_u \cdot f_v$ and $f_v \cdot f_v$ will be actually ignorable to determine the matrix Ω_m given in (1.49). A term which is not m-ignorable is called a *leading term* of order m.

For the computation of leading terms, we will use the following two convenient equivalence relations: Let $\mathcal{A}$ be the associative algebra generated by

$$X(j, k), \quad Y(j, k), \quad Z(j, k) \qquad (j, k = 0, 1, 2, \dots).$$

We denote by $\mathcal{A}_m$ the ideal of $\mathcal{A}$ generated by

$$\begin{aligned} &X(j,k) && (j+k \le m+1), \\ &Y(j,k) && (j+k \le m-1), \\ &Z(j,k) && (j+k \le m). \end{aligned}$$

If two elements δ_1, $\delta_2 \in \mathcal{A}$ satisfy $\delta_1 - \delta_2 \in \mathcal{A}_{m-1}$, then we write

$$\delta_1 \equiv_m \delta_2. \tag{3.1}$$

Let P, Q be two polynomials in

$$x,\ x_u,\ x_v, \quad y,\ y_u,\ y_v, \quad z,\ z_u,\ z_v,\ b'(y).$$

If all of the coefficients of $P|_m - Q|_m$ (as a polynomial in u, v) are contained in $\mathcal{A}_{m-1}$, we denote this by

$$P \equiv_m Q.$$

This notation is the same as the one used in (3.1), and this is rather useful for unifying the symbols. For example, if the term T satisfies $T \equiv_m 0$ if and only if the term T is m-ignorable.

3.2. The properties of terms containing $b'(y)$

In the right hand sides of (1.18), (1.19) and (1.20), terms containing b' appear, and they are

$$b'(y)yx_uy_u, \quad b'(y)xy_u^2, \quad b'(y)^2y_u^2 \qquad (\text{in } f_u \cdot f_u), \tag{3.2}$$

$$b'(y)yx_uy_v, \quad b'(y)yx_vy_u, \quad b'(y)xy_uy_v, \quad b'(y)^2y_uy_v \qquad (\text{in } f_u \cdot f_v), \tag{3.3}$$

$$b'(y)yx_vy_v, \quad b'(y)xy_v^2, \quad b'(y)^2y_v^2 \qquad (\text{in } f_v \cdot f_v), \tag{3.4}$$

respectively.

In this subsection, we show that the terms as in (3.2), (3.3) and (3.4) are all m-ignorable.

For the fixed characteristic function $b \in C_0^\infty(\boldsymbol{R})$, we set

$$\mathcal{B}'(Y) := [b'(y)] = \sum_{r=2}^{\infty} \frac{b_{r+1}}{r!} Y^r \qquad (Y := [y]).$$

Let $P_1, \ldots, P_N$ be power series in $\boldsymbol{R}[[u,v]]$, and assume that at least one of P_j's is $\mathcal{B}'(Y)$. In this situation,

$$\langle Y|\mathcal{B}'(Y)|P_1,\ldots,P_N\rangle_m := \langle \mathcal{B}'(Y)|P_1,\ldots,P_N\rangle_m - 1. \tag{3.5}$$

Then, this number gives

> an upper bound of the degree of the terms of Y which comes from the factor $\mathcal{B}'(Y)$ to compute the m-th order term of the product $P_1 \cdots P_N$. (3.6)

The expressions in (2.6) and (3.6) are similar. This is the reason why we used the same notation in (2.5) and (3.5).

Proposition 3.2 *For each $m \geq 3$, the terms given in* (3.2)*,* (3.3) *and* (3.4) *are all m-ignorable.*

Proof. Since $b(0) = b'(0) = b''(0) = 0$ and $y = y(u,v) \in \mathcal{O}_1$,

$$b'(y) \in \mathcal{O}_2. \tag{3.7}$$

We categorize the terms in (3.2), (3.3) and (3.4) into three classes. One is

$$\begin{aligned} &b'(y)yx_uy_u \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_0\mathcal{O}_1, \quad b'(y)yx_uy_v \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_0\mathcal{O}_0, \\ &b'(y)yx_vy_u \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_3\mathcal{O}_1, \quad b'(y)yx_vy_v \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_3\mathcal{O}_0. \end{aligned} \tag{3.8}$$

For example, we wrote $b'(y)yx_uy_u \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_0\mathcal{O}_1$. In fact, $b'(y) \in \mathcal{O}_2$ holds by (3.7), $x_u \in \mathcal{O}_0$ holds by (1.34), and the relations $y \in \mathcal{O}_1$, $y_u \in \mathcal{O}_1$ follow from (1.35). The other two classes are

$$\begin{gathered} b'(y)xy_u^2 \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1, \quad b'(y)xy_uy_v \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1\mathcal{O}_0, \\ b'(y)xy_v^2 \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_0\mathcal{O}_0, \end{gathered} \tag{3.9}$$

and

$$\begin{gathered} b'(y)^2y_u^2 \in \mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\mathcal{O}_1, \quad b'(y)^2y_uy_v \in \mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\mathcal{O}_0, \\ b'(y)^2y_v^2 \in \mathcal{O}_2\mathcal{O}_2\mathcal{O}_0\mathcal{O}_0, \end{gathered} \tag{3.10}$$

respectively. For example, we wrote

$$b'(y)xy_u^2 = b'(y)xy_uy_u \in \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1,$$

because $b'(y) \in \mathcal{O}_2$, $x \in \mathcal{O}_1$, $y_u \in \mathcal{O}_1$. First, we show that the term $b'(y)yx_uy_v$ in (3.8) is m-ignorable. Let

$$P_1 := [b'(y)] = \mathcal{B}'(Y), \quad P_2 := [y(u,v)], \quad P_3 := [x_u(u,v)], \quad P_4 := [y_v(u,v)].$$

