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Discrete Green Potentials with Finite Energy

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Abstract. For a hyperbolic infinite network, it is well-known that Green potentials with finite energy are Dirichlet potentials. Conversely, if a Dirichlet potential has non-positive Laplacian, then it is a Green potential with finite energy. In this paper, we study whether a Dirichlet potential can be expressed as a difference of two Green potentials with finite energy. Comparisons of the Dirichlet sum of a function and that of its Laplacian play important roles in our study. As a by-product, we obtain a Riesz decomposition of a function whose Laplacian is a Dirichlet function.

Key words: discrete potential theory, Dirichlet potential, Green potential, Riesz representation, discrete Laplacian.

1. Introduction with preliminaries

Let $\mathcal{N} = \{V, E, K, r\}$ be an infinite network which is connected and locally finite and has no self-loop, where V is the set of nodes, E is the set of arcs, and the resistance r is a strictly positive function on E. For $x \in V$ and for $e \in E$ the node-arc incidence matrix K is defined by $K(x, e) = 1$ if x is the initial node of e; $K(x, e) = -1$ if x is the terminal node of e; $K(x, e) = 0$ otherwise. Let $L(V)$ be the set of all real valued functions on V, $L^+(V)$ the set of all non-negative real valued functions on V, and $L_0(V)$ the set of all $u \in L(V)$ with finite support. We similarly define $L(E)$, $L^+(E)$, and $L_0(E)$. For $u \in L(V)$ we define the *discrete derivative* $\nabla u \in L(E)$ and the *Laplacian* $\Delta u \in L(V)$ as

$$\nabla u(e) = -r(e)^{-1} \sum_{x \in V} K(x, e)u(x),$$

$$\Delta u(x) = \sum_{e \in E} K(x, e)\nabla u(e).$$

For convenience we give specific forms. For $e \in E$ let $x^+ \in V$ be the initial node of e and $x^- \in V$ the terminal node of e. Then

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$$\nabla u(e) = \frac{u(x^-) - u(x^+)}{r(e)}.$$

For $x \in V$ let $\{e_1, \ldots, e_d\}$ be the set of edges adjacent to x and y_j the other node of e_j. Then

$$\Delta u(x) = \sum_{j=1}^{d} \frac{u(y_j) - u(x)}{r(e_j)}.$$

For $u, v \in L(V)$ and for $\varphi, \psi \in L(E)$, we put

$$\langle \varphi, \psi \rangle = \sum_{e \in E} r(e)\varphi(e)\psi(e),$$
$$(u, v) = \langle \nabla u, \nabla v \rangle,$$
$$\|u\| = (u, u)^{1/2},$$

and call (u, v) the *Dirichlet mutual sum* of u and v and $\|u\|$ the *Dirichlet seminorm* of u.

We define some classes of functions on V as

$$\mathbf{D} = \{u \in L(V) \mid \|u\| < \infty\},$$
$$\mathbf{H} = \{u \in \mathbf{D} \mid \Delta u \equiv 0\},$$
$$\mathbf{D}^{(2)} = \{u \in L(V) \mid \Delta u \in \mathbf{D}\},$$
$$\mathbf{H}^{(2)} = \{u \in \mathbf{D}^{(2)} \mid \Delta u \in \mathbf{H}\}.$$

The space $L^2(E) = \{\varphi \in L(E) \mid \langle \varphi, \varphi \rangle < \infty\}$ is a Hilbert space with respect to the inner product $\langle \varphi, \psi \rangle$; actually, this space is the same as the space l^2 of square-summable sequences. On the other hand, (u, v) is a degenerate bilinear form in $\mathbf{D}$; for example, $(1_V, u) = 0$ and $\|u + 1_V\| = \|u\|$ for $u \in \mathbf{D}$, where 1_V is the constant 1 on V. It is shown in [7, Theorem 1.1] that $\mathbf{D}$ is a Hilbert space with respect to the inner product $(u, v) + u(o)v(o)$ for a fixed node $o \in V$. We easily verify that a sequence $\{u_n\}_n \subset \mathbf{D}$ converges to u in $\mathbf{D}$ if and only if $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\{u_n\}_n$ converges pointwise to u. Denote by $\mathbf{D}_0$ the closure of $L_0(V)$ in $\mathbf{D}$. We call a function in $\mathbf{D}$ (resp. in $\mathbf{D}_0$) a *Dirichlet function* (resp. a *Dirichlet potential*). Let

$$\mathbf{D}_0^{(2)} = \{u \in L(V) \mid \Delta u \in \mathbf{D}_0\}.$$

We shall use the following results in [7]:

Lemma 1.1 *The following statements are equivalent to each other.*

(1) $\mathcal{N}$ *is hyperbolic*;
(2) $\mathbf{D}_0 \neq \mathbf{D}$;
(3) $1_V \notin \mathbf{D}_0$;
(4) $\mathbf{D}_0 \cap \mathbf{H} = \{0\}$;
(5) *For each* $a \in V$ *there exists a unique function* g_a *such that* $g_a \in \mathbf{D}_0$ *and* $\Delta g_a = -\delta_a$.

Here δ_a *denotes the characteristic function of* $\{a\}$.

We call g_a of the above lemma the *Green function* of $\mathcal{N}$ with pole at a. Note that $g_a \geq 0$ on V. It is well-known that the difference of two Green potentials (see Section 3 for the definition) with finite energy is a Dirichlet potential. The converse is not true in general; see [2, Example 4.9]. We study some conditions to assure that a Dirichlet potential can be represented as the difference of two Green potentials.

