

Topological bi- \mathcal{K} -equivalence of pairs of map germs

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Abstract. Let $P^k(n, p \times q)$ be the set of all pairs of real polynomial map germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$ with degree of $f_1, \dots, f_p, g_1, \dots, g_q$ less than or equal to $k \in \mathbb{N}$. The main result of this paper shows that the set of equivalence classes of $P^k(n, p \times q)$, with respect to bi- C^0 - \mathcal{K} -equivalence, is finite.

Key words: Topological contact equivalence, finiteness theorem, topological classification, pairs of map germs.

1. Introduction

The notions of bi-equivalence and bi-stability of couples of germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$ was introduced by J. P. Dufour in [3]. Motivated by Dufour's work, L. A. Favaro and his students in decade 80 introduced the notion of bi- \mathcal{K} -equivalence in order to reduce the problem of classification of pairs of germs for the problem of isomorphic classification of \mathbb{R} -algebras, similar to the approach taken by J. Mather [7]. An overview of this theory can be found in [2].

A pair of map germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$ can be seen as a divergent diagram

$$(\mathbb{R}^q, 0) \xleftarrow{g} (\mathbb{R}^n, 0) \xrightarrow{f} (\mathbb{R}^p, 0).$$

Divergent diagrams appear in several geometrical problems and it has many applications. Classifications of divergent diagrams can be found in the papers [3], [5], [12].

In [2] the authors introduced the notion of topological bi- \mathcal{K} -equivalence (or bi- C^0 - \mathcal{K} -equivalence) and listed some open questions about the classifi-

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cation problem with respect to this equivalence relation. In order to answer the questions, listed in [2], one has to study the finiteness property of the classification problems, induced by the equivalence relation. Consider the topological \mathcal{A} -equivalence (i.e., change of coordinates given by homeomorphisms in the source and in the target). Nakai [8] proved that the space $P^4(3, 2)$ has infinitely many equivalence classes. However, for the topological \mathcal{K} -equivalence (or C^0 - \mathcal{K} -equivalence) of real polynomial map germs the finiteness property holds (cf. [9], [1], [11]). The C^0 - \mathcal{K} -equivalence is a topological version of the classical contact equivalence (or \mathcal{K} -equivalence) introduced by J. Mather in [6].

In this paper we consider the set $P^k(n, p \times q)$ of all pairs of real polynomial map germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$ with degree of $f_1, \dots, f_p, g_1, \dots, g_q$ less than or equal to $k \in \mathbb{N}$. We show that the number of equivalence classes in $P^k(n, p \times q)$, with respect to bi- C^0 - \mathcal{K} -equivalence, is finite (Theorem 3.1).

2. C^0 - \mathcal{K} -equivalence and bi- C^0 - \mathcal{K} -equivalence

Definition 2.1 Two continuous map germs $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ are called C^0 - \mathcal{K} -equivalent if there exist germs of homeomorphisms $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$ and $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $H(\mathbb{R}^n \times \{0\}^p) = \mathbb{R}^n \times \{0\}^p$ and the following diagram is commutative:

$$\begin{array}{ccccc}
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\
 h \downarrow & & H \downarrow & & h \downarrow \\
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0)
 \end{array}$$

where $id_n : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ is the identity map germ of \mathbb{R}^n , $\pi_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n, 0)$ is the canonical projection germ and $\{0\}^p = (0, \dots, 0) \in \mathbb{R}^p$.

When $h = id_n$, we say that f and g are C^0 - \mathcal{C} -equivalent.

