A characterization for tropical polynomials being the minimum finishing time of project networks

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Abstract. A tropical polynomial is called R-polynomial if it can be realized as the minimum finishing time of a project network. R-polynomials satisfy the term extendability condition, and correspond to simple graphs. We give a characterization of R-polynomials in terms of simple graphs.

Key words: max-plus algebra, tropical algebra, discrete event system.

1. Introduction

Max-plus algebra (or *tropical algebra*) is the semiring with the operations 'max' as the addition and '+' as the multiplication. The algebra appears in various fields of mathematics and other studies, such as algebraic dynamical systems [1], computer science [6], and phylogenetics [4]. The algebra is also effectively used to analyze discrete event systems, such as project networks [3] and railway networks [2].

In this paper, we treat project networks. A *project network* consists of some activities, where each activity can be started after all the preceding activities have finished. We may regard the set of activities as an ordered set. By taking the Hasse diagram, a project network is represented as a directed acyclic graph. Each activity is endowed with a non-negative real number t_i, called *time cost*. We may consider that the time cost of an activity represents the time to complete the activity. The *minimum finishing time* of a project network is the minimum time taken for finishing all the activities in that network. The minimum finishing time is represented by a tropical polynomial of $t_1, \ldots, t_n$.

A tropical polynomial is called *realizable polynomial* or *R-polynomial* if it can be realized as the minimum finishing time of a project network. An R-polynomial satisfies the following three conditions (Proposition 2 in [3]):

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(1) the degree on each variable is exactly one, (2) the coefficient of each term is a unity and (3) no term is divisible by any other terms ('indivisibility'). A tropical polynomial satisfying those conditions is called *prerealizable polynomial* or *P-polynomial*. A *P*-polynomial is not always an *R*-polynomial. A simplest example of a *P*-polynomial which is not an *R*-polynomial is $t_1 t_2 + t_2 t_3 + t_3 t_1$ ([3]).

A characterization of *R*-polynomials using poset is known (Proposition 2.3), but it is not effective for judging whether a given *P*-polynomial is an *R*-polynomial. In this paper, we introduce another characterization by graph theory. We do this by two steps. We first show that every *R*-polynomial satisfies a term extendability condition, which we will define later, and prove the following theorem.

Theorem 1.1 *There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$. Via this correspondence, two simple graphs are isomorphic if and only if the corresponding P-polynomials coincide up to a permutation of variables.*

Secondly, by this theorem, we will give a characterization for *R*-polynomials in the context of graph theory. The following is our main theorem.

Theorem 1.2 *Let $f(t)$ be a P-polynomial of degree d with the term extendability. Then $f(t)$ is an R-polynomial if and only if there is a vertex coloring of the term graph $TG(f)$ with the color set $\{1, \ldots, d\}$ such that every increasing path of three vertices is a clique of $TG(f)$.*

By this theorem, we can give some examples of judging whether a given polynomial is an *R*-polynomial.

As for *P*-polynomials, we have a correspondence between the set of *P*-polynomials and the set of abstract complexes.

2. Project networks

In this section, we recall the relation between project networks and tropical algebra. For detail of this section, see [3].

Formally, a project network is an acyclic directed graph with no short-cuts, where a graph is said *with no short-cuts* if the following condition
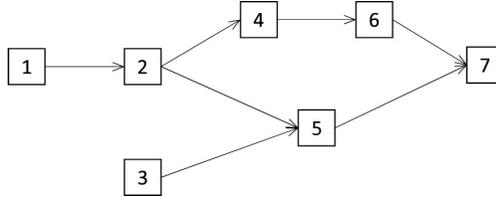
Figure 1.   project network

holds: if there are two distinct paths from activity a to activity b, then both paths consist of more than one arrow.