We said in [3] that $\mathcal{N}$ satisfies condition (LD) if there exists a constant $c > 0$ such that

$$\|\Delta f\| \leq c\|f\| \quad \text{for all } f \in L_0(V). \tag{LD}$$

Now we introduce the reverse inequality of condition (LD). We say that $\mathcal{N}$ satisfies condition (CLD) if there exists a constant $c > 0$ such that

$$\|f\| \leq c\|\Delta f\| \quad \text{for all } f \in L_0(V). \tag{CLD}$$

These conditions play important roles in our study. To assure condition (CLD), we recall Poincaré-Sobolev inequality (PS) and related inequalities (SPS) and (GPS) in Section 2. Our main results will be given in Section 3. We shall study some relations among these conditions more precisely in Section 4 by giving some examples.

2. Conditions related to the Laplacian

We say that $\mathcal{N}$ satisfies condition (PS) (the Poincaré-Sobolev inequality) if there exists a constant $c > 0$ such that

$$\sum_{x\in V} f(x)^2 \le c \sum_{e\in E} r(e)^2(\nabla f(e))^2 \quad \text{for all } f \in L_0(V). \tag{PS}$$

We say that $\mathcal{N}$ satisfies condition (SPS) (the strong Poincaré-Sobolev inequality) if there exists a constant $c > 0$ such that

$$\sum_{x\in V} f(x)^2 \le c\|f\|^2 \quad \text{for all } f \in L_0(V). \tag{SPS}$$

We said in [5] that $\mathcal{N}$ satisfies condition (GPS) (the generalized Poincaré-Sobolev inequality) if there exists a constant $c > 0$ such that

$$\sum_{x\in V} \rho(x) f(x)^2 \le c\|f\|^2 \quad \text{for all } f \in L_0(V), \tag{GPS}$$

where

$$\rho(x) = \sum_{e\in E} r(e)^{-1}|K(x,e)|.$$

If $0 < c_1 \le r(e) \le c_2 < \infty$ for all $e \in E$, then

$$c_1\|f\|^2 \le \sum_{e\in E} r(e)^2(\nabla f(e))^2 \le c_2\|f\|^2.$$

Thus we have

Proposition 2.1 *If* $\{r(e) \mid e \in E\}$ *is bounded above, then condition* (PS) *implies condition* (SPS). *If* $\{r(e) \mid e \in E\}$ *is bounded below from* 0*, then condition* (SPS) *implies condition* (PS).

If $0 < \rho_1 \le \rho(x) \le \rho_2 < \infty$ for all $x \in V$, then

$$\rho_1 \sum_{x\in V} f(x)^2 \le \sum_{x\in V} \rho(x) f(x)^2 \le \rho_2 \sum_{x\in V} f(x)^2.$$

Thus we have

Proposition 2.2 *If* $\{\rho(x) \mid x \in V\}$ *is bounded above, then condition* (SPS) *implies condition* (GPS). *If* $\{\rho(x) \mid x \in V\}$ *is bounded below from* 0*, then condition* (GPS) *implies condition* (SPS).

Remark 2.1 If $\{\rho(x) \mid x \in V\}$ is bounded above, then $\{r(e) \mid e \in E\}$ is bounded below from 0. If $\{r(e) \mid e \in E\}$ is bounded above, then $\{\rho(x) \mid x \in V\}$ is bounded below from 0.

Lemma 2.1 *If $\mathcal{N}$ satisfies condition* (SPS)*, then there exists $c > 0$ such that*

$$\sum_{x \in V} u(x)^2 \leq c\|u\|^2 \quad \text{for all } u \in \mathbf{D}_0.$$

Proof. Let $u \in \mathbf{D}_0$. There exists a sequence $\{f_n\}_n$ in $L_0(V)$ such that $\lim_{n\to\infty} \|u - f_n\| = 0$ and $\{f_n\}_n$ converges pointwise to u. Since $\mathcal{N}$ satisfies condition (SPS), there exists $c > 0$ such that $\sum_{x\in V} f_n(x)^2 \leq c\|f_n\|^2$ for all n. By Fatou's lemma, we have

$$\sum_{x \in V} u(x)^2 \leq \liminf_{n\to\infty} \sum_{x \in V} f_n(x)^2 \leq \lim_{n\to\infty} c\|f_n\|^2 = c\|u\|^2.$$

This completes the proof. □

Corollary 2.1 *If $\mathcal{N}$ satisfies condition* (SPS)*, then $\mathcal{N}$ is hyperbolic and there exists the biharmonic Green function for each $a \in V$.*

Proof. Since $\sum_{x\in V} 1_V(x)^2 = \infty$ and $\|1_V\| = 0$, Lemma 2.1 shows that $1_V \notin \mathbf{D}_0$. Lemma 1.1 implies that $\mathcal{N}$ is hyperbolic and there exists the Green function $g_a \in \mathbf{D}_0$ of $\mathcal{N}$ with pole at $a \in V$. By Lemma 2.1 again, there exists $c > 0$ such that $\sum_{x\in V} g_a(x)^2 \leq c\|g_a\|^2 < \infty$. By [8, Theorem 2.3] the biharmonic Green function exists. □

Proposition 2.3 *Condition* (SPS) *implies condition* (CLD)*.*

Proof. Interchanging the order of summation, we have

$$(f, g) = -\sum_{x \in V} f(x)\Delta g(x) \qquad \text{for } f, g \in L_0(V).$$

Let $f \in L_0(V)$. Then Δf is also in $L_0(V)$. Condition (SPS) implies

$$\|f\|^2 = -\sum_{x \in V} f(x)\Delta f(x) \leq \left(\sum_{x \in V} f(x)^2\right)^{1/2} \left(\sum_{x \in V} (\Delta f(x))^2\right)^{1/2}$$

$$\leq \left(c\|f\|^2\right)^{1/2}\left(c\|\Delta f\|^2\right)^{1/2} = c\|f\|\|\Delta f\|,$$

so that $\|f\| \leq c\|\Delta f\|$. □

We recall three lemmas.

