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Generalized Lucas Numbers of the form wx^2 and wV_mx^2

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Abstract. Let $P \geq 3$ be an integer. Let (V_n) denote generalized Lucas sequence defined by $V_0 = 2$, $V_1 = P$, and $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. In this study, when P is odd, we solve the equation $V_n = wx^2$ for some values of w. Moreover, when P is odd, we solve the equation $V_n = wkx^2$ with $k \mid P$ and $k > 1$ for $w = 3, 11, 13$. Lastly, we solve the equation $V_n = wV_mx^2$ for $w = 7, 11, 13$.

Key words: Generalized Lucas sequence, Generalized Fibonacci sequence, congruence, square terms in Lucas sequences.

1. Introduction

Let P and Q be nonzero integers such that $P^2+4Q > 0$. Generalized Fibonacci sequence $(U_n(P,Q))$ and Lucas sequence $(V_n(P,Q))$ are defined by $U_0(P,Q) = 0, U_1(P,Q) = 1$; $V_0(P,Q) = 2$, $V_1(P,Q) = P$, and $U_{n+1}(P,Q) = PU_n(P,Q) + QU_{n-1}(P,Q)$, $V_{n+1}(P,Q) = PV_n(P,Q) + QV_{n-1}(P,Q)$ for $n \geq 1$. The numbers $U_n(P,Q)$ and $V_n(P,Q)$ are called n-th generalized Fibonacci and Lucas numbers, respectively. Generalized Fibonacci and Lucas sequences for negative subscripts are defined as $U_{-n}(P,Q) = -U_n(P,Q)/(-Q)^n$ and $V_{-n}(P,Q) = V_n(P,Q)/(-Q)^n$ for $n \geq 1$. Since $U_n(-P,Q) = (-1)^{n-1}U_n(P,Q)$ and $V_n(-P,Q) = (-1)^nV_n(P,Q)$, it will be assumed that $P \geq 1$. For $P = Q = 1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n). For $P = 2$ and $Q = 1$, we have Pell and Pell-Lucas sequences (P_n) and (Q_n). For more information about generalized Fibonacci and Lucas sequences one can consult [7].

The terms in Lucas sequences of the form kx^2 have been investigated since 1962. When P is odd and $Q = \pm 1$, by using elementary argument many authors solved the equation $U_n = kx^2$ or $V_n = kx^2$ for specific integer values of k. The reader can consult [13] or [9] for a brief discussion of the subject. In [5], the authors solved $U_n = x^2$, $V_n = x^2$, $U_n = 2x^2$, and $V_n = 2x^2$ for odd relatively prime integers P and Q. In [8], the same authors solved $U_n = 3x^2$ for relatively prime odd integers P and Q. In [14],

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the authors solved $V_n = 3x^2$ and $V_n = 6x^2$ for relatively prime odd integers P and Q. Moreover, in [11], the authors solved $U_n = 6x^2$ for relatively prime odd integers P and Q. In [2], the authors solved $V_n(P, -1) = 5x^2$ and $U_n(P, -1) = 5x^2$ for odd integer $P \geq 3$. In [3], the authors solved $U_n = 7x^2$ and $V_n = 7x^2$ for odd integer $P \geq 1$ with $Q = 1$. In [1], the author solved $V_n = V_m x^2$ and $V_n = 2V_m x^2$ for odd value of P with $Q = \pm 1$. In [11], the author solved $V_n = V_m x^2$, $V_n = 2V_m x^2$, and $V_n = 6V_m x^2$ for relatively prime odd values of P and Q. In [2], the authors solved $V_n = 5V_m x^2$ for odd value of P with $Q = -1$.

In this study, we assume that $Q = -1$. We solve the equation $V_n = wx^2$ for some values of w. Moreover, we solve the equation $V_n = wkx^2$ with $k \mid P$ and $k > 1$ for $w = 3, 11, 13$. Lastly, we solve the equation $V_n = wV_m x^2$ for $w = 7, 11, 13$.

Throughout this study, $(*/*)$ will denote the Jacobi symbol. Our method is elementary and used by Cohn, Ribenboim, and McDaniel in [1] and [8], respectively.

2. Preliminaries

From now on, instead of $U_n(P, -1)$ and $V_n(P, -1)$, we sometimes write U_n and V_n, respectively. The following theorem is given in [12].

Theorem 2.1 *Let $n \in \mathbb{N} \cup \{0\}$ and m, $r \in \mathbb{Z}$. Then*

$$V_{2mn+r} \equiv (-1)^n V_r (\operatorname{mod} V_m) \tag{2.1}$$

and

$$V_{2mn+r} \equiv V_r (\operatorname{mod} U_m) \tag{2.2}$$

if $m \neq 0$.

