CONNECTIONS IN THE MANIFOLD ADMITTING CONTACT TRANSFORMATIONS

By

Tôyomon HOSOKAWA

The theory of connections in the manifold admitting the generalized transformations has been developed by the present author.[1] As an application of the theory, it is proposed now to consider some linear displacements in the general manifold preserving a contact transformation.

Consider an n-dimensional manifold X_n with coordinates x^ν ($\nu = a_1, a_2, \ldots, a_n$), and a covariant vector field of components p_λ osculating at each point of X_n. The new manifold obtained in this manner is called the general manifold T_n. However in this general manifold T_n there is no *a priori* basis for the comparison of the covariant vectors at different points. Hence we shall define the relation between an osculating covariant vector p_λ at a given point $P(x_0^\lambda)$ and $p_\lambda + dp_\lambda$ at any nearly point $P'(x_0^\lambda + dx^\lambda)$, by the following equations:

$$(1) \qquad dp_\lambda = \omega_{\lambda\mu} dx^\mu \qquad \lambda = a_1, a_2, \ldots a_n,$$

where parameters $\omega_{\lambda\mu}$ are arbitrary functions of x^ν as well as p_λ. Consequently we see that if at any point $P(x_0^\lambda)$ of X_n we let osculate a covariant vector p_λ, then we get an osculating covariant vector at every point of X_n, so that our manifold T_n is completely determined. The connection so defined is a generalization of that developed some-

(1) T. HOSOKAWA: Connections in the Manifold Admitting Generalized Transformations, Proc. of the Imperial Acad., vol. 8 (1932), p. 384-351.

what by several authors(1) as may be seen. The curves defined by equations (1) are called the *base paths.*

Let us now consider the transformations of the form

$$(2)\qquad 'x^\nu = 'x^\nu(x^\lambda ;\, p_\lambda)\,, \qquad 'p_\lambda = 'p_\lambda(x^\nu ;\, p_\nu)$$

in the $2n$ variables x^ν and p_λ, such that the following equations hold good

$$(3)\qquad d'x^\nu\,'p_\nu = \left(\frac{\partial' x^\nu}{\partial x^\lambda}dx^\lambda + \frac{\partial' x^\nu}{\partial p_\tau}dp_\tau\right)'p_\nu = dx^\nu p_\nu$$

for arbitrary values of the differentials dx^λ and dp_λ, followingly for arbitrary functions $\omega_{\lambda\mu}$.

A transformation (2) satisfying this condition is a contact transformation. From (3) are derived the equations

$$(4)\qquad 'p_\nu\frac{\partial' x^\nu}{\partial x^\lambda} = p_\lambda\,, \qquad 'p_\nu\frac{\partial' x^\nu}{\partial p_\lambda} = 0\,.$$

Then one may see that *a necessary and sufficient condition that a set of functions $'x^\nu(x;p)$ may determine a contact transformation (2) for which the $'p_\lambda(x;p)$ are uniquely determined is that the functions $'x^\nu(x;p)$ be homogeneous of degree zero in p's, that the Jacobian of the $'x^\nu(x;p)$ with respect to the x's be of rank n and that the identities*

$$\frac{\partial' x^\nu}{\partial x^\lambda}\frac{\partial' x^\mu}{\partial p_\lambda} - \frac{\partial' x^\nu}{\partial p_\lambda}\frac{\partial' x^\mu}{\partial x^\lambda} = 0$$

be satisfied.(2) Also *every contact transformation admits a unique inverse contact transformation:*(3)

$$(5)\qquad x^\lambda = x^\lambda('x^\nu ;\, 'p_\nu)\,, \qquad p_\lambda = p_\lambda('x^\nu ;\, 'p_\nu)\,.$$

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(1) T. LEVI-CIVITA: Nozione di parallelismo in una varietà qualunque e consequente specificazione geometrica della curvatura iemanniana, Rendiconti di Palermo, vol. 42 (1917), p. 173–205. L. BERWALD: Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus, Math. Zeit., vol. 25 (1926), p. 40–73. E. BORTOLLOTI: Differential invariants of direction and point displacements, Annals of Math., vol. 32 (1931), p. 361–377.