Lemma 2.2 ([4, Lemma 3.2]) *If $\mathcal{N}$ satisfies condition* (LD)*, then there exists a constant $c > 0$ such that*

$$\|\Delta u\| \leq c\|u\| \quad \text{for all } u \in \mathbf{D}_0.$$

Lemma 2.3 ([3, Lemma 6.1]) *If $\mathcal{N}$ satisfies condition* (LD)*, then $\Delta u \in \mathbf{D}_0$ for $u \in \mathbf{D}_0$.*

Lemma 2.4 ([4, Theorem 3.1]) *If $\mathcal{N}$ satisfies condition* (LD)*, then $\mathbf{D} \subset \mathbf{D}^{(2)}$.*

Proposition 2.4 *Assume that $\mathcal{N}$ satisfies conditions* (LD) *and* (CLD)*. Then there exists a constant $c > 0$ such that*

$$\|u\| \leq c\|\Delta u\| \quad \text{for all } u \in \mathbf{D}_0.$$

Proof. Let $u \in \mathbf{D}_0$. There exists a sequence $\{f_n\}_n \subset L_0(V)$ such that $\lim_{n\to\infty} \|u - f_n\| = 0$ and $\{f_n\}_n$ converges pointwise to u. Then $\|f_n\| \to \|u\|$ as $n \to \infty$. Since $\mathcal{N}$ satisfies condition (LD), Lemma 2.2 shows that there exists $c_1 > 0$ such that $\|\Delta u - \Delta f_n\| \leq c_1\|u - f_n\|$, so that $\|\Delta f_n\| \to \|\Delta u\|$ as $n \to \infty$. By condition (CLD), there exists a constant $c_2 > 0$ such that $\|f_n\| \leq c_2\|\Delta f_n\|$ for all n. Therefore $\|u\| \leq c_2\|\Delta u\|$. □

3. Representations of the space $\mathbf{D}_0$

In this section we always assume that $\mathcal{N}$ is hyperbolic. For $\mu \in L^+(V)$, we define the *Green potential* $G\mu \in L(V)$ of μ by

$$G\mu(x) = \sum_{y \in V} g_x(y)\mu(y),$$

where g_x denotes the Green function of $\mathcal{N}$ with pole at $x \in V$. For $\mu, \nu \in L^+(V)$, the *mutual Green energy* $G(\mu, \nu)$ is defined by

$$G(\mu, \nu) = \sum_{x \in V} G\mu(x)\nu(x).$$

Since $g_a(b) = g_b(a)$ for all $a, b \in V$, we have $G(\mu, \nu) = G(\nu, \mu)$. We call $G(\mu, \mu)$ the *Green energy* of μ. Let us put

$$\mathbf{E}(G) = \{\mu \in L^+(V) \mid G(\mu, \mu) < \infty\}.$$

We know the following

Lemma 3.1 ([7, Lemma 5.2]) *For $\mu \in \mathbf{E}(G)$, we have $\Delta(G\mu) = -\mu$.*

Lemma 3.2 ([7, Lemma 5.4]) *For $\mu, \nu \in L_0(V) \cap L^+(V)$, we have $(G\mu, G\nu) = G(\mu, \nu)$.*

Lemma 3.3 ([7, Theorem 5.2]) *For $\mu \in \mathbf{E}(G)$ we have $G\mu \in \mathbf{D}_0$ and $\Delta G\mu \leq 0$. Conversely, if $u \in \mathbf{D}_0$ satisfies $\Delta u \leq 0$, then there exists $\mu \in \mathbf{E}(G)$ such that $u = G\mu$.*

We shall prepare

Lemma 3.4 *For $\mu \in \mathbf{E}(G)$, there exists $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ such that $\{G\mu_n\}_n$ converges to $G\mu$ in $\mathbf{D}$ and $\{\mu_n\}_n$ converges pointwise to μ.*

Proof. Let $\{\mathcal{N}_n\}_n$ be an exhaustion of $\mathcal{N}$ with $\mathcal{N}_n = \{V_n, E_n, K_n, r_n\}$. We put $\mu_n = \mu$ on V_n and $\mu_n = 0$ on $V \setminus V_n$. Clearly, $\{\mu_n(x)\}_n$ increases monotonically and converges to $\mu(x)$ for each $x \in V$. We have $0 \leq G\mu_n(x) \leq G\mu_{n+1}(x) \leq G\mu(x)$ for $x \in V$. Fatou's lemma shows that

$$G\mu(x) \leq \liminf_{n \to \infty} G\mu_n(x) = \lim_{n \to \infty} G\mu_n(x) \leq G\mu(x).$$

This means that $\{G\mu_n\}_n$ converges pointwise to $G\mu$.