From (2.1), it follows that if a is odd, then

$$V_{2 \cdot 2^r a + m} \equiv -V_m (\operatorname{mod} V_{2^r}). \tag{2.3}$$

Since $8 \mid U_3$ when P is odd, we get

$$V_{6q+r} \equiv V_r (\operatorname{mod} 8). \tag{2.4}$$

When P is odd, we have $V_{2^r} \equiv 7 (\bmod 8)$ and thus,

$$\left(\frac{2}{V_{2^r}}\right) = 1 \tag{2.5}$$

and

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{2.6}$$

for $r \geq 1$. Moreover, we have

$$\left(\frac{3}{V_{2^r}}\right) = \begin{cases} 1 & \text{if } r \geq 1 \text{ and } 3 \nmid P, \\ -1 & \text{if } r = 1 \text{ and } 3 \mid P, \\ 1 & \text{if } r \geq 2 \text{ and } 3 \mid P. \end{cases} \tag{2.7}$$

Then it follows from (2.7) that

$$\left(\frac{3}{V_{2^r}}\right) = 1 \tag{2.8}$$

for $r \geq 2$.

When P is odd, we have

$$\left(\frac{P-1}{V_{2^r}}\right) = \left(\frac{P+1}{V_{2^r}}\right) = \left(\frac{P^2-1}{V_{2^r}}\right) = 1 \tag{2.9}$$

for $r \geq 1$ and

$$2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n. \tag{2.10}$$

Moreover, it can be seen that if n is odd, then

$$V_{2n} \equiv 2, 7 (\bmod 8). \tag{2.11}$$

and if n is odd and $3 \nmid n$, then

$$V_n \equiv P (\bmod 8). \tag{2.12}$$

The following identities are well known (see [7]).

$$V_{-n} = V_n. \tag{2.13}$$

$$V_{3n} = V_n(V_n^2 - 3) = V_n(V_{2n} - 1). \tag{2.14}$$

$$U_m \mid U_n \Leftrightarrow m \mid n. \tag{2.15}$$

$$V_m \mid V_n \Leftrightarrow m \mid n \text{ and } n/m \text{ is odd}. \tag{2.16}$$

$$U_{2n} = U_n V_n. \tag{2.17}$$

$$V_{2n} = V_n^2 - 2. \tag{2.18}$$

If $d = (m, n)$, then

$$(V_m, V_n) = \begin{cases} V_d & \text{if } m/n \text{ and } n/m \text{ odd}, \\ 1 \text{ or } 2 & \text{otherwise}. \end{cases} \tag{2.19}$$

Now we give the following theorems from [10].

Theorem 2.2 *Let P be odd. If $V_n = kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

Theorem 2.3 *Let P be odd. If $V_n = 2kx^2$ for some $k \mid P$ with $k > 1$, then $n = 3$.*

Lemma 1

$$V_n \equiv \begin{cases} 2(-1)^n (\operatorname{mod} P) & \textit{if } n \textit{ is even}, \\ 0 (\operatorname{mod} P) & \textit{if } n \textit{ is odd}. \end{cases}$$

3. Divisibility of V_n by Small Values of k

From now on, we will assume that n and m are positive integers.

Lemma 2 $3 \mid V_n$ *if and only if* $3 \mid P$ *and n is odd.*

Proof. If $3 \mid P$ and n is odd, then $3 \mid V_n$ by Lemma 1. Assume that $3 \mid V_n$. Let $3 \nmid P$. Then $3 \mid P^2 - 1$ and therefore $3 \mid U_3$. Let $n = 6q \pm r$ with $0 \le r \le 3$. Then by (2.2), $V_n \equiv V_{\pm r} (\operatorname{mod} U_3)$, which implies that $V_n \equiv V_r (\operatorname{mod} 3)$. It can be seen that $3 \nmid V_r$ for $0 \le r \le 3$. Thus, $3 \nmid V_n$. Therefore $3 \mid P$ and it is seen that n is odd by Lemma 1. □

Lemma 3 *$7 \mid V_n$ if and only if $7 \mid P$ and n is odd or $P^2 \equiv 2 (\mathrm{mod}\, 7)$ and $n = 2t$ for some odd integer t.*