The next lemma gives a sufficient condition for the C^0 - \mathcal{K} -equivalence:

Lemma 2.2 (Nishumura [10]) *Let U be a neighbourhood of the origin of \mathbb{R}^n and let $f, g : U \rightarrow \mathbb{R}^p$ be two continuous mappings. Suppose that there exists a family of continuous mappings $F_t : U \rightarrow \mathbb{R}^p$ ($t \in [0, 1]$) such that*

the following conditions hold:

- i) $F_0 = f$ and $F_1 = g$ or $\bar{g} = (g_1, \dots, g_{p-1}, -g_p)$.
- ii) $F_t^{-1}(0) = f^{-1}(0)$ for any $t \in [0, 1]$.
- iii) For any $t \in [0, 1]$, the vector $F_t(x)$ is not included in the set $\{\alpha F_0(x) \mid \alpha \in \mathbb{R}_-\}$ for any $x \in U \setminus f^{-1}(0)$, where $\mathbb{R}_- = \{s \in \mathbb{R} \mid s < 0\}$.

Then two germs F_t and $F_{t'}$ at 0 are C^0 - \mathcal{K} -equivalent for any $t, t' \in [0, 1]$. In particular, f and g are C^0 - \mathcal{K} -equivalent.

We apply the Lemma 2.2 and obtain the following result:

Theorem 2.3 Let $f = (f_1, \dots, f_p), g = (g_1, \dots, g_p) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be two real polynomial map germs. Consider the algebraic sets $X_i = f_i^{-1}(0), Y_j = g_j^{-1}(0)$, for all $1 \leq i, j \leq p$. Suppose there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that:

- 1) $h(X_1) = Y_1, \dots, h(X_p) = Y_p$ and
- 2) $sign(f_1(x)) = sign(g_1 \circ h(x))$, for all $x \in \mathbb{R}^n \setminus f_1^{-1}(0)$
- \vdots
- \vdots
- $sign(f_p(x)) = sign(g_p \circ h(x))$, for all $x \in \mathbb{R}^n \setminus f_p^{-1}(0)$.

Then, the germs f and g are C^0 - \mathcal{K} -equivalent.

Proof. For any $i = 1, \dots, p$, define the following homotopy $F_t : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0), t \in [0, 1]$,

$$F_t(x) = (1 - t)f_i(x) + t(g_i \circ h)(x).$$

Then, $F_0 = f_i$ and $F_1 = g_i \circ h$. Furthermore, $F_t^{-1}(0) = f_i^{-1}(0)$, for any $t \in [0, 1]$. In fact, if $x_0 \in F_t^{-1}(0)$ then

$$(1 - t)f_i(x_0) + t(g_i \circ h)(x_0) = 0.$$

But, since $sign(f_i(x)) = sign(g_i \circ h(x))$ for all $x \in \mathbb{R}^n \setminus f_i^{-1}(0)$, then

$$f_i(x_0) = g_i \circ h(x_0) = 0.$$

Hence, $x_0 \in f_i^{-1}(0)$. The converse is trivial.

Assertion. For any $t \in [0, 1]$, the vector $F_t(x)$ is not included in the set

$\{\alpha F_0(x) \mid \alpha \in \mathbb{R}_-\}$, for all $x \in \mathbb{R}^n \setminus f_i^{-1}(0)$.

Suppose that $F_{t_0}(x_0) \in \{\alpha F_0(x) \mid \alpha \in \mathbb{R}_-\}$ for some $x_0 \in \mathbb{R}^n \setminus f_i^{-1}(0)$ and for some $t_0 \in [0, 1]$. Then, there exists $\alpha \in \mathbb{R}_-$ such that

$$(1 - t_0)f_i(x_0) + t_0(g_i \circ h)(x_0) = \alpha f_i(x_0).$$

Therefore,

$$(1 - t_0 - \alpha)f_i(x_0) + t_0(g_i \circ h)(x_0) = 0.$$

Since $(1 - t_0 - \alpha), t_0 \geq 0$ and $sign(f_i(x)) = sign(g_i \circ h(x))$ we have that $x_0 \in f_i^{-1}(0)$ which is a contradiction. The *Assertion* is proved.

Using the Lemma 2.2, we conclude that f_i and $g_i \circ h$ are C^0 - \mathcal{K} -equivalent.