Then it is sufficient to show the m-th coefficient

$$Q(k,l) := (P_1P_2P_3P_4)(k,l) \qquad (k+l=m)$$

depends only on $X(s,t)$ $(s+t<m+1)$ and $Y(s,t)$ $(s+t<m-1)$. Since $b'(y) \in \mathcal{O}_2$, the highest order of $Y(s,t)$ in $Q(k,l)$ by the contribution of the first factor P_1 is computed by (cf. (3.5))

$$\begin{aligned}\langle Y|\mathcal{B}'(Y)|\mathcal{B}'(Y), Y, X_u, Y_v\rangle_m &= \langle \mathcal{B}'(Y)|\mathcal{B}'(Y), Y, X_u, Y_v\rangle_m \\ &= (m-1-0-0)-1 = m-2.\end{aligned}$$

Here we used the fact that $y \in \mathcal{O}_1$, $x_u \in \mathcal{O}_0$ and $y_v \in \mathcal{O}_0$. Since $m-2$ is less than $m-1$, $\mathcal{B}'(Y)$ does not effect the leading term (cf. Lemma 1.17). Similarly, we have that

$$\begin{aligned}\langle Y|\mathcal{B}'(Y), Y, X_u, Y_v\rangle_m &= m-2-0-0 = m-2 \ (<m-1), \\ \langle X; X_u|\mathcal{B}'(Y), Y, X_u, X_v\rangle_m &= (m-2-1-0)+1 = m-2 \ (<m+1), \\ \langle Y; Y_v|\mathcal{B}'(Y), Y, X_u, Y_v\rangle_m &= (m-2-1-0)+1 = m-2 \ (<m-1),\end{aligned}$$

and we can conclude that $b'(y)yx_uy_v$ is an m-ignorable term. Similarly, one can also prove that other three terms in (3.8) are also m-ignorable.

We next consider the term $b'(y)xy_v^2$ in (3.9). We have

$$\langle Y|\mathcal{B}'(Y)|\mathcal{B}'(Y), X, Y, Y_v, Y_v\rangle_m = (m-1-0-0)-1 = m-2 \ (<m-1).$$

Hence, the coefficients of Y appeared in $\mathcal{B}'(Y)$ do not effect the leading term. Similarly, the facts

$$\begin{aligned}\langle X|\mathcal{B}'(Y), X, Y_v, Y_v\rangle_m &= m-2-0-0 = m-2 \ (<m), \\ \langle Y; Y_v|\mathcal{B}'(Y), X, Y_v, Y_v\rangle_m &= (m-2-1-0)+1 = m-2 \ (<m-1),\end{aligned}$$

imply that the term $b'(y)xy_v^2$ is an m-ignorable term. Similarly, other two terms in (3.9) are also m-ignorable.

Finally, we consider the term $b'(y)^2y_v^2$ in (3.10). Since

$$\langle Y|\mathcal{B}'(Y)|\mathcal{B}'(Y),\mathcal{B}'(Y),Y_v,Y_v\rangle_m = (m-2-0-0)-1 = m-3\ (<m-1),$$

the coefficients of Y appearing in $b'(Y)$ do not effect the leading term. We have

$$\langle Y;Y_v|\mathcal{B}'(Y),\mathcal{B}'(Y),Y_v,Y_v\rangle_m = (m-2-2-0)+1 = m-3\ (<m-1),$$

and can conclude that $b'(y)^2y_v^2$ is an m-ignorable term. Similarly, other two terms in (3.10) are also m-ignorable. $\square$

The following terms

$$x^2y_u^2 \qquad (\text{in } f_u\cdot f_u), \tag{3.11}$$

$$x_ux_vy^2, \quad xyx_vy_u \qquad (\text{in } f_u\cdot f_v), \tag{3.12}$$

$$xyx_vy_v, \quad x_v^2, \quad x_v^2y^2 \qquad (\text{in } f_v\cdot f_v) \tag{3.13}$$

appear in (1.18), (1.19) and (1.20). We show the following:

Proposition 3.3 *For each $m\geq 3$, the terms given in* (3.11), (3.12) *and* (3.13) *are all m-ignorable terms.*

Proof. Since $x^2y_u^2 \in \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1$, we have

$$\langle X|X,X,Y_u,Y_u\rangle_m = m-1-1-1 = m-3\ (<m+1),$$
$$\langle Y;Y_u|X,X,Y_u,Y_u\rangle_m = (m-1-1-1)+1 = m-2\ (<m-1).$$

This implies that $x^2y_u^2$ is m-ignorable. On the other hand, both $x_ux_vy^2$ and $x_v^2y^2$ consist of two derivatives of x and y^2, and the former term has lower total order, so if we show $x_ux_vy^2$ is m-ignorable, then so is $x_v^2y^2$. In fact, since $x_ux_vy^2 \in \mathcal{O}_0\mathcal{O}_3\mathcal{O}_1\mathcal{O}_1$, we have

$$\begin{aligned}\langle X;X_u|X_u,X_v,Y,Y\rangle_m &\leq \langle X;X_v|X_u,X_v,Y,Y\rangle_m \\ &= m-0-1-1+1 = m-1\ (<m+1),\end{aligned}$$