Proof. Let $7 \mid P$ and n be odd. Then by Lemma 1, we get $7 \mid V_n$. Let $P^2 \equiv 2(\mathrm{mod}\, 7)$ and $n = 2t$ for some odd integer t. Then $7 \mid V_2$. Since $n = 4q + 2$, it follows that $V_n \equiv \pm V_2 (\mathrm{mod}\, V_2)$ by (2.1). Thus, we have $7 \mid V_n$. Now assume that $7 \mid V_n$. If $7 \mid P$, then n must be odd by Lemma 1. Let $7 \nmid P$. Then $P^2 \equiv 1, 2, 4(\mathrm{mod}\, 7)$. Let $P^2 \equiv 1(\mathrm{mod}\, 7)$. Then $7 \mid U_3$. We may write $n = 6q \pm r$ with $0 \le r \le 3$. Thus, $V_n = V_{6q+r} \equiv V_r (\mathrm{mod}\, U_3)$ by (2.2), which implies that $V_n \equiv V_r(\mathrm{mod}\, 7)$. Then we must have $7 \mid V_r$ for $0 \le r \le 3$, which is impossible. Let $P^2 \equiv 4(\mathrm{mod}\, 7)$ and $n = 14q \pm r$, $0 \le r \le 7$. Then $7 \mid U_7$ and thus, $V_n = V_{14q \pm r} \equiv V_{\pm r}(\mathrm{mod}\, U_7)$, which implies that $V_n \equiv V_r(\mathrm{mod}\, 7)$. This is impossible since $7 \nmid V_r$ for $0 \le r \le 7$. Let $P^2 \equiv 2(\mathrm{mod}\, 7)$. Then $7 \mid V_2$. Let $n = 2q + r$, $0 \le r \le 1$. If q is even, then $V_n = V_{2q+r} \equiv \pm V_r(\mathrm{mod}\, V_2)$ by (2.1). This is impossible since $7 \nmid V_r$ for $0 \le r \le 1$. Let q be odd. Then $q = 2t + 1$ and thus, by (2.1), we get

$$V_n = V_{2q+r} = V_{2(2t+1)+r} = \pm V_{r+2}(\mathrm{mod}\, V_2),$$

which implies that $V_n \equiv \pm V_{r+2}(\mathrm{mod}\, 7)$ since $7 \mid V_2$. But this is possible only if $r = 0$. Thus, $n = 2q$ with q odd. □

The ideas behind of the proof of the following lemmas are similar to that of the lemma above and we omit the proofs here.

Lemma 4 *$5 \mid V_n$ if and only if $5 \mid P$ and n is odd.*

Lemma 5 *$11 \mid V_n$ if and only if $11 \mid P$ and n is odd or $P^2 \equiv 3(\mathrm{mod}\, 11)$ and $n = 3t$ for some odd integer t.*

Lemma 6 *$13 \mid V_n$ if and only if $13 \mid P$ and n is odd or $P^2 \equiv 3(\mathrm{mod}\, 13)$ and $n = 3t$ for some odd integer t.*

4. Main Theorems

Theorem 4.1 *If P is odd and $11 \mid P$, then $V_n = 11x^2$ has the solution $n = 1$. If $P^2 \equiv 3(\mathrm{mod}\, 11)$, then the equation $V_n = 11x^2$ has no solutions.*

Proof. Assume that $V_n = 11x^2$ for some integer x. By Lemma 5, $11 \mid V_n$ if and only if $11 \mid P$ and n is odd or $P^2 \equiv 3(\mathrm{mod}\, 11)$ and $n = 3t$ for some odd integer t. Let $11 \mid P$ and n be odd. Then by Theorem 2.2, we get $n = 1$.

Now assume that $P^2 \equiv 3(\bmod 11)$ and $n = 3t$ for some odd integer t. Let $t = 4q \pm 1$. Then $n = 12q \pm 3$ and so

$$V_n \equiv V_{\pm 3} \equiv V_3(\bmod U_3)$$

by (2.2). Now assume that P is odd. Since $8 \mid U_3$, it follows that

$$11x^2 \equiv V_3 \equiv P(P^2 - 3)(\bmod 8).$$

Thus, $11x^2 \equiv -2P(\bmod 8)$, which implies that $x^2 \equiv -6P(\bmod 8)$. This is impossible since P is odd. Now assume that P is even. It can be seen that if n is odd, then $V_n \equiv P(\bmod P^2 - 4)$. Using the fact that n is odd, it follows that $11x^2 = V_n \equiv P(\bmod P^2 - 4)$. Since P is even, we get $4 \mid P^2 - 4$, which implies that $4 \mid P$. This shows that $P^2 - 1 \equiv 7(\bmod 8)$. Since $11x^2 \equiv P(P^2 - 3)(\bmod U_3)$, we get $11x^2 \equiv -2P(\bmod P^2 - 1)$. Then it follows that $(11/(P^2 - 1)) = (-2P/(P^2 - 1))$. Since $P^2 \equiv 3(\bmod 11)$, we get

$$\left(\frac{11}{P^2-1}\right) = -\left(\frac{P^2-1}{11}\right) = -\left(\frac{2}{11}\right) = 1.$$

But

$$\begin{aligned} 1 &= \left(\frac{11}{P^2-1}\right) = \left(\frac{-2P}{P^2-1}\right) = \left(\frac{-2}{P^2-1}\right)\left(\frac{P}{P^2-1}\right) = -\left(\frac{P}{P^2-1}\right) \\ &= -\left(\frac{2^r a}{P^2-1}\right) = (-1)\left(\frac{a}{P^2-1}\right) = (-1)(-1)^{(a-1)/2}\left(\frac{P^2-1}{a}\right) \\ &= (-1)(-1)^{(a-1)/2}\left(\frac{-1}{a}\right) = -1, \end{aligned}$$

a contradiction. □

From now on, we will assume that P is odd.