Hence, there exist germs of homeomorphisms $T_i : (\mathbb{R}^n \times \mathbb{R}, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ and $\phi_i : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $T_i(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id_n, f_i)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ \phi_i \downarrow & & T_i \downarrow & & \phi_i \downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{(id_n, g_i \circ h)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

Moreover, the homeomorphism T_i can be written in the following form: $T_i(x, z) = (\phi_i(x), H_i(x, z))$, where $H_i(x, 0) = 0$, where $(x, z) \in \mathbb{R}^n \times \mathbb{R}$.

In the proof of Lemma 2.2 Nishumura shows that the homeomorphism $\phi_i = id_n$ satisfies the previous conditions. Hence, we can assume $T_i(x, z) = (x, H_i(x, z))$. Therefore, f_i and $g_i \circ h$ are C^0 - \mathcal{C} -equivalent for any $i = 1, \dots, p$.

Consider the map germ $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$ given by

$$H(x, y) = (h(x), H_1(x, y_1), \dots, H_p(x, y_p)), \quad x \in \mathbb{R}^n, y = (y_1, \dots, y_p) \in \mathbb{R}^p,$$

where h, H_1, \dots, H_p are as we defined before. Clearly, H is a homeomorphism and $H(x, 0) = (h(x), H_1(x, 0), \dots, H_p(x, 0)) = (h(x), 0, \dots, 0)$. Moreover, the following diagram is commutative:

$$\begin{array}{ccccc}
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\
 \downarrow h & & \downarrow H & & \downarrow h \\
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0)
 \end{array}$$

That is why the germs f and g are C^0 - \mathcal{K} -equivalent. □

In this work we focus on a version of C^0 - \mathcal{K} -equivalence adapted for pairs of map germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$. For such pairs, the notion of *bi- C^0 - \mathcal{K} -equivalence* was introduced in [2], as following:

Definition 2.4 Two pairs of continuous map germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are said to be *topologically bi- \mathcal{K} -equivalent* (or *bi- C^0 - \mathcal{K} -equivalent*) if there exist germs of homeomorphisms

$$H : (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) \text{ and } h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$$

such that $H(\mathbb{R}^n \times \{0\}^p \times \{0\}^q) = \mathbb{R}^n \times \{0\}^p \times \{0\}^q$, the following diagram is commutative

$$\begin{array}{ccccc}
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, (f_1, f_2))} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\
 \downarrow h & & \downarrow H & & \downarrow h \\
 (\mathbb{R}^n, 0) & \xrightarrow{(id_n, (g_1, g_2))} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0)
 \end{array}$$

and moreover $H = (h \circ \pi_n, H_1 \circ \pi_{n,p}, H_2 \circ \pi_{n,q})$ where id_n is the identity map germ of \mathbb{R}^n ; π_n is the usual projection in \mathbb{R}^n ; $\pi_{n,p}$ is the usual projection in $\mathbb{R}^n \times \mathbb{R}^p$; $\pi_{n,q}$ is the usual projection in $\mathbb{R}^n \times \mathbb{R}^q$; $H_1 : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$; $H_2 : (\mathbb{R}^n \times \mathbb{R}^q, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^q, 0)$, $\{0\}^p = (0, \dots, 0) \in \mathbb{R}^p$ and $\{0\}^q = (0, \dots, 0) \in \mathbb{R}^q$.

When $h = id_n$ we say that (f_1, f_2) and (g_1, g_2) are *bi- C^0 - \mathcal{C} -equivalent*.

Remark 2.5

1. By Definition 2.4, the homeomorphism H can be written as follows:

$$H(x, y, z) = (h(x), H_1(x, y), H_2(x, z)) \text{ with } H_1(x, 0) = H_2(x, 0) = 0,$$

where $(x, y, z) \in (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0)$.