$$\langle Y | X_u, X_v, Y, Y \rangle_m = m - 0 - 3 - 1 = m - 4 \ (< m - 1).$$

So $x_u x_v y^2$ is m-ignorable. We next observe that both xyx_vy_u and xyx_vy_v consist of xy and derivatives of x, y. The term xyx_vy_v has lower total order. So if it is m-ignorable, then so is xyx_vy_u. The fact that $xyx_vy_v \in \mathcal{O}_1\mathcal{O}_1\mathcal{O}_3\mathcal{O}_0$ is m-ignorable follows from the following computations:

$$\begin{aligned}
\langle X | X, Y, X_v, Y_v \rangle_m &= \langle Y | X, Y, X_v, Y_v \rangle_m \\
&= m - 1 - 3 - 0 = m - 4 \ (< m - 1), \\
\langle X; X_v | X, Y, X_v, Y_v \rangle_m &= (m - 1 - 1 - 0) + 1 = m - 1 \ (< m + 1), \\
\langle Y; Y_v | X, Y, X_v, Y_v \rangle_m &= (m - 1 - 1 - 3) + 1 = m - 4 \ (< m - 1).
\end{aligned}$$

Finally, $x_v^2 \in \mathcal{O}_3\mathcal{O}_3$ is m-ignorable because

$$\langle X; X_v | X_v, X_v \rangle_m = m - 2 \ (< m + 1). \qquad \square$$

4. The existence of the formal power series solution

We fix a germ of the Whitney metric $d\sigma^2$ and $b \in C_0^\infty(\boldsymbol{R})$. Take a local coordinate system (u, v) satisfying (1.18), (1.19) and (1.20). Like as in the previous section, we suppose that the existence of $f(u, v)$ satisfying the properties of Lemma 1.14.

4.1. Leading terms of $(\boldsymbol{f_u} \cdot \boldsymbol{f_u})$, $(\boldsymbol{f_u} \cdot \boldsymbol{f_v})$ and $(\boldsymbol{f_v} \cdot \boldsymbol{f_v})$

Applying the computations in the previous section, we prove the following:

Proposition 4.1 *Let m be an integer greater than* 2*, and k, l non-negative integers such that*

$$k + l = m \geq 3. \tag{4.1}$$

Then the m-th order terms of the equalities (1.33) *reduce to the following relations*:

$$\begin{aligned}
\mathcal{E}(k, l) \equiv_m 2\bigl(&X(k + 1, l) + (k + 1)lY(k, l - 1) \\
&+ ka_{2,0}Z(k, l) + la_{1,1}Z(k + 1, l - 1)\bigr),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathcal{F}(k,l) \equiv_m\ & X(k,l+1) + mkY(k-1,l) + ka_{2,0}Z(k-1,l+1) \\ & + ma_{1,1}Z(k,l) + la_{0,2}Z(k+1,l-1),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathcal{G}(k,l) \equiv_m\ & 2\big(k(k-1)Y(k-2,l+1) \\ & + ka_{1,1}Z(k-1,l+1) + la_{0,2}Z(k,l)\big),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}Y(m,-1) &= Y(-1,m) = Y(-2,m+1) = Y(m+1,-2) \\ &= Z(-1,m+1) = Z(m+1,-1) = 0.\end{aligned}$$

Proof. Removing the m-ignorable terms (3.2) and (3.11) from $[f_u \cdot f_u]$ in (1.18), we have

$$[f_u \cdot f_u] \equiv_m L_1, \qquad L_1 := X_u^2 + Y^2X_u^2 + 2XYX_uY_u + Z_u^2.$$

Similarly, we get from (1.19), (1.20), (3.2) and (3.11) that

$$[f_u \cdot f_v] \equiv_m L_2, \qquad [f_v \cdot f_v] \equiv_m L_3,$$

where

$$L_2 := X_uX_v + XYX_uY_v + X^2Y_uY_v + Z_uZ_v, \qquad L_3 := X^2Y_v^2 + Z_v^2.$$

The first term of L_1 is X_u^2. We can write $X = u + \tilde{X}$ ($\tilde{X} \in \mathcal{O}_4$). Since

$$\langle \tilde{X}; \tilde{X}_u | \tilde{X}_u, \tilde{X}_u \rangle_m = m - 2 \ (< m+1),$$

we have

$$X_u^2 = (1 + \tilde{X}_u)^2 \equiv_m 2\tilde{X}_u$$

and

$$\begin{aligned}X_u^2(k,l) \equiv_m 2\tilde{X}_u(k,l) \equiv_m 2X_u(k,l) = 2X(k+1,l)& \\ (k+l = m \geq 3),&\end{aligned} \tag{4.5}$$

where we have applied Lemma 2.1. The second term of L_1 is $Y^2X_u^2$. Since

$$\langle X; X_u | Y, Y, X_u, X_u \rangle_m = m - 1 - 1 - 0 + 1 = m - 1 \ (< m + 1)$$

and

$$\langle Y | Y, Y, X_u, X_u \rangle_m = m - 1 - 0 - 0 = m - 1,$$

the m-th order terms of $Y^2 X_u^2$ might not be m-ignorable. In fact, it can be written in terms of the $(m-1)$-st order coefficients of Y as follows. Since $X_u = 1 + \tilde{X}_u$ ($\tilde{X}_u \in \mathcal{O}_3$), we have $Y^2 X_u^2 \equiv_m Y^2$. Since $Y = v + \tilde{Y}$ ($\tilde{Y} \in \mathcal{O}_2$) and $\langle \tilde{Y} | \tilde{Y}, \tilde{Y} \rangle_m = m - 2$ $(< m - 1)$, $\tilde{Y}^2$ is an m-ignorable term. Thus, we have