Theorem 4.2 *Let $V_n = 7x^2$ for some integer x. Then $n = 1$ or 2.*

Proof. Assume that $V_n = 7x^2$ for some integer x. By Lemma 3, $7 \mid V_n$ if and only if $7 \mid P$ and n is odd or $P^2 \equiv 2(\bmod 7)$ and $n = 2t$ for some odd integer t. Let $7 \mid P$. Then $n = 1$ by Theorem 2.2. Assume that

$P^2 \equiv 2 (\bmod 7)$ and $n = 2t$ for some odd integer t. Let $t > 1$. Then $t = 4q \pm 1$ for some $q > 0$ and so $n = 2t = 2 \cdot 2^r a \pm 2$ with a odd and $r \geq 2$. Therefore we get $7x^2 \equiv -V_{\pm 2} \equiv -V_2 (\bmod V_{2^r})$ by (2.3). This shows that

$$\left(\frac{7}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{V_2}{V_{2^r}}\right) = -\left(\frac{V_2}{V_{2^r}}\right) \tag{4.1}$$

by (2.6). Let $r = 2$. Then

$$\left(\frac{7}{V_4}\right) = -\left(\frac{V_2}{V_4}\right) = \left(\frac{V_4}{V_2}\right) = \left(\frac{V_2^2 - 2}{V_2}\right) = \left(\frac{-2}{V_2}\right) = -1.$$

Thus, we get

$$-1 = \left(\frac{7}{V_4}\right) = -\left(\frac{V_4}{7}\right) = -\left(\frac{V_2^2 - 2}{7}\right) = -\left(\frac{-2}{7}\right) = 1,$$

which is a contradiction. Now let $r \geq 3$. Then $V_{2^r} \equiv 2 (\bmod 7)$ and $V_{2^r} \equiv 2 (\bmod V_2)$. Thus,

$$\left(\frac{7}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{7}\right) = -\left(\frac{2}{7}\right) = -1$$

and

$$\left(\frac{V_2}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{V_2}\right) = -\left(\frac{2}{V_2}\right) = -1.$$

But this is impossible by (4.1). Thus, $t = 1$ and therefore $n = 2$. □

Theorem 4.3 *The equation $V_n = 13x^2$ has the solution $n = 1$ if $13 \mid P$ and has no solutions if $P^2 \equiv 3 (\bmod 13)$.*

Proof. Let $V_n = 13x^2$ for some integer x. By Lemma 6, $13 \mid V_n$ if and only if $13 \mid P$ and n is odd or $P^2 \equiv 3 (\bmod 13)$ and $n = 3t$ for some odd integer t. Assume that $13 \mid P$. Then by Theorem 2.2, we get $n = 1$. Now assume that $P^2 \equiv 3 (\bmod 13)$ and $n = 3t$ for some odd integer t. Then $n = 3t = 6q + 3$ and so by (2.4), we get

$$13x^2 \equiv V_3 = P(P^2 - 3) (\bmod 8).$$

Since $P^2 \equiv 1 \pmod 8$, it follows that $x^2 \equiv -2P \pmod 8$. However, this is impossible since P is odd. □

Theorem 4.4 *If $V_n = 3kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

Proof. Let $V_n = 3kx^2$ for some $k \mid P$ with $k > 1$. Since $3 \mid V_n$, we get $3 \mid P$ and n is odd by Lemma 2. Let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Then $V_n \equiv V_1, V_3, V_5 \pmod 8$ by (2.4). Thus we get $3kx^2 \equiv P, -2P \pmod 8$. Let $P = kM$. Then $3kMx^2 \equiv PM, -2PM \pmod 8$. That is, $3Px^2 \equiv PM, -2PM \pmod 8$. This implies that $3x^2 \equiv M, -2M \pmod 8$ since P is odd. Thus, we get $x^2 \equiv 3M, 2M \pmod 8$. This shows that $M \equiv 3 \pmod 8$ since M is odd. Let $n > 1$. Then $n = 4q \pm 1$ for some $q > 0$. Thus, we can write $n = 2 \cdot 2^r a \pm 1$ with a odd and $r \geq 1$. Then by (2.3), we get $3kx^2 = V_n \equiv -V_{\pm 1} \pmod{V_{2^r}}$, which implies that $3kx^2 \equiv -P \pmod{V_{2^r}}$. Since $(k, V_{2^r}) = 1$, we get $3x^2 \equiv -M \pmod{V_{2^r}}$. This shows that