2. If (f_1, f_2) and (g_1, g_2) are bi- C^0 - \mathcal{K} -equivalent, then the germs $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are C^0 - \mathcal{K} -equivalent.
3. If (f_1, f_2) and (g_1, g_2) are bi- C^0 - \mathcal{K} -equivalent, then the germs f_1 and g_1 are C^0 - \mathcal{K} -equivalent and also f_2 and g_2 are C^0 - \mathcal{K} -equivalent.

3. Finiteness theorem

The finiteness problem of a set of polynomial mappings or map germs with respect to some topological equivalence relation has a long history. In the references, we cite some examples of topological classifications, where the finiteness property holds (cf. [1], [9], [11]) and also we cite an examples where it does not hold (cf. [8]). The main theorem of this paper is the following:

Theorem 3.1 (Finiteness theorem) *Let $P^k(n, p \times q)$ be the set of all pairs of real polynomial map germs $(f, g) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p \times \mathbb{R}^q, 0)$, $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_q)$, with degree of $f_1, \dots, f_p, g_1, \dots, g_q$ less than or equal to $k \in \mathbb{N}$. Then the set of the equivalence classes of $P^k(n, p \times q)$, with respect to bi- C^0 - \mathcal{K} -equivalence, is finite.*

Before proving the Theorem 3.1 we need some preliminary results.

Definition 3.2 Let $V_1, \dots, V_p, W_1, \dots, W_p$ be algebraic subsets of \mathbb{R}^n . A finite sequence of algebraic sets (V_1, \dots, V_p) is said to be *strongly topologically equivalent* to a sequence (W_1, \dots, W_p) if there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $h(V_1) = W_1, \dots, h(V_p) = W_p$.

Definition 3.3 Let $P^k(n, p)$ be the set of all real polynomial map germs $f = (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ with degree of f_1, \dots, f_p less than or equal to $k \in \mathbb{N}$. Two map germs $f, g \in P^k(n, p)$ are said to be \mathcal{V} -equivalent if the sequence of algebraic sets (X_1, \dots, X_p) is strongly topologically equivalent to the sequence of algebraic sets (Y_1, \dots, Y_p) , where $X_i = f_i^{-1}(0)$ and $Y_j = g_j^{-1}(0)$ for all $i, j = 1, \dots, p$.

Theorem 3.4 *The set of the equivalence classes of $P^k(n, p)$, with respect to \mathcal{V} -equivalence, is finite.*

Lemma 3.5 *Let $f = (f_1, \dots, f_p), g = (g_1, \dots, g_p) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ in $P^k(n, p)$. Suppose there exists a homeomorphism $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, 0)$ such that:*

- i) The sets $\mathbb{R}^n \times \{0\}^p$, $\mathbb{R}^n \times \mathbb{R}^{p-1} \times \{0\}$, $\mathbb{R}^n \times \mathbb{R}^{p-2} \times \{0\} \times \mathbb{R}$, \dots , $\mathbb{R}^n \times \{0\} \times \mathbb{R}^{p-1}$ are invariant under H . In other words, H satisfies the following properties

$$\begin{aligned} H(\mathbb{R}^n \times \{0\}^p) &= \mathbb{R}^n \times \{0\}^p, H(\mathbb{R}^n \times \mathbb{R}^{p-1} \times \{0\}) \\ &= \mathbb{R}^n \times \mathbb{R}^{p-1} \times \{0\}, \dots, H(\mathbb{R}^n \times \{0\} \times \mathbb{R}^{p-1}) \\ &= \mathbb{R}^n \times \{0\} \times \mathbb{R}^{p-1}. \end{aligned}$$

- ii) $H(\text{graph}(f)) = \text{graph}(g)$, where $\text{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid y = f(x)\}$.

Then, the germs f and g are \mathcal{V} -equivalent.

Proof. Let $\pi_n : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n, 0)$ be the usual canonical projection germ and let $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ be defined by $h(x) = \pi_n(H(x, f(x)))$.