Let $m < n$. We have

$$G(\mu_m, \mu_m) \leq G(\mu_m, \mu_n) \leq G(\mu_n, \mu_n) \leq G(\mu, \mu).$$

Lemma 3.2 implies that $\{\|G\mu_n\|\}_n$ converges and that

$$\|G\mu_m\|^2 \leq G(\mu_m, \mu_n).$$

We have

$$\|G\mu_n - G\mu_m\|^2 = \|G\mu_n\|^2 - 2(G\mu_n, G\mu_m) + \|G\mu_m\|^2$$
$$\leq \|G\mu_n\|^2 - \|G\mu_m\|^2,$$

so that $\{G\mu_n\}_n$ is a Cauchy sequence in $\mathbf{D}$. Therefore $\{G\mu_n\}_n$ converges to some v in $\mathbf{D}$. Since $\{G\mu_n\}_n$ converges pointwise to both $G\mu$ and v, we conclude that $\{G\mu_n\}_n$ converges to $G\mu$ in $\mathbf{D}$. □

Proposition 3.1 *Let $\{\mu_n\}_n \subset \mathbf{E}(G)$. If $\{G\mu_n\}_n$ converges to a function u in $\mathbf{D}$, then there exists $\mu \in \mathbf{E}(G)$ such that $u = G\mu$.*

Proof. Lemma 3.3 implies that $\{G\mu_n\}_n \subset \mathbf{D}_0$, so that $u \in \mathbf{D}_0$. Lemma 3.1 implies

$$\Delta u(x) = \lim_{n\to\infty} \Delta G\mu_n(x) = -\lim_{n\to\infty} \mu_n(x) \leq 0.$$

Lemma 3.3 shows that $u = G\mu$ for some $\mu \in \mathbf{E}(G)$. □

We show the following

Lemma 3.5 *$\mathbf{D}_0$ is a Hilbert space with respect to the inner product $(\cdot, \cdot)$.*

Proof. Suppose that $u \in \mathbf{D}_0$ and $\|u\| = 0$. Then u is a constant function. Since $\mathcal{N}$ is hyperbolic, Lemma 1.1 shows that $u = 0$. Thus $(\cdot, \cdot)$ is an inner product on $\mathbf{D}_0$. Assume that $\{u_n\}_n$ is a Cauchy sequence in $\mathbf{D}_0$ with respect to the norm $\|\cdot\|$, i.e., $\|u_n - u_m\| \to 0$ as $n, m \to \infty$. Then $\{\|u_n\|\}_n$ is bounded. We show that $\{u_n\}_n$ is bounded. Suppose that, on the contrary, there exists $x_0 \in V$ such that $|u_n(x_0)| \to \infty$ by choosing a subsequence if necessary. Put $v_n = u_n/u_n(x_0)$. Then $v_n \in \mathbf{D}_0$, $v_n(x_0) = 1$, and

$$\|v_n - 1_V\| = \|v_n\| = \frac{\|u_n\|}{|u_n(x_0)|} \to 0$$

as $n \to \infty$, so that $\{v_n\}$ converges to 1_V in $\mathbf{D}$. This means $1_V \in \mathbf{D}_0$, which contradicts Lemma 1.1. Therefore $\{u_n\}_n$ is bounded. By the diagonal process, we may assume that $\{u_n\}_n$ converges pointwise to $u \in L(V)$. Hence $\{u_n\}_n$ is a Cauchy sequence in $\mathbf{D}$ and converges to u in $\mathbf{D}$. We conclude that $u \in \mathbf{D}_0$. □

Corollary 3.1 *Let $\{u_n\}_n \subset \mathbf{D}_0$ and $u \in \mathbf{D}_0$. If $\|u_n - u\| \to 0$ as $n \to \infty$, then $\{u_n\}_n$ converges to u in $\mathbf{D}$.*

Lemma 3.6 *Let $\{u_n\}_n$ be a sequence in* $\mathbf{D}$ *such that $\{\|u_n\|\}_n$ is bounded and that $\{u_n\}_n$ converges pointwise to a function $u \in \mathbf{D}$. Then* $\lim_{n\to\infty}(u_n, v) = (u, v)$ *for $v \in \mathbf{D}$.*

Proof. We take M with $\|u_n\| \le M$ for all n. Fatou's lemma shows that

$$\|u\| \le \liminf_{n\to\infty} \|u_n\| \le M.$$

Let $v \in \mathbf{D}$ and $\varepsilon > 0$. We find a finite subset E_0 of E such that $\sum_{e\in E\setminus E_0} r(e)|\nabla v(e)|^2 < \varepsilon$. Since $\{\nabla u_n(e)\}_n$ converges to $\nabla u(e)$ for each $e \in E$ and E_0 is a finite set, $\sum_{e\in E_0} r(e)|\nabla u_n(e) - \nabla u(e)|^2 < \varepsilon$ for sufficiently large n. We have

$$\begin{aligned}
&\sum_{e\in E_0} r(e)|\nabla v(e)||\nabla u_n(e) - \nabla u(e)| \\
&\quad \le \left(\sum_{e\in E_0} r(e)|\nabla v(e)|^2\right)^{1/2} \left(\sum_{e\in E_0} r(e)|\nabla u_n(e) - \nabla u(e)|^2\right)^{1/2} \\
&\quad \le \|v\|\varepsilon^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{e\in E\setminus E_0} r(e)|\nabla v(e)||\nabla u_n(e) - \nabla u(e)| \\
&\quad \le \left(\sum_{e\in E\setminus E_0} r(e)|\nabla v(e)|^2\right)^{1/2} \left(\sum_{e\in E\setminus E_0} r(e)|\nabla u_n(e) - \nabla u(e)|^2\right)^{1/2} \\
&\quad \le \varepsilon^{1/2}\left(\sum_{e\in E\setminus E_0} r(e)\cdot 2(|\nabla u_n(e)|^2 + |\nabla u(e)|^2)\right)^{1/2} \\
&\quad \le \varepsilon^{1/2}(2\|u_n\|^2 + 2\|u\|^2)^{1/2} \\
&\quad \le \varepsilon^{1/2}(4M^2)^{1/2} = 2\varepsilon^{1/2}M.
\end{aligned}$$