Proposition 2.1 (Proposition 1 in [3])   *Let X be a finite set. There is a one-to-one correspondence between the set of partial orders on X and the set of simple directed graphs with vertex set X without cycles or short-cuts.*

The correspondence in Proposition 2.1 is given as follows. For a given partial order of X, we take its Hasse diagram as the corresponding graph. For a given project network with the vertex set X, we define the corresponding partial order on X so that, for each arrow, its head is greater than its tail.

Each activity is endowed with a non-negative real number t_i, called *time cost*. We may consider that the time cost of an activity represents the time to complete the activity. The *minimum finishing time* of a project network is the minimum time taken for finishing all the activities in that network. Then the minimum finishing time is a function of $t_1, \ldots, t_n$, which has following properties.

Proposition 2.2 (Proposition 2 in [3])   *The minimum finishing time $f(t) = f(t_1, \ldots, t_n)$ can be written as a tropical polynomial of $t_1, \ldots, t_n$ satisfying the following three conditions*:

(1) *the degree on each variable is exactly one,*
(2) *the coefficient of each term is a unity,*
(3) *no term is divisible by any other terms. ('indivisibility')*

A tropical polynomial is called a *realizable polynomial* or an *R-polynomial* if it can be obtained as the minimum finishing time of a project network. Also, a tropical polynomial is called a *prerealizable polynomial* or a *P-polynomial* if it satisfies the condition (1)–(3). A P-polynomial is not always an R-polynomial.

For a set of variables $\{t_i\}_{i\in\Lambda}$ and a subset $I \subset \Lambda$, we denote the monomial $\prod_{i\in I} t_i$ by t_I. We know the following characterization of R-polynomials.

Proposition 2.3 (Proposition 3 in [3])   *Let $f(t) = \sum_{I\in\mathcal{I}} t_I$ be a tropical polynomial in n variables. Then $f(t)$ is an R-polynomial if and only if there exist a poset structure on the index set $[n]$ such that*

$$I \text{ is a maximal totally ordered subset } \Leftrightarrow t_I \text{ is a term of } f(t).$$

If we want to check whether a given P-polynomial is an R-polynomial by this characterization, for example, we may list up the all poset structure on $[n]$. However, the calculation amount is not realistic. Thus we introduce another approach in the later section.

3.   Term extendability

In this section, we introduce our key condition, called *term extendability*, which holds for every R-polynomial. For a given P-polynomial, checking for term extendability is easier than checking whether the polynomial is an R-polynomial. Many P-polynomials are excluded from the candidates for R-polynomials by restricting via this condition. Furthermore, in the next section, we will construct a correspondence between P-polynomials with term extendability and simple graphs. The correspondence is important for our new characterization. Unfortunately, there is a P-polynomial that has term extendability but is not an R-polynomial. We will see some examples of such polynomials in this section.

In the later of this section, we will estimate the number of terms of R-polynomials by using term extendability condition. In addition, we will show that if the number of terms is smaller than 5, then the term extendability condition is sufficient for a P-polynomial to be an R-polynomial.

First, we give a definition and a proposition. Let $f(t) = f(t_1,\ldots,t_n)$ be a P-polynomial. For $i, j \in [n]$, we say that i and j are *comparable* in $f(t)$ if $f(t)$ has a term which is divisible by $t_i t_j$. Note that if $f(t)$ is an R-polynomial, then i and j are comparable if and only if i and j are comparable in the usual sense in the poset of the corresponding project network.

Proposition 3.1   *Let $f(t) = f(t_1,\ldots,t_n)$ be an R-polynomial and $I \subset [n]$ be a subset. Suppose that any two elements of I are comparable. Then $f(t)$*

has a term which is divisible by t_I.

Proof. Let N be the corresponding project network to $f(t)$. Since any two distinct elements of I are comparable, the set I forms a totally ordered vertex set of N. Then there is a maximal totally ordered vertex set J of N containing I. Therefore t_J is a term of $f(t)$, which is divisible by t_I. $\square$

Now we define the term extendability. Let $f(t) = f(t_1, \ldots, t_n)$ be a P-polynomial. Then $f(t)$ has term extendability if, for any subset $I \subset [n]$ such that any two distinct elements of I are comparable in $f(t)$, there is a term of $f(t)$ divisible by t_I.