$$Y^2 \equiv_m v^2 + 2v\tilde{Y} + \tilde{Y}^2 \equiv_m 2v\tilde{Y},$$

and so

$$(Y^2 X_u^2)(k, l) \equiv_m 2(v\tilde{Y})(k, l) \equiv_m 2(vY)(k, l) \equiv_m 2lY(k, l - 1), \tag{4.6}$$

where we have applied Corollary 2.4. We examine the third term XYX_uY_u of L_1. Since

$$\begin{aligned}
\langle X | X, Y, X_u, Y_u \rangle_m &= m - 1 - 0 - 1 = m - 2 \ (< m + 1), \\
\langle Y | X, Y, X_u, Y_u \rangle_m &= m - 1 - 0 - 1 = m - 2 \ (< m - 1), \\
\langle X; X_u | X, Y, X_u, Y_u \rangle_m &= (m - 1 - 1 - 1) + 1 = m - 2 \ (< m + 1), \\
\langle Y; Y_u | X, Y, X_u, Y_u \rangle_m &= (m - 1 - 1 - 0) + 1 = m - 1,
\end{aligned}$$

the m-th order terms of XYX_uY_u can be written in terms of the coefficients of Y_u modulo $\mathcal{A}_{m-1}$. Thus

$$XYX_uY_u = (u + \tilde{X})(v + \tilde{Y})(1 + \tilde{X}_u)Y_u \equiv_m uvY_u$$

and

$$(XYX_uY_u)(k, l) \equiv_m (uvY_u)(k, l) = klY_u(k-1, l-1) \equiv_m klY(k, l-1). \tag{4.7}$$

The fourth term of L_1 is Z_u^2. Since $Z_u^2 \in \mathcal{O}_1\mathcal{O}_1$ (cf. (1.36)), we have

$$\langle Z; Z_u | Z_u, Z_u \rangle_m = (m - 1) + 1 = m.$$

Hence the m-th order terms of Z_u^2 can be written in terms of the coefficients of Z_u modulo $\mathcal{A}_{m-1}$. If we write (cf. (1.36))

$$Z = \frac{1}{2}(a_{2,0}u^2 + 2a_{1,1}uv + a_{0,2}v^2) + \tilde{Z} \qquad (\tilde{Z} \in \mathcal{O}_3),$$

then $\tilde{Z}_u^2$ is an m-ignorable term, and

$$Z_u^2 \equiv_m (a_{2,0}u + a_{1,1}v + \tilde{Z}_u)^2 \equiv_m 2\bigl(a_{2,0}u\tilde{Z}_u + a_{1,1}v\tilde{Z}_u\bigr).$$

Since $Z(k,l) = \tilde{Z}(k,l)$ for $k+l \geq 3$, we have

$$\begin{aligned} Z_u^2(k,l) &\equiv_m 2a_{2,0}(uZ_u)(k,l) + 2a_{1,1}(vZ_u)(k,l) \\ &= 2a_{2,0}kZ_u(k-1,l) + 2a_{1,1}lZ_u(k,l-1) \\ &= 2ka_{2,0}Z(k,l) + 2la_{1,1}Z(k+1,l-1). \end{aligned} \tag{4.8}$$

By (4.5), (4.6), (4.7) and (4.8), we have (4.2).

We next prove (4.3). Since

$$\begin{aligned} \langle X; X_u | X_u, X_v \rangle_m &= m - 3 + 1 = m - 2 \ (< m+1), \\ \langle X; X_v | X_u, X_v \rangle_m &= m - 0 + 1 = m + 1, \end{aligned}$$

X_v contributes to the leading term. Thus

$$X_u X_v \equiv_m (1 + \tilde{X}_u)X_v \equiv_m X_v$$

and

$$(X_u X_v)(k,l) \equiv_m X_v(k,l) = X(k,l+1). \tag{4.9}$$

On the other hand, since

$$\begin{aligned} \langle X | X, Y, X_u, Y_v \rangle_m &= m - 1 - 0 - 0 = m - 1 \ (< m+1), \\ \langle Y | X, Y, X_u, Y_v \rangle_m &= m - 1 - 0 - 0 = m - 1, \\ \langle X; X_u | X, Y, X_u, Y_v \rangle_m &= (m - 1 - 1 - 0) + 1 = m - 1 \ (< m+1), \\ \langle Y; Y_u | X, Y, X_u, Y_u \rangle_m &= (m - 1 - 1 - 0) + 1 = m - 1, \end{aligned}$$

the coefficients of the factors Y and Y_v appear in the leading terms. Thus

$$\begin{aligned} XYX_uY_v &\equiv_m (u+\tilde{X})(v+\tilde{Y})(1+\tilde{X}_u)(1+\tilde{Y}_v) \\ &\equiv_m u(v+\tilde{Y})(1+\tilde{Y}_v) \equiv_m uv\tilde{Y}_v + u\tilde{Y} \end{aligned}$$

and

$$\begin{aligned} (XYX_uY_v)(k,l) &\equiv_m (uvY_v)(k,l) + (uY)(k,l) \\ &\equiv_m klY_v(k-1,l-1) + kY(k-1,l) \\ &\equiv_m k(l+1)Y(k-1,l). \end{aligned} \tag{4.10}$$