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -\left(\frac{M}{V_{2^r}}\right). \tag{4.2}$$

Let $r = 1$. Then

$$\left(\frac{3}{V_2}\right) = -\left(\frac{M}{V_2}\right) = \left(\frac{V_2}{M}\right) = \left(\frac{P^2-2}{M}\right) = \left(\frac{-2}{M}\right) = 1.$$

Since $3 \mid P$, we get $(3/V_2) = -1$ by (2.7). But this is impossible since $(3/V_2) = 1$. Let $r \geq 2$. Then $(3/V_{2^r}) = 1$ by (2.8) and $V_{2^r} \equiv 2 \pmod M$. Thus,

$$1 = \left(\frac{3}{V_{2^r}}\right) = -\left(\frac{M}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{M}\right) = \left(\frac{2}{M}\right) = -1,$$

which is impossible. □

Theorem 4.5 *If $V_n = 11kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

Proof. Let $V_n = 11kx^2$ for some $k \mid P$ with $k > 1$. Since $11 \mid V_n$, n is odd by Lemma 5. Let $P = kM$. Similarly, it can be seen that $M \equiv 3 \pmod 8$. Since $11 \mid V_n$, it follows that $11 \mid P$ and n is odd or $P^2 \equiv 3 \pmod{11}$ and $n = 3t$ with t odd. Let $n > 1$. Then $n = 4q \pm 1$ for some $q > 0$ and so $n = 2 \cdot 2^r a \pm 1$ with a odd and $r \geq 1$. Thus, $11kx^2 = V_n \equiv -V_1 \pmod{V_{2^r}}$

by (2.3). This shows that $11x^2 \equiv -M (\bmod V_{2^r})$, which implies that

$$\left(\frac{11}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -\left(\frac{M}{V_{2^r}}\right). \tag{4.3}$$

Now let $r = 1$. If $11 \mid P$ or $P^2 \equiv 3 (\bmod 11)$, then it can be seen that $(11/V_2) = (M/V_2)$. This is impossible by (4.3). Let $r \geq 2$. If $P^2 \equiv 3 (\bmod 11)$, then it can be seen that $V_{2^r} \equiv -1 (\bmod 11)$ and $V_{2^r} \equiv 2 (\bmod M)$. If $11 \mid P$, then $V_{2^r} \equiv 2 (\bmod 11)$ and $V_{2^r} \equiv 2 (\bmod M)$. In both cases, it is seen that $(11/V_{2^r}) = (M/V_{2^r})$, which is impossible by (4.3). Therefore $n = 1$. □

Since the proof of the following theorem is similar to those of the above theorems, we omit it.

Theorem 4.6 *If $V_n = 13kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

Theorem 4.7 *Let $P^2 \equiv 3 (\bmod 13)$. Let $m = 2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5} > 1$ with $a_j = 0$ or 1 for $1 \leq j \leq 5$. If $V_n = 13mx^2$ for some integer x, then $n = 3$.*

Proof. Assume that $V_n = 13mx^2$ for some integer x. Since $P^2 \equiv 3 (\bmod 13)$ and $13 \mid V_n$, we get $n = 3t$ for some odd integer t by Lemma 6. Thus, n is odd. If $7 \mid m$, then $7 \mid V_n$ and so it follows that $7 \mid P$ by Lemma 3. Therefore we get $7 \mid V_t$ since t is odd. It is clear that $3^{a_2}5^{a_3} \mid V_t$ by Lemmas 2 and 4. Let $m_2 = 3^{a_2}5^{a_3}7^{a_4}$. Then it follows that $m_2 \mid V_t$. Suppose that $P^2 \equiv 3 (\bmod 11)$. Then by (2.14), we get

$$2^{a_1} \cdot 11 \cdot 13 \cdot m_2 \cdot x^2 = V_n = V_{3t} = V_t(V_t^2 - 3) = V_t(V_{2t} - 1),$$

which implies that $2^{a_1} \cdot 11 \cdot 13 \cdot x^2 = (V_t/m_2)(V_t^2 - 3)$. Since $(V_t, V_t^2 - 3) = 1$ or 3, it follows that $V_{2t} - 1 = wa^2$ for some integers a and w where $w = 2^a3^b11^c13^d$ with $a, b, c, d \in \{0, 1\}$. Assume now that $t > 1$ and therefore $2t = 2(4q \pm 1) = 2 \cdot 2^r a \pm 2$ with a odd and $r \geq 2$. Thus, it follows that $wa^2 + 1 = V_{2t} \equiv -V_2 (\bmod V_{2^r})$ by (2.1). This implies that $wa^2 \equiv -(P^2 - 1) (\bmod V_{2^r})$. Therefore

$$\left(\frac{w}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2 - 1}{V_{2^r}}\right).$$