Proposition 3.1 means that every R-polynomial has term extendability.

Example 3.2 The P-polynomial of the form $(t_1 t_2 + t_2 t_3 + t_3 t_1) f(t) + g(t)$ ($f(t), g(t)$ be P-polynomials) does not have term extendability. Indeed, suppose that $h(t) := (t_1 t_2 + t_2 t_3 + t_3 t_1) f(t) + g(t)$ has term extendability. Let t_I be a term of $f(t)$. (If $f(t)$ is constant, let $I = \emptyset$). Consider the set $I' := I \cup \{1, 2, 3\}$. Any two element of I' are comparable, so $h(t)$ has a term divisible by $t_{I'}$. Since $h(t)$ also has a term $t_I t_1 t_2$, this contradicts the indivisibility for $h(t)$. We conclude that $h(t)$ is not an R-polynomial.

There is an example that $h(t) f(t) + g(t)$ has term extendability although $h(t)$ does not have.

Example 3.3 The polynomial $h(t) = t_1 t_2 t_4 + t_1 t_3 t_5 + t_2 t_3 t_6$ does not satisfy term extendability for $I = \{1, 2, 3\}$, while the polynomial $h(t) + t_1 t_2 t_3 = t_1 t_2 t_4 + t_1 t_3 t_5 + t_2 t_3 t_6 + t_1 t_2 t_3$ satisfies term extendability.

This polynomial is in fact not an R-polynomial, but $h(t) + t_1 t_2 t_3 + t_2 t_4 t_6 = t_1 t_2 t_4 + t_1 t_3 t_5 + t_2 t_3 t_6 + t_1 t_2 t_3 + t_2 t_4 t_6$ is an R-polynomial. We will show that in Example 4.12.

Next we estimate the number of terms of R-polynomials.

Proposition 3.4 *Let $f(t)$ be a P-polynomial having term extendability. If t_I, t_J and t_K are distinct three terms of $f(t)$, then $I \cup J \neq I \cup K$.*

Proof. Suppose that $I \cup J = I \cup K$. We use term extendability for the set $J \cup K$. To do this we show that every two distinct elements of $J \cup K$ are comparable.

Let $i, j \in J \cup K$. If $i, j \in J$ or $i, j \in K$, then i and j are obviously

comparable. If $i \in J \smallsetminus K$ and $j \in K \smallsetminus J$, we have $i \in I \cup J = I \cup K$. Then $i \in I$. Similarly, $j \in I$. Hence i and j are comparable.

By the term extendability, $f(t)$ has a term divisible by $t_{J \cup K}$. Since $J \subsetneq J \cup K$, this contradicts the indivisibility for $f(t)$. □

Corollary 3.5 *Let $f(t) = f(t_1, \ldots, t_n)$ be a P-polynomial having term extendability. Let d be the degree of $f(t)$. Then $f(t)$ has at most $\sum_{i=0}^{\min\{d, n-d\}} \binom{n-d}{i}$ terms.*

Proof. Let t_{I_0} be a term of $f(t)$ of degree d. Consider the map $t_I \mapsto I_0 \cup I$ from the set of terms of $f(t)$ to the set $\{J \subset [n] \mid I_0 \subset J \text{ and } \#J \leq 2d\}$. By Proposition 3.4, this map is injective. Then the number of terms of $f(t)$ is at most $\#\{J \subset [n] \mid I_0 \subset J \text{ and } \#J \leq 2d\} = \sum_{i=0}^{\min\{d,n-d\}} \binom{n-d}{i}$. □