The third term of L_2 is $X^2Y_uY_v$. Since

$$\begin{aligned} \langle X|X,X,Y_u,Y_v\rangle_m &= m-1-1-0 = m-2 \ (< m+1), \\ \langle Y;Y_u|X,X,Y_u,Y_v\rangle_m &= m-1-1-0+1 = m-1, \\ \langle Y;Y_v|X,X,Y_u,Y_v\rangle_m &= m-1-1-1+1 = m-2 \ (< m-1), \end{aligned}$$

only the factor Y_u affects the computation of the leading term. So we have

$$X^2Y_uY_v \equiv_m (u+\tilde{X})^2Y_u(1+\tilde{Y}_v) \equiv_m u^2Y_u$$

and

$$\begin{aligned} (X^2Y_uY_v)(k,l) &\equiv_m (u^2Y_u)(k,l) \\ &\equiv_m k(k-1)Y_u(k-2,l) \equiv_m k(k-1)Y(k-1,l). \end{aligned} \tag{4.11}$$

The fourth term of L_2 is $Z_uZ_v \in \mathcal{O}_1\mathcal{O}_1$. Since

$$\langle Z;Z_u|Z_u,Z_v\rangle_m = \langle Z;Z_v|Z_u,Z_v\rangle_m = m-1+1 = m,$$

both Z_u and Z_v contribute to the leading terms, and

$$\begin{aligned} Z_uZ_v &= (a_{2,0}u + a_{1,1}v + \tilde{Z}_u)(a_{1,1}u + a_{0,2}v + \tilde{Z}_v) \\ &\equiv_m (a_{1,1}u + a_{0,2}v)\tilde{Z}_u + (a_{2,0}u + a_{1,1}v)\tilde{Z}_v. \end{aligned}$$

So we have

$$
\begin{aligned}
(Z_u Z_v)(k,l) &\equiv_m \big((a_{1,1}u + a_{0,2}v)Z_u\big)(k,l) + \big((a_{2,0}u + a_{1,1}v)Z_v\big)(k,l) \\
&\equiv_m ka_{1,1}Z(k,l) + la_{0,2}Z(k+1,l-1) \\
&\qquad + ka_{2,0}Z(k-1,l+1) + la_{1,1}Z(k,l) \\
&= ka_{2,0}Z(k-1,l+1) + ma_{1,1}Z(k,l) + la_{0,2}Z(k+1,l-1).
\end{aligned} \tag{4.12}
$$

By (4.9), (4.10), (4.11) and (4.12), we obtain (4.3).

Finally, we consider L_3. The first term of L_3 is $X^2Y_v^2$. Since

$$
\begin{aligned}
\langle X | X, X, Y_v, Y_v \rangle_m &= m - 1 - 0 - 0 = m - 1 \ (< m+1), \\
\langle Y; Y_v | X, X, Y_v, Y_v \rangle_m &= m - 1 - 1 - 0 + 1 = m - 1,
\end{aligned}
$$

the coefficients of Y_v affect the leading term of $X^2Y_v^2$. We have

$$
(X^2Y_v^2) \equiv_m (u + \tilde{X})^2(1 + \tilde{Y}_v)^2 \equiv_m 2u^2\tilde{Y}_v
$$

and

$$
(X^2Y_v^2)(k,l) \equiv_m 2\big(u^2Y_v\big)(k,l) \equiv_m 2k(k-1)Y(k-2,l+1). \tag{4.13}
$$

The second term of L_3 is Z_v^2. Since

$$
Z_v^2 \equiv_m (a_{1,1}u + a_{0,2}v + \tilde{Z}_v)^2 \equiv_m 2(a_{1,1}u + a_{0,2}v)\tilde{Z}_v,
$$

we have

$$
\begin{aligned}
Z_v^2(k,l) &\equiv_m 2\big((a_{1,1}u + a_{0,2}v)Z_v\big)(k,l) \\
&\equiv_m 2ka_{1,1}Z(k-1,l+1) + 2la_{0,2}Z(k,l).
\end{aligned} \tag{4.14}
$$

By (4.13) and (4.14), we obtain (4.4). □

4.2. Proof of Proposition 1.20

We prove the assertion by induction. By Lemma 1.21, we have already determined the coefficients

$$
\hat{X}(i,l) \quad (0 \le i+l \le 3), \quad \hat{Y}(j,l) \quad (0 \le j+l \le 1), \quad \hat{Z}(k,l) \quad (0 \le k+l \le 2).
$$

For the sake of simplicity, we set

$$\hat{X}_i := \hat{X}(i, m-i+1), \quad \hat{Y}_j := \hat{Y}(k, m-j-1), \quad \hat{Z}_k := \hat{Z}(k, m-k). \tag{4.15}$$

We say that $W = \hat{X}_i$, $\hat{Y}_j$, $\hat{Z}_k$ is *m-fixed* if it is uniquely expressed in terms of

$$\begin{aligned} \hat{X}(i,l) &\qquad (0 \le i+l \le m), \\ \hat{Y}(j,l) &\qquad (0 \le j+l \le m-2), \\ \hat{Z}(k,l) &\qquad (0 \le k+l \le m-1), \\ \mathcal{E}(i,l),\ \mathcal{F}(i,l),\ \mathcal{G}(i,l) &\qquad (0 \le i+l \le m), \end{aligned}$$

using (4.2), (4.3) and (4.4). To prove the assertion, it is sufficient to prove that

$$\hat{X}_i \ \ (i = 0, \dots, m+1), \quad \hat{Y}_j \ \ (j = 0, \dots, m-1), \quad \hat{Z}_k \ \ (k = 0, \dots, m)$$

are all m-fixed. (We remark that this conclusion is equivalent that the matrix Ω_m as in (1.49) is non-singular, although we do not use Ω_m in this proof explicitly.) By (4.2), (4.3) and (4.4), we can write