Claim 1. The germ h at 0 is a germ of homeomorphism.

Since g is a polynomial map germ, the projection $\pi_n|_{\text{graph}(g)}$ is a homeomorphism. By the same argument, the map $x \mapsto (x, f(x))$ is also a homeomorphism. The map H is homeomorphism by hypothesis and H maps the $\text{graph}(f)$ onto $\text{graph}(g)$. Hence, the composition h is a homeomorphism map germ. The Claim 1 is proved.

Claim 2. $h(X_j) = Y_j$, where $X_j = f_j^{-1}(0)$ and $Y_j = g_j^{-1}(0)$, for all $j = 1, \dots, p$.

Take a point $x \in X_j$. Then the point $(x, f(x))$ belongs to the set $\text{graph}(f) \cap \mathbb{R}^n \times \mathbb{R}^{j-1} \times \{0\} \times \mathbb{R}^{p-j}$. Since H maps $\text{graph}(f)$ to $\text{graph}(g)$ and the space $\mathbb{R}^n \times \mathbb{R}^{j-1} \times \{0\} \times \mathbb{R}^{p-j}$ is invariant under H , we obtain that $(h(x), g \circ h(x))$ belongs to $\text{graph}(g) \cap \mathbb{R}^n \times \mathbb{R}^{j-1} \times \{0\} \times \mathbb{R}^{p-j}$. It means that $g_j \circ h(x) = 0$, i.e., $h(x) \in Y_j$. The converse is analogous, just using the same procedure for the map h^{-1} . The Claim 2 is proved.

It follows from Claim 1 and Claim 2 that $f = (f_1, \dots, f_p)$ and $g = (g_1, \dots, g_p)$ are \mathcal{V} -equivalent. \square

Remark 3.6 The equivalence of sets which appears in Lemma 3.5 is called *topological equivalence of $\text{graph}(f)$ and $\text{graph}(g)$ with respect to the family*

of algebraic sets $\mathbb{R}^n \times \{0\}^p$, $\mathbb{R}^n \times \mathbb{R}^{p-1} \times \{0\}$, $\mathbb{R}^n \times \mathbb{R}^{p-2} \times \{0\} \times \mathbb{R}$, \dots , $\mathbb{R}^n \times \{0\} \times \mathbb{R}^{p-1}$.

Proof of Theorem 3.4. By Hardt's semialgebraic triviality theorem (cf. [4]), the number of equivalence classes with respect to topological equivalence of $\text{graph}(f)$ and $\text{graph}(g)$ with respect to the family of algebraic sets described in the Lemma 3.5 is finite. Notice that to apply the Hardt's theorem it is essential that the degrees of the coordinate function germs are limited. Then, by Lemma 3.5, it follows that the number of equivalence classes of $P^k(n, p)$, with respect to \mathcal{V} -equivalence, is also finite. \square

Let $(V_1, \dots, V_p), (W_1, \dots, W_p)$ be two sequences of algebraic subsets of \mathbb{R}^n . We say that they have *limited complexity* if the degree of polynomials $f_1, \dots, f_p, g_1, \dots, g_p : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ are limited, where $f_i^{-1}(0) = V_i$ and $g_i^{-1}(0) = W_i$, $i = 1, \dots, p$.

Corollary 3.7 *Let \mathcal{F} be the set of all sequences of algebraic subsets of \mathbb{R}^n with limited complexity. Then the set of equivalence classes of \mathcal{F} , with respect to strongly topological equivalence, is finite.*

4. Proof of Theorem 3.1

In this section we present the proof of our main result. We start from the following lemma:

Lemma 4.1 *Let $f_i, g_i : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be real polynomial function germs with degree of f_i, g_i less than or equal to $K \in \mathbb{N}$ and let X_i, Y_i be algebraic subsets of \mathbb{R}^n , such that $f_i^{-1}(0) = X_i$ and $g_i^{-1}(0) = Y_i$ for all $i \in \mathbb{N}$. Suppose that for each $i, j \in \mathbb{N}$, (X_i, Y_i) and (X_j, Y_j) are strongly topologically equivalent. Then, there exist $s, t \in \mathbb{N}$, $s \neq t$, and a homeomorphism $h_{st} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that:*

$$\begin{cases} \text{sign}(f_s(x)) = \text{sign}(f_t \circ h_{st}(x)) & \forall x \in \mathbb{R}^n \setminus f_s^{-1}(0), \\ \text{sign}(g_s(x)) = \text{sign}(g_t \circ h_{st}(x)) & \forall x \in \mathbb{R}^n \setminus g_s^{-1}(0). \end{cases}$$

Proof. The construction, used in the proof is a local construction, but for simplicity of the notations we do not use the symbol of germs. Since (X_i, Y_i) and (X_j, Y_j) are strongly topologically equivalent, each subset X_i has the same topological type of X_j and each Y_i has the same topological type of

Y_j , for all $i, j \in \mathbb{N}$. Hence, $\mathbb{R}^n \setminus X_i$ (resp. $\mathbb{R}^n \setminus Y_i$) has the same number of connected components of $\mathbb{R}^n \setminus X_j$ (resp. $\mathbb{R}^n \setminus Y_j$) and this number is finite. Denote by C_{iv} be a connected component of $\mathbb{R}^n \setminus X_i$ for each $i \in \mathbb{N}$ (resp. D_{iv} be a connected component of $\mathbb{R}^n \setminus Y_i$ for each $i \in \mathbb{N}$). We can associate to each connected component C_{iv} (resp. D_{iv}) a sign in the following way: if $f_i > 0$ in C_{iv} we associate the sign $+$ (resp. if $g_i > 0$ in D_{iv} we associate the sign $+$). If $f_i < 0$ in C_{iv} we associate the sign $-$ (resp. if $g_i < 0$ in D_{iv} we associate the sign $-$), where $f_i^{-1}(0) = X_i$ and $g_i^{-1}(0) = Y_i$. Suppose, for instance, we have the following table of signs for the connected components C_{iv} 's:

$C_{11} \rightarrow +$	$C_{21} \rightarrow +$	$C_{31} \rightarrow -$	\cdots	$C_{k1} \rightarrow -$	\cdots
$C_{12} \rightarrow +$	$C_{22} \rightarrow +$	$C_{32} \rightarrow -$	\cdots	$C_{k2} \rightarrow +$	\cdots
$C_{13} \rightarrow +$	$C_{23} \rightarrow -$	$C_{33} \rightarrow +$	\cdots	$C_{k3} \rightarrow -$	\cdots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$C_{1v} \rightarrow +$	$C_{2v} \rightarrow -$	$C_{3v} \rightarrow -$	\cdots	$C_{kv} \rightarrow +$	\cdots

As for each X_i the set $\mathbb{R}^n \setminus X_i$ has v connected components C_{iv} , then there exist only 2^v distinct possibilities for their signs. Since the number of algebraic subsets X_i 's is infinite and we have a finite combination for the signs, then at least one permutation of the signs repeats infinitely many times. Choose this codification. Taking subsequence, we may assume that the signs are preserved on each connected component.

Repeating the same process for the sequence $Y_1, Y_2, \dots, Y_k, \dots$ and for the respective connected components D_{iv} of $\mathbb{R}^n \setminus Y_i$, we can also assume that the signs are preserved in each connected component and at least one permutation of the signs repeats infinitely many times. Choose this codification.