Note that this estimate is best bound if $\min\{d, n - d\} = n - d$, i.e. $2d \geq n$. Indeed, in that case, the number of terms of $f(t)$ is at most $\sum_{i=0}^{n-d} \binom{n-d}{i} = 2^{n-d}$. It can be attained by the minimum finishing time of the project network in Figure 3.
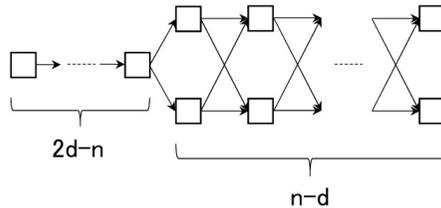

Figure 2.

Proposition 3.6 *Let $f(t) = f(t_1, \ldots, t_n)$ be a P-polynomial. If $\deg(f(t)) \geq n - 2$, then $f(t)$ is an R-polynomial if and only if $f(t)$ has term extendability.*

Proof. If $\deg(f(t)) = n$, we have $f(t) = t_1 \cdots t_n$ and so $f(t)$ is an R-polynomial.

If $\deg(f(t)) = n - 1$, then $f(t)$ is a binomial by Corollary 3.5. Note that $f(t)$ is not a monomial because every variable appears at least once. Let $f(t) = t_I + t_J$. Then $f(t)$ is the minimum finishing time of the project network in Figure 3, so $f(t)$ is an R-polynomial.

If $\deg(f(t)) = n - 2$, we may assume that $f(t)$ has a term $t_{[n-2]}$. By the indivisibility, the term other than $t_{[n-2]}$ is divisible by t_{n-1} or t_n. Moreover,
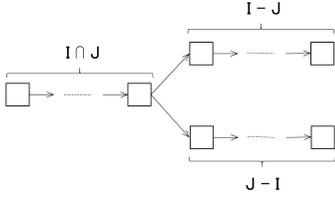
Figure 3.
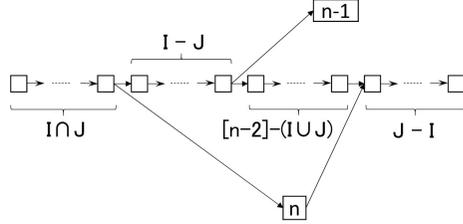
Figure 4.

there is at most one term of the form $t_I t_{n-1}$ $(I \subset [n-2])$. Indeed, if $t_I t_{n-1}$ and $t_J t_{n-1}$ $(I, J \subset [n-2])$ are the terms of $f(t)$, we have $[n-2] \cup (I \cup \{n-1\}) = [n-2] \cup (J \cup \{n-1\})$, which contradicts Proposition 3.4. It is similar for the term of the form $t_I t_n$ and $t_I t_{n-1} t_n$ $(I \subset [n-2])$. Thus there are following 4 cases:

If $f(t)$ is of the form $t_{[n-2]} + t_I t_{n-1} t_n$ $(I \subset [n-2])$, then $f(t)$ is a binomial. Therefore we can show that $f(t)$ is an R-polynomial by the same argument with the case $\deg(f(t)) = n - 1$.

If $f(t)$ is of the form $t_{[n-2]} + t_I t_{n-1} + t_J t_n$ $(I, J \subset [n-2])$, then $f(t)$ is the minimum finishing time of the project network in Figure 4. Therefore $f(t)$ is an R-polynomial.

Suppose $f(t)$ is of the form $t_{[n-2]} + t_I t_{n-1} + t_J t_{n-1} t_n$ $(I, J \subset [n-2])$. By the term extendability, there must be a term of $f(t)$ which is divisible by $t_{I \cup J} t_{n-1}$. If the term is $t_J t_{n-1} t_n$, we have $I \subset J$, which contradicts the indivisibility. Thus the term is $t_I t_{n-1}$, so we have $I \supset J$. Then $f(t)$ is the minimum finishing time of the project network in Figure 5. Hence $f(t)$ is an R-polynomial.