$$\hat{X}_{k+1} + (k+1)(m-k)\hat{Y}_k + ka_{2,0}\hat{Z}_k + (m-k)a_{1,1}\hat{Z}_{k+1} = \tilde{\mathcal{E}}_k, \tag{4.16}$$

$$\hat{X}_k + mk\hat{Y}_{k-1} + ka_{2,0}\hat{Z}_{k-1} + ma_{1,1}\hat{Z}_k + (m-k)a_{0,2}\hat{Z}_{k+1} = \tilde{\mathcal{F}}_k, \tag{4.17}$$

$$k(k-1)\hat{Y}_{k-2} + ka_{1,1}\hat{Z}_{k-1} + (m-k)a_{0,2}\hat{Z}_k = \tilde{\mathcal{G}}_k, \tag{4.18}$$

for $k = 0, \dots, m$, where $\tilde{\mathcal{E}}_k$, $\tilde{\mathcal{F}}_k$ and $\tilde{\mathcal{G}}_k$ are all previously m-fixed terms, by the inductive assumption.

If we set $k = 0$ in (4.18), we have

$$\hat{Z}_0 = \frac{\tilde{\mathcal{G}}_0}{ma_{0,2}}, \tag{4.19}$$

where we used the fact that $a_{0,2} > 0$. If we next set $k = 1$ in (4.18), then we have

$$a_{1,1}\hat{Z}_0 + (m-1)a_{0,2}\hat{Z}_1 = \tilde{\mathcal{G}}_1$$

and

$$\hat{Z}_1 = \frac{\tilde{\mathcal{G}}_1 - a_{1,1}\hat{Z}_0}{(m-1)a_{0,2}}. \tag{4.20}$$

Hence $\hat{Z}_1$ is m-fixed (cf. (4.19)). On the other hand, (4.18) for $2 \le k \le m$ can be rewritten as

$$(1+k)(2+k)\hat{Y}_k + a_{1,1}(2+k)\hat{Z}_{k+1} + a_{0,2}(-2-k+m)\hat{Z}_{k+2} = \tilde{\mathcal{G}}_{k+2} \tag{4.21}$$

for $k = 0, \dots, m-2$. If we set $k = 0$ in (4.17), then we have

$$\hat{X}_0 + ma_{1,1}\hat{Z}_0 + ma_{0,2}\hat{Z}_1 = \tilde{\mathcal{F}}_0. \tag{4.22}$$

Thus $\hat{X}_0$ can be m-fixed. On the other hand, (4.17) for $1 \le k \le m$ can be rewritten as

$$\begin{aligned} &\hat{X}_{k+1} + m(k+1)\hat{Y}_k + (k+1)a_{2,0}\hat{Z}_k + ma_{1,1}\hat{Z}_{k+1} + (m-k-1)a_{0,2}\hat{Z}_{k+2} \\ &\quad = \tilde{\mathcal{F}}_{k+1}, \end{aligned} \tag{4.23}$$

where $k = 0, \dots, m-1$. Subtracting (4.16) from (4.23), we have

$$k(k+1)\hat{Y}_k + a_{2,0}\hat{Z}_k + ka_{1,1}\hat{Z}_{k+1} + (m-k-1)a_{0,2}\hat{Z}_{k+2} = \tilde{\mathcal{F}}_{k+1} - \tilde{\mathcal{E}}_k \tag{4.24}$$

for $k = 0, \dots, m-1$. By (4.24) and (4.21), we have

$$\hat{Z}_{k+2} = \frac{1}{a_{0,2}(2m-k-2)}\big(-a_{2,0}(2+k)\hat{Z}_k + (k+2)(\tilde{\mathcal{F}}_{k+1} - \tilde{\mathcal{E}}_k) - k\tilde{\mathcal{G}}_{k+2}\big) \tag{4.25}$$

for $k = 0, \dots, m-2$. Thus $\hat{Z}_2, \dots, \hat{Z}_m$ are m-fixed. Then $\hat{Y}_0, \dots, \hat{Y}_{m-1}$ are m-fixed by (4.24), and $\hat{X}_1, \dots, \hat{X}_m$ are also m-fixed by (4.23). Finally, if we set $k = m$ in (4.16), then we have

$$\hat{X}_{m+1} + ma_{2,0}\hat{Z}_m = \tilde{\mathcal{E}}_m, \tag{4.26}$$

and $\hat{X}_{m+1}$ is m-fixed. We then get the desired m-th formal solution

$$f^m = \big(\mathcal{X}_{m+1}, \mathcal{X}_{m+1}\mathcal{Y}_{m-1} + \mathcal{B}_{m+1}(\mathcal{Y}_{m-1}), \mathcal{Z}_m\big)$$

satisfying (1.33), since the terms $\tilde{\mathcal{E}}_k$, $\tilde{\mathcal{F}}_k$ and $\tilde{\mathcal{G}}_k$ as in (4.16), (4.17) and (4.18) can be computed using the coefficients of $\mathcal{X}_m$, $\mathcal{Y}_{m-2}$, $\mathcal{Z}_{m-1}$ and $\mathcal{E}, \mathcal{F}, \mathcal{G}$. The uniqueness of f^m and the relation $\pi_{m-1}(f^m) = \pi_{m-1}(f^{m-1})$ are now obvious from our construction.

Remark 4.2 If needed, using the above proof, we can explicitly write down the lower order terms $\tilde{\mathcal{E}}_k$, $\tilde{\mathcal{F}}_k$ and $\tilde{\mathcal{G}}_k$ as in (4.16), (4.17) and (4.18), and write down the non-singular matrix Ω_m and the vector η_m, that would give a recursive formula for the highest order coefficients of the m-th formal solution f^m in terms of its lower order coefficients for each $m \geq 3$. However we omit such formulas here, as they are complicated.