Then

$$\left(\frac{w}{V_{2^r}}\right) = -1 \tag{4.4}$$

by (2.6) and (2.9). Since $V_2 = P^2 - 2 \equiv 1 (\text{mod } 11)$, we get $V_{2^r} \equiv -1 (\text{mod } 11)$ for $r \geq 2$. Thus,

$$\left(\frac{11}{V_{2^r}}\right) = -\left(\frac{V_{2^r}}{11}\right) = -\left(\frac{-1}{11}\right) = 1.$$

Moreover, since $V_2 = P^2 - 2 \equiv 1 (\text{mod } 13)$, we get $V_{2^r} \equiv -1 (\text{mod } 13)$ for $r \geq 2$. Thus,

$$\left(\frac{13}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{13}\right) = \left(\frac{-1}{13}\right) = 1.$$

Moreover, we get $(2/V_{2^r}) = (3/V_{2^r}) = 1$ by (2.5) and (2.8), respectively. Then it follows that $(w/V_{2^r}) = 1$, which is impossible by (4.4). Now suppose that $11 \mid P$. Then $11 \mid V_t$ by Lemma 5. Let $m_1 = 3^{a_2} 5^{a_3} 7^{a_4} 11^{a_5}$. Then it follows that $m_1 \mid V_t$ and so

$$2^{a_1} \cdot 13 \cdot m_1 \cdot x^2 = V_n = V_{3t} = V_t(V_t^2 - 3) = V_t(V_{2t} - 1),$$

which implies that $2^{a_1} \cdot 13 \cdot x^2 = (V_t/m_1)(V_t^2 - 3)$. Since $(V_t, V_t^2 - 3) = 1$ or 3, it follows that $V_{2t} - 1 = wa^2$ for some integers a and w where $w = 2^a 3^b 13^c$ with $a, b, c \in \{0, 1\}$. In a similar way, if $t > 1$, then a contradiction follows. So we get $t = 1$ and therefore $n = 3$. □

Since the proof of the following theorem is similar to that of the above theorem, we omit it.

Theorem 4.8 *Let $P^2 \equiv 3 (\text{mod } 11)$. Let $m = 2^{a_1} 3^{a_2} 5^{a_3} 7^{a_4} 13^{a_5} > 1$ with $a_j = 0$ or 1 for $1 \leq j \leq 5$. If $V_n = 11mx^2$ for some integer x, then $n = 3$.*

Corollary 1 *Let $m = 3^{a_1} 5^{a_2} 7^{a_3} 11^{a_4} 13^{a_5} > 1$ with $a_j = 0$ or 1 for $1 \leq j \leq 5$. If $V_n = 2mx^2$, then $n = 3$.*

Proof. Assume that $V_n = 2mx^2$ for some integer x. If $m \mid P$, then we get $n = 3$ by Theorem 2.3. If $a_4 = 1$ and $P^2 \equiv 3 (\text{mod } 11)$ or $a_5 = 1$ and $P^2 \equiv 3 (\text{mod } 13)$, then by Theorems 4.8 and 4.7, we get $n = 3$. □

By using Theorems 4.7, 4.8, and 2.2, we can give the following corollaries.

Corollary 2 *Let $m = 3^{a_1}5^{a_2}7^{a_3}11^{a_4}13^{a_5} > 1$ with $a_j = 0$ or 1 for $1 \leq j \leq 5$. Suppose that $a_4 \neq 0$ or $a_5 \neq 0$. If $V_n = mx^2$ for some integer x, then $n = 1$ or 3.*

Corollary 3 *If $V_n = 14x^2$ for some integer x, then $n = 3$.*

Proof. Let $V_n = 14x^2$ for some integer x. If $7 \mid P$, then $n = 3$ by Theorem 2.3. Now assume that $P^2 \equiv 2 \pmod 7$ and $n = 2t$ for some odd t. Moreover, since $2 \mid V_n$, it follows that $3 \mid n$ by (2.10). Thus, $n = 6k$ for some odd integer k. Then by (2.4), we get

$$14x^2 = V_n \equiv V_0 = 2 \pmod 8,$$

which is impossible. □

Theorem 4.9 *Let $A \mid P$ with $A > 1$ odd. Then $V_n = AV_mx^2$ has no solutions.*