(2) L. P. EISENHART: Continuous Groups of Transformations, Princeton University Press, (1933), p. 242.

(3) L. P. EISENHART: loc. cit., p. 249.

By differentiation of the first set of (2) and (5), we have

$$(6)\qquad d'x^\nu=\left(\frac{\partial' x^\nu}{\partial x^\lambda}+\frac{\partial' x^\nu}{\partial p_\tau}\omega_{\tau\lambda}\right)dx^\lambda,\quad dx^\lambda=\left(\frac{\partial x^\lambda}{\partial' x^\mu}+\frac{\partial x^\lambda}{\partial' p_\tau}\bar{\omega}_{\tau\mu}\right)d'x^\mu,$$

where

$$d'p_\nu=\bar{\omega}_{\nu\mu}\,d'x^\mu\,.$$

Any set of n quantities $v^\nu(x;p)$, which are transformed by the transformation (2) into n new quantities $'v^\nu('x;'p)$ in such a way that

$$(7)\qquad 'v^\nu=u^\nu_\lambda v^\lambda,$$

will be called a *contravariant vector*; a *covariant vector* is a set of n quantities w_λ which are transformed by (2)

$$(8)\qquad 'w_\mu=v^\lambda_\mu w_\lambda,$$

where

$$(9)\qquad u^\nu_\lambda=\frac{\partial' x^\nu}{\partial x^\lambda}+\frac{\partial' x^\nu}{\partial p_\sigma}\omega_{\sigma\lambda},\quad v^\lambda_\mu=\frac{\partial x^\lambda}{\partial' x^\mu}+\frac{\partial x^\lambda}{\partial' p_\sigma}\bar{\omega}_{\sigma\mu}\,.$$

Let it now be assumed that the following relations are satisfied:

$$(10)\qquad \frac{\partial x^\nu}{\partial' p_\sigma}\bar{\omega}_{\sigma\varkappa}\omega_{\lambda\nu}-\frac{\partial p_\lambda}{\partial' p_\sigma}\bar{\omega}_{\sigma\varkappa}+\frac{\partial x^\nu}{\partial' x^\varkappa}\omega_{\lambda\nu}=\frac{\partial p_\lambda}{\partial' x^\varkappa}$$

and

$$(11)\qquad \frac{\partial x^\nu}{\partial p_\sigma}\omega_{\sigma\varkappa}\,\bar{\omega}_{\lambda\nu}-\frac{\partial' p_\lambda}{\partial p_\sigma}\omega_{\sigma\varkappa}+\frac{\partial' x^\mu}{\partial x^\varkappa}\omega_{\lambda\mu}=\frac{\partial' p_\lambda}{\partial x^\varkappa}\,.$$

But from (9) is obtained

$$u^\nu_\lambda v^\lambda_\mu=\frac{\partial' x^\nu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial' x^\mu}+\frac{\partial' x^\nu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial' p_\sigma}\bar{\omega}_{\sigma\mu}+\frac{\partial x^\varkappa}{\partial' x^\mu}\frac{\partial' x^\nu}{\partial p_\lambda}\omega_{\lambda\varkappa}+\frac{\partial' x^\nu}{\partial p_\lambda}\frac{\partial x^\varkappa}{\partial' p_\sigma}\omega_{\lambda\varkappa}\bar{\omega}_{\sigma\mu}\,,$$

and on the other hand

$$\frac{\partial' x^\nu}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial' p_\mu}+\frac{\partial' x^\nu}{\partial p_\sigma}\frac{\partial p_\sigma}{\partial' p_\mu}=0$$

holds good. Therefore from (10), is obtained

(12) $$u^{\nu}_{\lambda} v^{\lambda}_{\mu} = \delta^{\nu}_{\mu} ,$$

and in like manner from (11)

(13) $$v^{\nu}_{\lambda} u^{\lambda}_{\mu} = \delta^{\nu}_{\mu} ,$$

where the δ's are KRONECKER's deltas.