Suppose $f(t)$ is of the form $t_{[n-2]} + t_I t_{n-1} + t_J t_n + t_K t_{n-1} t_n$ $(I, J, K \subset [n-2])$. In the same way with the above case, we have $I \supset K$ and $J \supset K$, and hence $I \cap J \supset K$. If $I \cap J \neq K$, there is $i \in (I \cap J) \setminus K$. By the term extendability, there is a term of $f(t)$ which is divisible by $t_i t_{n-1} t_n$. However, any terms of $f(t)$ are not divisible by $t_i t_{n-1} t_n$. Thus we have $I \cap J = K$. Then $f(t)$ is the minimum finishing time of the project network in Figure 6. Hence $f(t)$ is an R-polynomial. $\square$

Corollary 3.7 *For $n \leq 4$, $f(t)$ is an R-polynomial if and only if $f(t)$ has term extendability.*

Remark 3.8 For $n = 5$, the polynomial $t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_5 + t_5 t_1$ is a
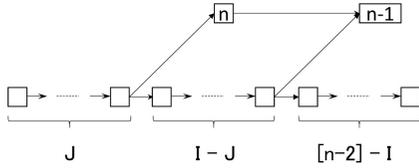
Figure 5.
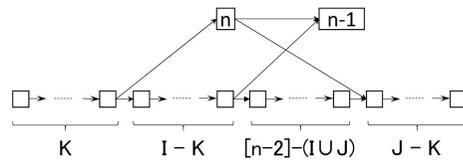
Figure 6.

counterexample. We will show that this polynomial is not an R-polynomial in Example 4.11.

We remark that we may associate an abstract complex with a P-polynomial as follows. Let $f(t_1, \ldots, t_n) = \sum_{I \in \mathcal{I}} t_I$ be a P-polynomial. Then the set

$$\{J \subset [n] \mid J \text{ is a subset of some } I \in \mathcal{I}\}$$

forms an abstract complex. Conversely, for a given abstract complex with the vertex set $[n]$, the tropical polynomial $\sum_{I:\text{maximal face}} t_I$ is a P-polynomial. Then the following proposition is clear.

Proposition 3.9 *Let $\mathcal{P}_n$ be the set of P-polynomials with the variables $t_1, \ldots, t_n$ and $\mathcal{A}_n$ be the set of abstract complexes with the vertex set $[n]$. Then the above constructions give bijections between $\mathcal{P}_n$ and $\mathcal{A}_n$, which are inverse of each other. Moreover, a P-polynomial has term extendability if and only if the corresponding complex is flag complex, i.e. for any $I \subset [n]$, if $\{i, j\}$ is a simplex for all $i, j \in I$, then I is a simplex.*

4. Term graphs

In this section, we show Theorem 4.7, the main theorem of this paper. The theorem gives us a characterization for R-polynomials. As a preparation, we show the following theorem.

Theorem 4.1 *There is a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$. Via this correspondence, two simple graphs are isomorphic if and only if the corresponding P-polynomials coincide up to a permutation of variables.*

Remark 4.2 This theorem follows from Proposition 3.9 and a well-known

fact that there is a bijection between the set of flag complexes and the set of 'clique complexes' (see [5]). Here, we directly construct a bijection in Theorem 4.1.

First of all, let us associate a simple graph to a given P-polynomial.

Definition 4.3 Let $f(t) = f(t_1, \ldots, t_n)$ be a P-polynomial. We define the *term graph* of $f(t)$ as the simple graph $TG(f) = (V, E)$, where $V = [n]$ is the vertex set and E is the edge set which consists of the pairs that are comparable in $f(t)$.

It is clear by definition that if t_I is a term of $f(t)$, then I forms a clique in $TG(f)$.

By the following lemma, a P-polynomial $f(t)$ can be reconstructed by the term graph $TG(f)$ if $f(t)$ has term extendability.