Proof. Assume that $V_n = AV_mx^2$ for some $A \mid P$ with $A > 1$ odd. Since $A \mid V_n$ and $A \mid P$, n is odd by Lemma 1. Moreover, we get $n = mt$ for some odd integers m and t by (2.16). Assume that $3 \mid t$. Then $t = 3s$ for some positive integer s. Thus,

$$AV_mx^2 = V_n = V_{mt} = V_{3ms} = V_{ms}(V_{ms}^2 - 3)$$

and it follows that

$$\frac{V_{ms}}{V_m}(V_{ms}^2 - 3) = Ax^2$$

since $V_m \mid V_{ms}$ by (2.16). It can be easily seen that $(A, V_{ms}^2 - 3) = 1$ or 3. Assume that $(A, V_{ms}^2 - 3) = 1$. Then it follows that

$$\frac{V_{ms}}{AV_m}(V_{ms}^2 - 3) = x^2.$$

Clearly, $d = \left(V_{ms}/AV_m, V_{ms}^2 - 3\right) = 1$ or 3. Then it follows that $V_{ms}^2 - 3 = a^2$ or $V_{ms}^2 - 3 = 3a^2$ for some integer a. The first one is impossible. If $V_{ms}^2 - 3 = 3a^2$, then $3(V_{ms}/3)^2 = 1 + a^2$, which is impossible. Assume that $(A, V_{ms}^2 - 3) = 3$. Then there exist relatively prime integers A_1, B_1 such that $A = 3A_1$ and $V_{ms}^2 - 3 = 3B_1$. And so,

$$3A_1x^2 = Ax^2 = \frac{V_{ms}}{V_m}(V_{ms}^2 - 3) = \frac{V_{ms}}{V_m}3B_1,$$

i.e.,

$$\frac{V_{ms}}{A_1V_m}B_1 = x^2$$

since $(A_1, B_1) = 1$. Clearly, $d = (V_{ms}/A_1V_m, B_1) = 1$ or 3. Then it follows that $B_1 = a^2$ or $3a^2$ for some integer a. Since $V_{ms}^2 - 3 = 3B_1$, we get $V_{ms}^2 - 3 = 3a^2$ or $V_{ms}^2 - 3 = 9a^2$. In a similar way, it is seen that both cases are impossible. Therefore $3 \nmid t$. Now assume that $3 \mid m$. Since t is odd, we can write $t = 4q \pm 1$ for some $q \geq 0$. Thus, $V_n = V_{4qm \pm m} \equiv V_{\pm m} (\operatorname{mod} U_{2m})$ by (2.2), which implies that $AV_mx^2 \equiv V_m (\operatorname{mod} U_mV_m)$ by (2.17). This shows that $Ax^2 \equiv 1 (\operatorname{mod} U_m)$. Since $3 \mid m$, we get $U_3 \mid U_m$ and therefore $8 \mid U_m$ by (2.15). Then it follows that $Ax^2 \equiv 1 (\operatorname{mod} 8)$. Assume that $3 \nmid m$. Then $3 \nmid n$ since $3 \nmid t$. Therefore $V_n \equiv P (\operatorname{mod} 8)$ and $V_m \equiv P (\operatorname{mod} 8)$ by (2.12). Thus, we see that $APx^2 \equiv P (\operatorname{mod} 8)$, which implies that $Ax^2 \equiv 1 (\operatorname{mod} 8)$. Consequently, we get $Ax^2 \equiv 1 (\operatorname{mod} 8)$ in both cases. This shows that $A \equiv 1 (\operatorname{mod} 8)$. Assume that $t > 1$. Then $t = 4q \pm 1$ for some $q > 0$. Thus, $n = mt = 4qm \pm m = 2 \cdot 2^r a \pm m$ with a odd and $r \geq 1$. Then by (2.3), we get

$$AV_mx^2 \equiv -V_{\pm m} (\operatorname{mod} V_{2^r}),$$

which implies that $AV_mx^2 \equiv -V_m (\operatorname{mod} V_{2^r})$. Then it follows that $Ax^2 \equiv -1 (\operatorname{mod} V_{2^r})$ since $(V_m, V_{2^r}) = 1$ by (2.19). Thus, $(A/V_{2^r}) = (-1/V_{2^r}) = -1$. Since $V_{2^r} \equiv \pm 2 (\operatorname{mod} A)$ for $r \geq 1$, we get

$$-1 = \left(\frac{A}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{A}\right) = \left(\frac{\pm 2}{A}\right) = 1,$$

a contradiction. Thus, $t = 1$ and therefore $n = m$. But this is impossible since $A > 1$. □

Theorem 4.10 *Let $A > 3$ be odd and $P^2 \equiv 3 (\operatorname{mod} A)$. Then the equation $V_n = AV_mx^2$ has a solution only when $m = 1$ and $n = 3$.*