We have

$$|(v, u_n - u)| \le \sum_{e \in E} r(e)|\nabla v(e)||\nabla u_n(e) - \nabla u(e)|$$
$$\le \|v\|\varepsilon^{1/2} + 2\varepsilon^{1/2}M.$$

Since ε is arbitrary, we have $\lim_{n\to\infty} |(v, u_n - u)| = 0$. □

Lemma 3.7 *If $\mu \in \mathbf{D}_0 \cap L^+(V)$, then there exists $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ which converges to μ in $\mathbf{D}$.*

Proof. Since $\mu \in \mathbf{D}_0$, there exists a sequence $\{f_n\}_n$ in $L_0(V)$ which converges to μ in $\mathbf{D}$. Let $\mu_n = \max\{f_n, 0\}$. Then $\|\mu_n\| \le \|f_n\|$. Since $\mu \ge 0$, $\{\mu_n\}_n$ converges pointwise to μ. By Fatou's lemma, we have

$$\|\mu\| \le \liminf_{n\to\infty} \|\mu_n\| \le \limsup_{n\to\infty} \|\mu_n\| \le \lim_{n\to\infty} \|f_n\| = \|\mu\|.$$

This implies that $\{\|\mu_n\|\}_n$ is bounded. Lemma 3.6 shows that $(\mu_n, v) \to (\mu, v)$ as $n \to \infty$ for every $v \in \mathbf{D}$. Thus we have

$$\|\mu - \mu_n\|^2 = \|\mu\|^2 - 2(\mu, \mu_n) + \|\mu_n\|^2 \to 0$$

as $n \to \infty$. Therefore $\{\mu_n\}_n$ converges to μ in $\mathbf{D}$. □

Theorem 3.1 *If condition* (LD) *is satisfied, then $\mathbf{E}(G) \subset \mathbf{D}_0 \cap L^+(V)$ holds.*

Proof. By definition, $\mathbf{E}(G) \subset L^+(V)$. Let $\mu \in \mathbf{E}(G)$. By Lemma 3.4, there exists $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ such that $\{G\mu_n\}_n$ converges to $G\mu$ in $\mathbf{D}$ and $\{\mu_n\}_n$ converges pointwise to μ. By Lemmas 3.1, 3.3, and 2.2

$$\|\mu - \mu_n\| = \|\Delta G\mu_n - \Delta G\mu\| \le c\|G\mu_n - G\mu\| \to 0$$

as $n \to \infty$. We see that $\{\mu_n\}_n$ converges to μ in $\mathbf{D}$. Thus $\mu \in \mathbf{D}_0$. □

Theorem 3.2 *If conditions* (LD) *and* (CLD) *are fulfilled, then $\mathbf{E}(G) = \mathbf{D}_0 \cap L^+(V)$.*

Proof. By virtue of Theorem 3.1, it suffices to show that $\mathbf{D}_0 \cap L^+(V) \subset \mathbf{E}(G)$. Let $\mu \in \mathbf{D}_0 \cap L^+(V)$. By Lemma 3.7, there exists $\{\mu_n\}_n \subset L_0(V) \cap L^+(V)$ which converges to μ in $\mathbf{D}$. Lemma 3.3 implies $\{G\mu_n\}_n \subset \mathbf{D}_0$. Proposition 2.4 and Lemma 3.1 show that

$$\|G\mu_n - G\mu_m\| \leq c\|\Delta(G\mu_n - G\mu_m)\| = c\|\mu_m - \mu_n\| \to 0$$

as $n, m \to \infty$. By Lemma 3.5 and its corollary, we see that there exists $u \in \mathbf{D}_0$ such that $\{G\mu_n\}_n$ converges to u in $\mathbf{D}$. Lemma 3.1 implies

$$\Delta u(x) = \lim_{n\to\infty} \Delta G\mu_n(x) = -\lim_{n\to\infty} \mu_n(x) = -\mu(x)$$

for each $x \in V$, so that $\Delta u = -\mu$. By Fatou's lemma and Lemma 3.2, we have

$$G(\mu, \mu) \leq \liminf_{n\to\infty} G(\mu_n, \mu_n) = \lim_{n\to\infty} \|G\mu_n\|^2 = \|u\|^2 < \infty.$$

Namely $\mu \in \mathbf{E}(G)$. □

For any $u \in L(V)$, we define Gu by $Gu = Gu^+ - Gu^-$ if both Gu^+ and Gu^- converge, where $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

Theorem 3.3 *If $\mathcal{N}$ satisfies conditions* (LD) *and* (CLD), *then* $\mathbf{D}_0 = \mathbf{E}(G) - \mathbf{E}(G)$.

Proof. By Theorem 3.1 $\mathbf{E}(G) - \mathbf{E}(G) \subset \mathbf{D}_0$. For $u \in \mathbf{D}_0$ we have $u = u^+ - u^-$ and $u^+, u^- \in \mathbf{D}_0 \cap L^+(V) = \mathbf{E}(G)$ by Theorem 3.2. □

Theorem 3.4 *Let $u \in \mathbf{D}_0$. If conditions* (LD) *and* (CLD) *are fulfilled, then $Gu \in \mathbf{D}_0$ and $\Delta Gu = -u$.*

Proof. Theorem 3.2 shows that $u^+, u^- \in \mathbf{D}_0 \cap L^+(V) = \mathbf{E}(G)$. By Lemma 3.3 we have that $Gu = Gu^+ - Gu^- \in \mathbf{D}_0$. Lemma 3.1 shows that $\Delta Gu = \Delta Gu^+ - \Delta Gu^- = -u^+ + u^- = -u$. □