Lemma 4.4 *Let $f(t) = f(t_1, \ldots, t_n)$ be a P-polynomial having term extendability. Then for any subset $I \subset [n]$, the monomial t_I is a term of $f(t)$ if and only if the set I is a maximal clique of $TG(f)$.*

Proof. We show this by the induction for $\#I$. Let d be the maximum size of the cliques of $TG(f)$. If $\#I > d$, then I is not a clique of $TG(f)$, and then t_I is not a term of $f(t)$. Thus we may assume $\#I \leq d$.

We consider the case $\#I = d$ at first. If t_I is a term of $f(t)$, then I is a clique of $TG(f)$, and the maximality follows from the definition of d. Conversely, if I is a maximal clique of $TG(f)$, then any two distinct elements of I are comparable. Therefore, by the term extendability, $f(t)$ has a term $t_{I'}$ which is divisible by t_I. Then I' is a clique of $TG(f)$ containing I. By the maximality of I, we have $I' = I$. Hence t_I is a term of $f(t)$.

Next we assume that $\#I < d$ and the statement holds for any $J \subset [n]$ such that $\#J > \#I$. If t_I is a term of $f(t)$, then I is a clique of $TG(f)$. If I is not a maximal clique, then there is a maximal clique I' containing I properly. By the assumption of induction, $t_{I'}$ is a term of $f(t)$, which contradicts the indivisibility of $f(t)$. Hence I is maximal. Conversely, if I is a maximal clique of $TG(f)$, we can show that t_I is a term of $f(t)$ by the proof similar to the above case. □

This lemma means that the map $f(t) \mapsto TG(f)$ between the two sets in Theorem 4.1 is injective.

For showing the surjectivity, we construct the inverse map. For a given

simple graph G with the vertex set $[n]$, we associate the following tropical polynomial f_G;

$$f_G(t) = \sum_{I:\text{maximal clique of } G} t_I.$$

Lemma 4.5 *The polynomial $f_G(t)$ has term extendability.*

Proof. Let $I \subset [n]$ be a subset such that any two distinct elements of I are comparable in $f_G(t)$. Let i and j be distinct elements of I. Since i and j are comparable, then $f_G(t)$ has a term which is divisible by $t_i t_j$. Thus the original graph G has a clique including i and j, and then i and j are adjacent in G. Hence any two distinct elements of I are adjacent in G, i.e. I forms a clique of G. Let I' be a maximal clique including I. Then the term $t_{I'}$ of $f_G(t)$ is divisible by t_I. □

Proof of Theorem 4.1. By Lemma 4.4 and Lemma 4.5, we obtain a one-to-one correspondence between the set of P-polynomials $f(t) = f(t_1, \ldots, t_n)$ having term extendability and the set of simple graphs with the vertex set $[n]$. The remaining part is clear. □

Finally we describe a characterization for R-polynomials. To do this, we use vertex colorings of term graphs.

Definition 4.6 Let G be a simple graph. Assume that there is a vertex coloring of G with the color set $\{1, \ldots, d\}$. Then the sequence of vertices $v_1, \ldots, v_m$ of G is an *increasing path* if v_i and v_{i+1} are adjacent for $i = 1, \ldots, m-1$ and the colors of them are increasing.

Theorem 4.7 *Let $f(t)$ be a P-polynomial of degree d having term extendability. Then $f(t)$ is an R-polynomial if and only if there is a vertex coloring of the term graph $TG(f)$ with the color set $\{1, \ldots, d\}$ such that every increasing path of three vertices is a clique of $TG(f)$.*

Remark 4.8 The condition that every increasing path *of three vertices* is a clique of $TG(f)$ is equivalent to the condition that every increasing path is a clique of $TG(f)$. Indeed, assume that every increasing path of 3 vertices is a clique and let $v_1, \ldots, v_m$ is an increasing path. Then, for $k \leq m-2$, the sequence v_k, v_{k+1}, v_{k+2} forms an increasing path. Thus $\{v_k, v_{k+1}, v_{k+2}\}$ is a clique, and then v_k and v_{k+2} are adjacent. Hence the

sequence v_k, v_{k+2}, v_{k+3} forms an increasing path for $k \leq m-3$. By repeating this argument, every pair of two distinct vertices in $v_1, \ldots, v_m$ are adjacent. It means that $\{v_1, \ldots, v_m\}$ is a clique.