Proof. Assume that $V_n = AV_mx^2$ and $P^2 \equiv 3(\bmod A)$ with $A > 3$ odd. Since $V_m \mid V_n$, we get $n = mt$ for some odd integer t by (2.16). Since $A \mid V_3$, by using (2.1), it can be shown that $n = 3k_1$ for some odd positive integer k_1. This shows that m is odd. Let $3 \mid m$. Then $U_3 \mid U_m$ and therefore $8 \mid U_m$ by (2.15). Since t is odd, $n = mt = m(4q \pm 1) = 4qm \pm m$ for some integer q. Therefore by using (2.2), we get

$$V_n = V_{4qm \pm m} = V_{\pm m}(\bmod U_{2m}),$$

which implies that

$$AV_mx^2 \equiv V_m(\bmod U_mV_m)$$

by (2.17). It follows that $Ax^2 \equiv 1(\bmod U_m)$ and so $Ax^2 \equiv 1(\bmod 8)$ since $8 \mid U_m$. Therefore $A \equiv 1(\bmod 8)$. Let $t > 1$. Then $n = m(4q \pm 1) = 2 \cdot 2^r a \pm m$ with a odd and $r \geq 1$. Therefore by using (2.3), we get

$$AV_mx^2 = V_n \equiv -V_{\pm m}(\bmod V_{2^r}),$$

which shows that

$$Ax^2 \equiv -1(\bmod V_{2^r})$$

since $(V_m, V_{2^r}) = 1$ by (2.19). Thus,

$$\left(\frac{A}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right) = -1$$

by (2.6). Then by using the fact that $V_{2^r} \equiv \pm 1(\bmod A)$, when $P^2 \equiv 3(\bmod A)$, we get

$$-1 = \left(\frac{A}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{A}\right) = \left(\frac{\pm 1}{A}\right) = 1,$$

a contradiction. Therefore $t = 1$ and so $n = m$, which is impossible since $A > 3$. Now let $3 \nmid m$. Then $3 \mid t$ and so $t = 3s$ for some odd integer s. Thus, $n = mt = 3ms$. Therefore by using (2.14), we get

$$AV_mx^2 = V_n = V_{3ms} = V_{ms}(V_{ms}^2 - 3),$$

i.e.,

$$\frac{V_{ms}}{V_m}(V_{ms}^2-3)=Ax^2.$$

Clearly, $(V_{ms}/V_m, V_{ms}^2-3)=1$ or 3. Then by using (2.14), it is seen that

$$V_{ms}^2-3=V_{2ms}-1=kx^2 \text{ or } 3kx^2 \text{ with } k \mid A.$$

Let $ms>1$. Since ms is odd, we get $2ms=2(4q\pm 1)=2\cdot 2^r a\pm 2$ with a odd and $r\geq 2$. Therefore

$$wx^2=V_{2ms}-1\equiv -V_{\pm 2}-1(\operatorname{mod} V_{2^r})$$

by (2.3), which implies that

$$wx^2\equiv -(P^2-1)(\operatorname{mod} V_{2^r})$$

where $w=k$ or $3k$ with $k\mid A$. This shows that

$$\left(\frac{w}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-1}{V_{2^r}}\right),$$

which implies that

$$\left(\frac{w}{V_{2^r}}\right)=-1 \tag{4.5}$$

by (2.6) and (2.9), respectively. Since $r\geq 2$, we get $(3/V_{2^r})=1$ by (2.8). Now we show that $(k/V_{2^r})=1$. Clearly, $(k/V_{2^r})=1$ if $k=1$. Let $k>1$. Then $V_{2^r}\equiv -1(\operatorname{mod} k)$ and thus, we get

$$\left(\frac{k}{V_{2^r}}\right)=(-1)^{(k-1)/2}\left(\frac{V_{2^r}}{k}\right)=(-1)^{(k-1)/2}\left(\frac{-1}{k}\right)=1.$$

As a consequence, we have $(k/V_{2^r})=1$ for $k\mid A$. This shows that $(w/V_{2^r})=1$, which is impossible by (4.5). Thus, $ms=1$ and so $m=1$ and $n=3$. □

Corollary 4 *The equation $V_n=11V_mx^2$ has no solutions.*

Proof. Assume that $V_n = 11V_mx^2$ for some integer x. Then by Theorems 4.9 and 4.10, we get $n = 3$ and $m = 1$. Thus, it follows that $V_3 = 11Px^2$, which implies that $P^2-3 = 11x^2$. This is impossible since $11x^2 \equiv -2 (\mathrm{mod}\, 8)$ in this case. □

By using Theorems 4.9 and 4.10, we can give the following corollaries.

Corollary 5 *The equation $V_n = 13V_mx^2$ has no solutions.*

Corollary 6 *The equation $V_n = 7V_mx^2$ has no solutions.*

Proof. Assume that $V_n = 7V_mx^2$ for some integer x. If $7 \mid P$, then $V_n = 7V_mx^2$ has no solutions by Theorem 4.9. Assume that $P^2 \equiv 2 \pmod 7$. Then $n = 2t$ for some odd integer t by Lemma 3. Since $V_m \mid V_n$, $n = ms$ for some odd integer s by (2.16). Then it follows that $m = 2q$ for some odd integer q. Thus, we get $V_m \equiv 2, 7 \pmod 8$ and $V_n \equiv 2, 7 \pmod 8$ by (2.11). This is impossible since $V_n = 7V_mx^2$. □

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