Corollary 3.2 $\{Gu \mid u \in \mathbf{D}_0\} \subset \mathbf{D}_0$ *if conditions* (LD) *and* (CLD) *are fulfilled.*

Theorem 3.5 *Assume that conditions* (LD) *and* (CLD) *are fulfilled. Then $G\Delta u = -u$ for $u \in \mathbf{D}_0$.*

Proof. Let $u \in \mathbf{D}_0$. Lemma 2.3 shows that $v := \Delta u \in \mathbf{D}_0$. Theorem 3.4 shows that $Gv \in \mathbf{D}_0$ and $\Delta(u+Gv) = v-v = 0$. Therefore $u+Gv \in \mathbf{D}_0 \cap \mathbf{H}$. Lemma 1.1 implies that $u = -Gv$. □

Corollary 3.3 *If conditions* (LD) *and* (CLD) *are fulfilled,* $\mathbf{D}_0 = \{G\mu - G\nu \mid \mu, \nu \in \mathbf{E}(G)\}$.

Proof. Lemma 3.3 implies that $\{G\mu - G\nu \mid \mu, \nu \in \mathbf{E}(G)\} \subset \mathbf{D}_0$. We show the converse. Let $u \in \mathbf{D}_0$. We put $v = -\Delta u$. Theorem 3.5 shows that $u = Gv = Gv^+ - Gv^-$. □

As an application of our results, we shall give a version of Riesz decomposition of $u \in \mathbf{D}^{(2)}$:

Theorem 3.6 *Assume that conditions* (LD) *and* (CLD) *hold. Then, for every* $u \in \mathbf{D}^{(2)}$, *there exist a unique* $v \in \mathbf{D}_0$ *and a unique* $w \in \mathbf{H}^{(2)}$ *such that* $u = Gv + w$.

Proof. Let $u \in \mathbf{D}^{(2)}$. We use Royden's decomposition for $\Delta u \in \mathbf{D}$, i.e.,

$$\Delta u = -v + h \quad \text{with} \quad v \in \mathbf{D}_0 \quad \text{and} \quad h \in \mathbf{H}.$$

Theorem 3.4 and Lemma 2.4 show that $Gv \in \mathbf{D}_0 \subset \mathbf{D} \subset \mathbf{D}^{(2)}$ and $\Delta Gv = -v \in \mathbf{D}_0$. We see that $Gv \in \mathbf{D}_0^{(2)}$. Let $w = u - Gv \in \mathbf{D}^{(2)}$. We have

$$\Delta w = \Delta u - \Delta Gv = \Delta u + v = h,$$

so that $w \in \mathbf{H}^{(2)}$.

To show the uniqueness, we assume that $u = Gv_1 + w_1 = Gv_2 + w_2$ with $v_1, v_2 \in \mathbf{D}_0$ and $w_1, w_2 \in \mathbf{H}^{(2)}$. Theorem 3.4 shows that $w_1 - w_2 = Gv_2 - Gv_1 \in \mathbf{D}_0$. Lemma 2.3 implies $\Delta(w_1 - w_2) \in \mathbf{D}_0$. On the other hand, since $w_1 - w_2 \in \mathbf{H}^{(2)}$, we have $\Delta(w_1 - w_2) \in \mathbf{H}$. Lemma 1.1 shows that $\Delta(w_1 - w_2) = 0$, so that $w_1 - w_2 \in \mathbf{D}_0 \cap \mathbf{H}$. Again by Lemma 1.1 we have $w_1 = w_2$, so that $Gv_1 = Gv_2$. Theorem 3.4 gives $v_1 = -\Delta Gv_1 = -\Delta Gv_2 = v_2$. □

Corollary 3.4 $\mathbf{D}^{(2)} = \mathbf{D}_0^{(2)} + \mathbf{H}^{(2)}$ *if conditions* (LD) *and* (CLD) *are fulfilled.*

Proof. Clearly $\mathbf{D}_0^{(2)} + \mathbf{H}^{(2)} \subset \mathbf{D}^{(2)}$. We show the converse. Let $u \in \mathbf{D}^{(2)}$. By Theorem 3.6 we take $v \in \mathbf{D}_0$ and $w \in \mathbf{H}^{(2)}$ such that $u = Gv + w$. Theorem 3.4 shows that $\Delta Gv = -v \in \mathbf{D}_0$, so that $Gv \in \mathbf{D}_0^{(2)}$. □

4. Supplementary remarks

We shall study conditions (PS), (SPS), and (GPS) introduced in Section 2 by giving examples. For a finite subset A of V, denote by ∂A the set of arcs $e \in E$ whose one endpoint belongs to A and another does to $V \setminus A$. Let $|A|$ and $|\partial A|$ be the cardinality of the sets A and ∂A, respectively.