The *length* of a path of a project network is the number of arrows in the path.

Lemma 4.9 *Let N be a project network with the vertex set $[n]$. Let d be the maximum length of paths of N. We define the subsets $V_0, \ldots, V_d \subset [n]$ as follows*:

$$V_0 := \{v \in [n] \mid v \text{ is minimal in } [n]\},$$

$$V_k := \left\{ v \in [n] \mid v \text{ is minimal in } [n] \smallsetminus \bigcup_{l=0}^{k-1} V_l \right\} \ (k = 1, \ldots, d).$$

Then $V_0, \ldots, V_d$ satisfy the followings:

(1) *The set $[n]$ is the disjoint union of $V_0, \ldots, V_d$.*
(2) *Each V_k is non-empty.*
(3) *For each path of N and each $k = 0, \ldots, d$, the intersection of the path and V_k is empty or singleton.*

Proof. (1) Suppose that the set $[n] \smallsetminus \bigcup_{k=0}^{d} V_k$ is not empty and let i be a minimal vertex of $[n] \smallsetminus \bigcup_{k=0}^{d} V_k$. We claim that there is a vertex $v_d \in V_d$ such that $v_d < i$.

Indeed, let m be the number

$$\max\{k \mid 0 \leq k \leq d, \text{ there is a vertex } j \in V_k \text{ such that } j < i\}.$$

By the minimality of i, i is a minimal vertex of $[n] \smallsetminus \bigcup_{k=0}^{m} V_k$. Thus, if $m < d$, then $i \in V_{m+1}$. It contradicts the definition of i. Hence $m = d$, and there is a vertex $v_d \in V_d$ such that $v_d < i$.

By the same proof, there are vertices $v_{d-1}, \ldots, v_0$ of N such that $v_k \in V_k$ $(k = 0, \ldots, d-1)$ and $v_0 < \cdots < v_d$. Then there is a path through $v_0, \ldots, v_d, i$, which contradicts the definition of d.

(2) We denote $[n] \smallsetminus \bigcup_{l=0}^{k-1} V_l$ by W_k. Let $(v_0, \ldots, v_d)$ be a maximal path of N and $v_i \in V_{k_i}$. We claim that $k_i < k_{i+1}$. Otherwise, we have $v_i \in W_{k_i} \subset W_{k_{i+1}}$. Hence $v_i, v_{i+1} \in W_{k_{i+1}}$ and $v_i < v_{i+1}$, but v_{i+1} is a minimal vertex of $W_{k_{i+1}}$ because $v_{i+1} \in V_{k_{i+1}}$. It is a contradiction. Thus

we have $0 \leq k_0 < \cdots < k_d \leq d$, which means that $k_i = i$. Hence $v_k \in V_k \neq \emptyset$. (3) is clear. $\qquad\square$

Proof of Theorem 4.7.   If $f(t)$ is an R-polynomial, let N be the corresponding project network with the vertex set $[n]$. The maximum length of paths of N is d. Take $V_1, \ldots, V_d$ as Lemma 4.9 for N. Note that the vertex sets of $TG(f)$ and N are same, namely, are $[n]$. For each $k = 1, \ldots, d$, color the vertices in V_k with k.