5. New intrinsic invariants of cross caps

Let $\mathcal{W}$ be the set of germs of Whitney metrics at their singularities. Two metric germs $d\sigma_i^2$ $(i = 1, 2)$ in $\mathcal{W}$ are called *isometric* if there exists a local diffeomorphism germ φ such that $d\sigma_2^2$ is the pull-back of $d\sigma_1^2$ by φ. A map

$$I : \mathcal{W} \to \boldsymbol{R}$$

is called an *invariant of Whitney metrics* if it takes a common value for all metrics in each isometric class. For a cross cap singularity, we can take a canonical coordinate system (x, y) such that $f(x, y)$ is expressed as (cf. (1.2))

$$\begin{gathered} f(x,y) = (x,\ xy + b(y),\ z(x,y)), \\ [b(y)] = \sum_{i=3}^{\infty} \frac{b_i}{i!} y^i, \quad [z(x,y)] = \sum_{j+k\geq 2}^{\infty} \frac{a_{j,k}}{j!k!} x^j y^k. \end{gathered} \tag{5.1}$$

As shown in [5, Theorem 6], the coefficients $a_{2,0}$, $a_{1,1}$ and $a_{0,2}$ are intrinsic invariants. By (1.11), (1.12) and (1.13), one can observe that these three invariants are determined by the second order jets of E, F and G. So one might expect that the coefficients of the Taylor expansions of the functions E, F, G are all intrinsic invariants of cross caps. However, for example,

$$E_{vvv}(0,0) = 6a_{1,1}a_{1,2}$$

is not an intrinsic invariant of f, since $a_{1,2}$ is changed by an isometric deformation of cross caps (cf. [5, Theorem 4]). In this section, we construct a family of intrinsic invariants $\{\alpha_{i,j}\}_{i+j\geq 2}$ of Whitney metrics ($\alpha_{2,0}, \alpha_{1,1}$ and $\alpha_{0,2}$ have been already defined in [5]). When the metric is induced from a cross cap expressed by the canonical coordinate, then $a_{2,0}$, $a_{1,1}$ and $a_{0,2}$ as in (5.1) coincide with $\alpha_{2,0}, \alpha_{1,1}$ and $\alpha_{0,2}$ for the induced Whitney's metric.

Let $d\sigma^2$ be a Whitney metric defined on a 2-manifold M^2, and $p \in M^2$ a singular point of the metric. Applying Theorem 1.11 for $b = 0$, there exists a C^∞ map germ f into $\boldsymbol{R}^3$ defined on a neighborhood U of p having a cross cap singularity at p satisfying the following two properties:

(1) the first fundamental form $d\sigma_f^2$ of f is jet-equivalent (cf. Definition 1.10) to $d\sigma^2$ at p,
(2) the characteristic function of f is a flat function at p, that is, the Taylor expansion at p is the zero power series.

If f is real analytic, it is a normal cross cap (cf. Definition 1.3). However, we do not assume here the real analyticity of $d\sigma_f^2$ and f. Taking the normal form of f, we may assume that f is expressed as

$$f(x,y) = (x,\ xy,\ z(x,y)),$$

where

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(x,y)$$

are smooth functions defined on a neighborhood of $p = (0,0)$. For each pair of integers (i,j) satisfying $i + j \geq 2$ and $i, j \geq 0$, there exists a unique assignment

$$d\sigma^2 \mapsto \alpha_{i,j}^{d\sigma^2} \in \boldsymbol{R}$$

such that

$$[z] = \sum_{n=2}^{\infty}\sum_{i=0}^{n} \frac{\alpha_{i,n-i}^{d\sigma^2}}{i!(n-i)!} x^i y^{n-i}.$$

So we may regard the series

$$\alpha(d\sigma^2, p) := \{\alpha_{i,j}^{d\sigma^2}\}_{i+j\geq 2,\ i,j\geq 0}$$

as a family of invariants of $d\sigma^2$. By Theorem 1.11, we get the following assertion:

Theorem 5.1 *Let $d\sigma_1^2$ and $d\sigma_2^2$ be Whitney metrics on M^2 having a singularity at the same point $p \in M^2$. Then the two metrics are jet-equivalent if and only if $\alpha(d\sigma_1^2, p) = \alpha(d\sigma_2^2, p)$.*

In other words, α is a family of complete invariants distinguishing the jet-equivalence classes of Whitney metrics at p. This family of invariants also induces a family of intrinsic invariants for cross caps in an arbitrarily given Riemannian 3-manifold (N^3, g) as follows. Let $f : M^2 \to N^3$ be a C^∞ map which admits only cross cap singularities. Then the induced metric $d\sigma_f^2$ gives a Whitney metric. Let $p \in M^2$ be a cross cap singularity of f. Then we set

$$A(f, p) := \alpha(d\sigma_f^2, p),$$

which can be considered as a family of intrinsic invariants of a germs of cross cap singularities. When (N^3, g) is the Euclidean 3-space, we can give an explicit algorithm to compute the invariants as follows:

1. Take the $(m+1)$-st $(m \geq 2)$ canonical coordinate system (u, v) centered at p, that is, f has the following Taylor expansion at $p = (0, 0)$:

$$[f] = \left(u, uv + \sum_{n=3}^{m+1} \frac{b_n v^n}{n!}, \sum_{n=2}^{m+1} \sum_{i=0}^{n} \frac{a_{i,n-i}}{i!(n-i)!} u^i v^{n-i}\right) + O_{m+2}(u, v).$$