We say that the *strong isoperimetric inequality* (SI) holds if there exists a constant $c > 0$ such that

$$|A| \leq c|\partial A| \quad \text{for all finite subsets } A \subset V. \tag{SI}$$

We recall $\rho(x) = \sum_{e \in E} r(e)^{-1}|K(x,e)|$ for $x \in V$. Let

$$\rho(A) = \sum_{x \in A} \rho(x), \quad R(\partial A) = \sum_{e \in \partial A} r(e)^{-1}.$$

A *generalized strong isoperimetric inequality* (GSI) is defined in [5] as that there exists a constant $c > 0$ such that

$$\rho(A) \leq cR(\partial A) \quad \text{for all finite subsets } A \subset V. \tag{GSI}$$

We have

Lemma 4.1 ([5, Theorem 2.1]) *Condition* (GPS) *holds if and only if condition* (GSI) *holds.*

Lemma 4.2 ([1, Proposition 4.4]) *Suppose that $\mathcal{N}$ has a bounded degree; i.e.,* $\sup_{x \in V} \sum_{e \in E} |K(x,e)| < \infty$. *Then condition* (PS) *holds if and only if condition* (SI) *holds.*

Example 4.1 ((SPS) and (GPS), but not (PS)) Let $\mathcal{N} = \{V, E, K, r\}$ be a half linear network; i.e., let $V = \{x_0, x_1, x_2, \dots\}$ and $E = \{e_1, e_2, \dots\}$. Let $K(x_{n-1}, e_n) = 1$ and $K(x_n, e_n) = -1$ for $n = 1, 2, \dots$; let $K(x,e) = 0$ for any other pair $(x,e) \in V \times E$. Let $r(e_n) = 2^{-n}$.

Let $X_n = \{x_0, x_1, \dots, x_n\}$. Then $|X_n| = n+1$ and $|\partial X_n| = 1$. Lemma 4.2 shows that condition (PS) does not hold.

Let A be a non-empty finite subset of V. Let n be the smallest number with $A \subset X_n$. We claim $\rho(A) \leq 3R(\partial A)$ by induction on n. If $n = 0$, then A is a singleton, and so $\rho(A) = R(\partial A)$. We assume that $\rho(B) \leq 3R(\partial B)$ for $B \subset X_{n-1}$ and suppose that $x_n \in A \subset X_n$. Let $A_1 = A \setminus \{x_n\}$ and $A_2 = \{x_n\}$. By the induction hypothesis $\rho(A_1) \leq 3R(\partial A_1)$. Since A_2 is a singleton, we have $\rho(A_2) = R(\partial A_2)$. Also we have $\rho(A) = \rho(A_1) + \rho(A_2)$ and $R(\partial A) \geq R(\partial A_1) + R(\partial A_2) - 2r(e_n)^{-1}$. Since

$$\rho(A_2) = \rho(x_n) = r(e_n)^{-1} + r(e_{n+1})^{-1} = 2^n + 2^{n+1} = 3 \cdot 2^n = 3r(e_n)^{-1},$$

we have

$$\begin{aligned}\rho(A) = \rho(A_1) + \rho(A_2) &\leq 3R(\partial A_1) + 3R(\partial A_2) - 2\rho(A_2) \\ &= 3(R(\partial A_1) + R(\partial A_2) - 2r(e_n)^{-1}) \leq 3R(\partial A).\end{aligned}$$

Lemma 4.1 and Proposition 2.2 show that conditions (GPS) and (SPS) hold. □

Lemma 4.3 *Let $\mathcal{N} = \{V, E, K, r\}$ be a binary tree network, i.e.,*

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x_i^{(n)} \mid i = 0, 1, \dots, 2^n - 1\},$$

$$E = \bigcup_{n=1}^{\infty} E_n, \qquad E_n = \{e_i^{(n)} \mid i = 0, 1, \dots, 2^n - 1\}.$$

We define $K(x_i^{(n)}, e_i^{(n)}) = -1$ for $n = 1, 2, \dots$ and for $i = 0, 1, \dots, 2^n - 1$; $K(x_i^{(n)}, e_i^{(n+1)}) = K(x_i^{(n)}, e_{i+2^n}^{(n+1)}) = 1$ for $n = 0, 1, \dots$ and for $i = 0, 1, \dots, 2^n - 1$; $K(x, e) = 0$ for any other pair $(x, e) \in V \times E$. Let A be a non-empty finite subset of V. Then $|\partial A| \geq |A| + 1$. Especially condition (PS) *holds.*

Proof. Let $m = |A|$. We show $|\partial A| \geq m + 1$ by induction on m. If $m = 1$, then $|\partial A|$ is two or three, and so $|\partial A| \geq m+1$. Assume that $|\partial B| \geq |B|+1$ if $|B| < m$ and suppose $|A| = m$. Let n be the largest number with $V_n \cap A \neq \emptyset$ and let $x_i^{(n)} \in A$. Let $B = A \setminus \{x_i^{(n)}\}$. Since $|B| = m - 1$, we have $|\partial B| \geq m$. Since $\partial B \setminus \partial A \subset \{e_i^{(n)}\}$ and $\partial A \setminus \partial B \supset \{e_i^{(n+1)}, e_{i+2^n}^{(n+1)}\}$, we have $|\partial A| \geq |\partial B| + 1 \geq m + 1$. Lemma 4.2 shows that condition (PS) holds. □

Example 4.2 ((PS) and (SPS), but not (GPS)) Let V, E, and K be the same as those of Lemma 4.3. Let $r(e_0^{(n)}) = n^{-1}$ and $r(e_i^{(n)}) = 1$ for $n = 1, 2, \dots$ and for $i = 1, 2, \dots, 2^n - 1$.

Lemma 4.3 and Proposition 2.1 show that conditions (PS) and (SPS) hold.

Let $A_n = \{x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(n)}\}$. Then

$$\rho(A_n) = \sum_{k=0}^{n} \rho(x_0^{(k)}) = 2 + \sum_{k=1}^{n} \big(k + (k+1) + 1\big)$$
$$= 2 + \sum_{k=1}^{n} (2k+2) = n^2 + 3n + 2,$$
$$R(\partial A_n) = \sum_{k=1}^{n+1} r\big(e_{2^k-1}^{(k)}\big)^{-1} + r\big(e_0^{(n+1)}\big)^{-1}$$
$$= (n+1)\cdot 1 + (n+1) = 2n+2$$

Lemma 4.1 shows that condition (GPS) does not hold. □

Example 4.3 (hyperbolic, (PS), and (GPS), but not (SPS)) Let V, E, and K be the same as those of Lemma 4.3. Let $1 < a < 2$ and $r(e_i^{(n)}) = a^n$ for each n and i.