Let v_1, v_2, v_3 be an increasing path of $TG(f)$. For $k = 1, 2$, v_k and v_{k+1} are adjacent in $TG(f)$, so t_{v_k} and $t_{v_{k+1}}$ are comparable in $f(t)$. Hence v_k and v_{k+1} are comparable in N. Since the color of v_{k+1} is greater than that of v_k, we have $v_k < v_{k+1}$. Therefore v_1, v_2, v_3 is totally ordered in N. Then $f(t)$ has a term divisible by $t_{v_1} t_{v_2} t_{v_3}$, which means that the set $\{v_1, v_2, v_3\}$ is a clique of $TG(f)$.

Conversely, if there is a vertex coloring of the term graph $TG(f)$ by d colors $1, \ldots, d$ such that every increasing path is a clique of $TG(f)$, we may define the partial order of $[n]$ by the following way: For $i, j \in [n]$, i and j are comparable if and only if i and j are adjacent in $TG(f)$. The order of them is induced by the order of their colors.

Using this order, we can define the project network N on $[n]$. Let $g(t)$ be the minimum finishing time of N. We claim that $g(t) = f(t)$. Let $I \subset [n]$ be a subset. Then

> t_I is a term of $g(t)$
>> $\Leftrightarrow I$ is the vertex set of a maximal path of N
>> $\Leftrightarrow I$ is the vertex set of a maximal increasing path of $TG(f)$
>> $\Leftrightarrow I$ forms a maximal clique of $TG(f)$
>> $\Leftrightarrow t_I$ is a term of $f(t)$.

Hence $g(t) = f(t)$. $\qquad\square$

Corollary 4.10   *Let $f(t) = f(t_1, \ldots, t_n)$ be a homogeneous P-polynomial of degree 2. Then $f(t)$ is R-polynomial if and only if the term graph $TG(f)$ is a bipartite graph.*

Example 4.11   The polynomial $f(t) = t_1 t_2 + t_2 t_3 + t_3 t_4 + t_4 t_5 + t_5 t_1$ is not an R-polynomial. Indeed, the term graph $TG(f)$ is just a pentagon, which is not a bipartite graph.
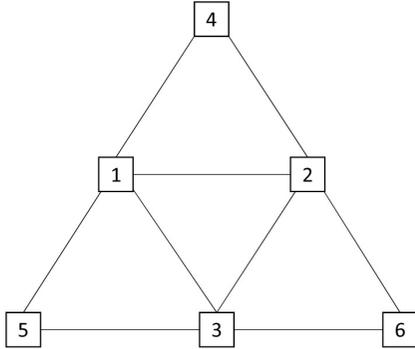
Figure 7.
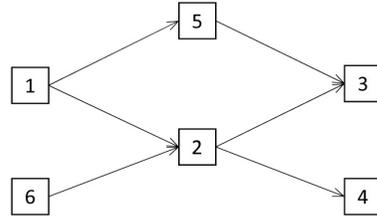
Figure 8.

Example 4.12 $g(t) := t_1t_2t_4 + t_1t_3t_5 + t_2t_3t_6 + t_1t_2t_3$ is not an R-polynomial, but $g(t) + t_2t_4t_6$ is an R-polynomial.

Indeed, the term graph $TG(g)$ is the graph in Figure 7. If $g(t)$ is an R-polynomial, there is a vertex coloring with the colors c_1, c_2 and c_3 ($c_1 < c_2 < c_3$). By symmetry, we may assume that the colors of the vertex $1, 2, 3$ are c_1, c_2, c_3 respectively. Since the vertex set $\{1, 2, 6\}$ is not a clique in $TG(g)$, the sequence $(1, 2, 6)$ is not an increasing path. Then the color of 6 is less than c_2, and hence the color of 6 is c_1. Similarly the color of 4 is c_3. Therefore the sequence $(6, 2, 4)$ is an increasing path, but the set $6, 2, 4$ is not a clique. This is a contradiction.

On the other hand, $g(t) + t_2t_4t_6$ is the minimum finishing time of the project network in Figure 8. Hence $f(t)$ is an R-polynomial.

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