In other words, we can assume without loss of generality that there exist indices $s, t \in \mathbb{N}$, $s \neq t$, and a homeomorphism $h_{st} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that

$$\begin{cases} \text{sign}(f_s(x)) = \text{sign}(f_t \circ h_{st}(x)) & \forall x \in \mathbb{R}^n \setminus f_s^{-1}(0), \\ \text{sign}(g_s(x)) = \text{sign}(g_t \circ h_{st}(x)) & \forall x \in \mathbb{R}^n \setminus g_s^{-1}(0). \end{cases} \quad \square$$

Proof of Theorem 2.3. Suppose, by absurd, that the number of equivalence

classes is not finite. Then we can take an infinite number of representatives

$$(f_1, g_1), (f_2, g_2), \dots, (f_i, g_i), \dots, (f_j, g_j), \dots$$

such that they are not bi- C^0 - \mathcal{K} -equivalent each two of them.

For any $i, j \in \mathbb{N}$, let $f_i = (f_{i1}, \dots, f_{ip}) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ and $g_j = (g_{j1}, \dots, g_{jq}) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^q, 0)$ and consider the algebraic subsets of \mathbb{R}^n :

$$X_{ir} = f_{ir}^{-1}(0), \quad Y_{jk} = g_{jk}^{-1}(0), \quad 1 \leq r \leq p, \quad 1 \leq k \leq q.$$

Consider the following infinite set formed by sequences of algebraic subsets of \mathbb{R}^n :

$$\{(X_{11}, \dots, X_{1p}, Y_{11}, \dots, Y_{1q}), \dots, (X_{n1}, \dots, X_{np}, Y_{n1}, \dots, Y_{nq}), \dots\}$$

Since strongly topological equivalence admits the finiteness property (Corollary 3.7), taking a subsequence, we can assume that:

$$(X_{i1}, \dots, X_{ip}, Y_{i1}, \dots, Y_{iq}), (X_{j1}, \dots, X_{jp}, Y_{j1}, \dots, Y_{jq})$$

are strongly topologically equivalent for all $i, j \in \mathbb{N}$. By Lemma 4.1 let us assume that for some fixed $i \neq j$ the following conditions hold:

$$\left\{ \begin{array}{ll} \text{sign}(f_{i1}(x)) = \text{sign}(f_{j1} \circ h_{ij}(x)) & \forall x \in \mathbb{R}^n \setminus f_{i1}^{-1}(0) \\ \vdots & \vdots \\ \text{sign}(f_{ip}(x)) = \text{sign}(f_{jp} \circ h_{ij}(x)) & \forall x \in \mathbb{R}^n \setminus f_{ip}^{-1}(0) \\ \text{sign}(g_{i1}(x)) = \text{sign}(g_{j1} \circ h_{ij}(x)) & \forall x \in \mathbb{R}^n \setminus g_{i1}^{-1}(0) \\ \vdots & \vdots \\ \text{sign}(g_{iq}(x)) = \text{sign}(g_{jq} \circ h_{ij}(x)) & \forall x \in \mathbb{R}^n \setminus g_{iq}^{-1}(0). \end{array} \right.$$

By simplicity, denotes h_{ij} by h . We can rewrite the previous expressions as the following:

- i) $h(X_{is}) = X_{js}$, where $X_{is} = f_{is}^{-1}(0)$, $X_{js} = f_{js}^{-1}(0)$ for all $1 \leq s \leq p$ and
- ii) $\text{sign}(f_{is}(x)) = \text{sign}(f_{js} \circ h(x))$ for all $x \in \mathbb{R}^n - f_{is}^{-1}(0)$ and $1 \leq s \leq p$.

Then, by Theorem 2.3 the germs f_i and $f_j \circ h$ are C^0 - \mathcal{C} -equivalent.

Analogously, we also can conclude that the germs g_i and $g_j \circ h$ are C^0 - \mathcal{C} -equivalent.

Hence the pair (f_i, g_i) is bi- C^0 - \mathcal{C} -equivalent to $(f_j \circ h, g_j \circ h) = (f_j, g_j) \circ h$. Then, (f_i, g_i) is bi- C^0 - \mathcal{K} -equivalent to (f_j, g_j) , which is an absurd. \square

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