   Such a coordinate system can be taken using Fukui-Hasegawa's algorithm given in [2].
2. Using this coordinate system (u, v), we can determine the coefficients of the following expansion up to $(m+1)$-st order terms because of the expression $f = (u, 0, 0) + O_2(u, v)$:

$$[E] = \sum_{i+j \le m+1} \frac{E(i,j)}{i!j!} u^i v^j + O_{m+2}(u,v),$$

$$[F] = \sum_{i+j \le m+1} \frac{F(i,j)}{i!j!} u^i v^j + O_{m+2}(u,v),$$

$$[G] = \sum_{i+j \le m+1} \frac{G(i,j)}{i!j!} u^i v^j + O_{m+2}(u,v),$$

where $d\sigma_f^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$.

3. Setting $b = 0$, we compute

$$X(k,l) \qquad (0 \le k+l \le m+2),$$
$$Y(k,l) \qquad (0 \le k+l \le m),$$
$$Z(k,l) \qquad (0 \le k+l \le m+1),$$

according to the algorithm given in the proof of Theorem 1.11.

4. We formally set

$$u := \sum_{i+j \le m} \frac{U(i,j)}{i!j!} x^i y^j + O_{m+1}(x,y),$$

$$v := \sum_{i+j \le m} \frac{V(i,j)}{i!j!} x^i y^j + O_{m+1}(x,y),$$

and substitute them into the expansions

$$x = \sum_{i+j \le m} \frac{X(i,j)}{i!j!} u^i v^j + O_{m+1}(u,v),$$

$$y = \sum_{i+j \le m} \frac{Y(i,j)}{i!j!} u^i v^j + O_{m+1}(u,v).$$

Then we can determine all of the coefficients

$$U(k,l), \quad V(k,l) \qquad (0 \le k+l \le m).$$

5. Using them, we can finally determine all of the coefficients of the expansion

$$[Z] = \sum_{i+j \le m} \frac{A_{i,j}}{i!j!} x^i y^j + O_{m+1}(x, y), \tag{5.2}$$

where $\{A_{i,j}\}_{i+j \ge 2} = A(f, p)$.

However, the uniqueness of the expression (5.2) was already shown, and one can alternatively compute $\{A_{i,j}\}_{i+j \le m}$ via any suitable method. We remark that the normal cross cap shown in the right-hand side of Figure 2 is drawn using the invariants $A_{i,j}$ for $0 \le i + j \le 11$.

One can get the following tables of intrinsic invariants;

$$A_{2,0} = a_{2,0}, \qquad A_{1,1} = a_{1,1}, \qquad A_{0,2} = a_{0,2},$$

$$A_{3,0} = -\frac{b_3 a_{1,1}^2 a_{2,0} + b_3 a_{2,0} - 2a_{3,0} a_{0,2}^2}{2a_{0,2}^2}, \quad A_{2,1} = -\frac{b_3 a_{1,1} a_{2,0} - 6a_{0,2} a_{2,1}}{6a_{0,2}},$$

$$A_{1,2} = \frac{b_3 a_{1,1}^2 + 2a_{0,2} a_{1,2} + b_3}{2a_{0,2}}, \quad A_{0,3} = \frac{3b_3 a_{1,1} + 2a_{0,3}}{2}.$$

The numerators of the above invariants have been computed in [5]. The authors also computed the fourth order invariants $A_{i,j}$ $(i+j = 4)$, which are more complicated. For example, $A_{0,4}$ has the simplest expression amongst them, which is given by

$$A_{0,4} = \frac{\begin{array}{l} 4a_{0,2}(4b_4 a_{1,1} + 3a_{0,4}) \\ \quad + 3b_3\left(b_3(15a_{1,1}^2 - 4a_{0,2}a_{2,0} + 7) + 4(a_{0,3}a_{1,1} + 4a_{0,2}a_{1,2})\right) \end{array}}{12a_{0,2}}.$$

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References

[1] Bruce J. W. and West J. M., *Functions on a cross cap.* Math. Proc. Camb. Phil. Soc., **123** (1998), 19–39.

[2] Fukui T. and Hasegawa M., *Fronts of Whitney umbrella—a differential geometric approach via blowing up.* J. Singul., **4** (2012), 35–67.

[3] Golubitsky M. and Guillemin V., *Stable mappings and their singularities*, Graduate Texts in Mathematics, vol. 14, Springer Verlag, 1973.

[4] Hasegawa M., Honda A., Naokawa K., Saji K., Umehara M. and Yamada K., *Intrinsic properties of surfaces with singularities*. Internat. J. Math., **26** (4) (2015), 1540008, 34 pp.

[5] Hasegawa M., Honda A., Naokawa K., Umehara M. and Yamada K., *Intrinsic invariants of cross caps*. Selecta Math., **20** (2014), 769–785.

[6] Honda A., Naokawa K., Umehara M. and Yamada K., *Isometric deformations and symmetries of wave fronts at non-degenerate singular points*, preprint, arXiv:1710.02999.

[7] Kossowski M., *Realizing a singular first fundamental form as a nonimmersed surface in Euclidean* 3*-space*. J. Geom., **81** (2004), 101–113.

[8] Naokawa K., Umehara M. and Yamada K., *Isometric deformations of cuspidal edges*. Tohoku Math. J., **68** (2015), 73–90.

[9] West J., *The differential geometry of the cross-cap*. Ph.D. thesis, University of Liverpool, 1995.

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