Lemma 4.3 shows that condition (PS) holds.

Let $\varphi(e_i^{(n)}) = 2^{-n}$. Then

$$\langle \varphi, \varphi \rangle = \sum_{n=1}^{\infty} 2^n a^n (2^{-n})^2 = \sum_{n=1}^{\infty} (a/2)^n < \infty,$$

so that $\{V, E, K, r\}$ is hyperbolic by [6, Theorem 4.3].

Let f_n be the characteristic function of $\bigcup_{k=0}^{n} V_k$. Then

$$\sum_{x \in V} f_n(x)^2 = \sum_{k=0}^{n} 2^k \cdot 1^2 = 2^{n+1} - 1,$$
$$\|f_n\|^2 = \sum_{i=0}^{2^{n+1}-1} r(e_i^{(n+1)})\big(r(e_i^{(n+1)})^{-1}\big)^2 = (2/a)^{n+1}.$$

This means that condition (SPS) does not hold.

For $n = 1, 2, \dots$ let

$$\rho_n := \rho(x_i^{(n)}) = (a^n)^{-1} + 2(a^{n+1})^{-1} = (a+2)a^{-n-1}.$$

Let $c = (2+a)/(2-a)$. Note that $(c-1)\rho_n = 2ca^{-n}$. Let A be a non-empty finite subset of V and n the smallest number with $A \subset \bigcup_{k=0}^{n} V_k$. We show that $\rho(A) \le cR(\partial A)$ by induction on n. If $n = 0$, then A is a singleton, so that $\rho(A) = R(\partial A)$. We assume that $\rho(B) \le cR(\partial B)$ for $B \subset \bigcup_{k=0}^{n-1} V_k$ and suppose that $A \subset \bigcup_{k=0}^{n} V_k$. Let $A_1 = A \cap (\bigcup_{k=0}^{n-1} V_k)$ and $A_2 = A \cap V_n$. We may assume that $A_2 \ne \emptyset$. By the induction hypothesis $\rho(A_1) \le cR(\partial A_1)$. Let $Q = \partial A_1 \cap \partial A_2$, $q = |Q|$ and $p = |A_2|$. Then $q \le p$. Also we have

$$\rho(A) = \rho(A_1) + \rho(A_2) = \rho(A_1) + p\rho_n$$

and

$$\begin{aligned} R(\partial A) &= R(\partial A_1) + R(\partial A_2) - 2\sum_{e \in Q} r(e)^{-1} = R(\partial A_1) + p\rho_n - 2qa^{-n} \\ &\ge R(\partial A_1) + p\rho_n - 2pa^{-n}. \end{aligned}$$

We obtain

$$\begin{aligned} cR(\partial A) - \rho(A) &\ge (cR(\partial A_1) - \rho(A_1)) + cp\rho_n - 2cpa^{-n} - p\rho_n \\ &\ge (c-1)p\rho_n - 2cpa^{-n} = 0. \end{aligned}$$

Lemma 4.1 shows that condition (GPS) holds. □

Next we consider conditions (LD) and (CLD).

Example 4.4 ((LD) and (CLD)) Let V, E, and K be the same as those of Lemma 4.3. Let $r(e_i^{(n)}) = 1$ for each n and i.

By [3, Proposition 6.1] condition (LD) holds.

Lemma 4.3, Propositions 2.1 and 2.3 show that condition (CLD) holds. □

Finally we address an open question and give a partially affirmative answer.

Question 4.1 *Does condition* (CLD) *imply condition* (SPS)*?*

Proposition 4.1 *Assume that $\mathcal{N} = \{V, E, K, r\}$ is a hyperbolic network and that $\alpha := \inf\{g_a(a) \mid a \in V\} > 0$. Also assume that there exists a constant $c > 0$ such that $\|u\| \le c\|\Delta u\|$ for all $u \in \mathbf{D}_0$. Then condition* (SPS) *holds.*

Proof. Let $f \in L_0(V)$. First assume that $f \ge 0$. Lemma 3.3 shows that

$u := Gf \in \mathbf{D}_0$. Using Lemmas 3.2 and 3.1, we have

$$G(f, f) = \|Gf\|^2 = \|u\|^2 \leq c^2 \|\Delta u\|^2 = c^2 \|f\|^2$$

and

$$G(f, f) = \sum_{x \in V} \sum_{y \in V} g_x(y) f(x) f(y) \geq \alpha \sum_{x \in V} f(x)^2.$$

Thus $\sum_{x \in V} f(x)^2 \leq c^2 \alpha^{-1} \|f\|^2$.

In the general case let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Then $f = f^+ - f^-$ and $f^+(x) f^-(x) = 0$ for each $x \in V$. Also, by [7, Lemma 1.4], we have $\|f^+\| \leq \|f\|$ and $\|f^-\| \leq \|f\|$. Therefore

$$\begin{aligned}\sum_{x \in V} f(x)^2 &= \sum_{x \in V} (f^+(x)^2 + f^-(x)^2) \leq c^2 \alpha^{-1} (\|f^+\|^2 + \|f^-\|^2) \\ &\leq 2c^2 \alpha^{-1} \|f\|^2,\end{aligned}$$

and condition (SPS) holds. □

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