By means of these new definitions it is to be seen that the p_λ is a covariant vector, because from (4) we get

$$'p_\nu \frac{\partial' x^\nu}{\partial x^\lambda} + 'p_\nu \frac{\partial' x^\nu}{\partial p_\sigma} \omega_{\sigma\lambda} = p_\lambda$$

i. e.

$$u^{\nu}_{\lambda}\, 'p_\nu = p_\lambda ,$$

which becomes by (2)

$$'p_\mu = v^{\lambda}_{\mu} p_\lambda .$$

The equations (6) show that the differential dx^λ is a contravariant vector.

A tensor of the higher order is defined by the following equations:

$$'v^{\nu_1 \dots \nu_t}_{\lambda_1 \dots \lambda_s} = v^{\alpha_1 \dots \alpha_t}_{\beta_1 \dots \beta_s} u^{\nu_1}_{\alpha_1} \dots u^{\nu_t}_{\alpha_t} v^{\beta_1}_{\lambda_1} \dots v^{\beta_s}_{\lambda_s} .$$

When a quantity is invariant by the transformation (2), it is called a *scalar*. Then from (13) it can be shown that $v^\nu w_\nu$ is a scalar.

Now let "metrics" be introduced in our manifold. The metrics must be an invariance by means of the transformation (2). We consider one parameter continuous group G_1 of the contact transformations. An infinitesimal transformation of the group G_1 is defined by equations of the form

(14) $$'x^\lambda = x^\lambda + \frac{\partial C}{\partial p_\lambda} \delta t , \qquad 'p_\lambda = p_\lambda - \frac{\partial C}{\partial x_\lambda} \delta t ,$$

where

$$C = p_\lambda \frac{\partial C}{\partial p_\lambda} \text{[1]} . \tag{15}$$

The function C is called the characteristic function of the contact transformation, and is an invariant function by means of the contact transformation (2).

From (14) are derived

$$\frac{dx^\lambda}{dt} = \frac{\partial C}{\partial p_\lambda} , \qquad \frac{dp_\lambda}{dt} = -\frac{\partial C}{\partial x^\lambda} , \tag{16}$$

and by integration of the above equations we get the finite equations of G_1:

$$'x^\lambda = {'x^\lambda}(x\,;\; p,\, t) , \qquad 'p_\lambda = {'p_\lambda}(x\,;\; p,\, t) .$$

If equations (16) are transformed by means of a contact transformation (2), we obtain

$$\frac{d'x^\lambda}{dt} = \frac{\partial \bar{C}}{\partial' p_\lambda} , \qquad \frac{d'p_\lambda}{dt} = -\frac{\partial \bar{C}}{\partial' x^\lambda} ,$$

where $\bar{C}$ is the transform of the characteristic function of the group G_1[2]. Accordingly we see that the $\dfrac{\partial C}{\partial p_\lambda}$ is a contravariant vector.

In particular we put

$$C = \sqrt{g^{\lambda\mu} p_\lambda p_\mu} , \tag{17}$$

where the $g^{\lambda\mu}$'s are functions of the x's as well as p's, and are homogeneous of zero-th degree in the p's, and the rank of the matrix of the $g^{\lambda\mu}$'s is n. But it is evident that the $g^{\lambda\mu}$'s are components of a contravariant tensor of the second order. We shall take $g^{\lambda\mu}$ as the *fundamental tensor* of the metrics.

(1) L. P. EISENHART: loc. cit., p. 252.
(2) L. P. EISENHART: loc. cit., p. 254.

If the functions $g^{\lambda\mu}$ be defined by the following equations:

$$(18) \qquad g^{\lambda\mu} = \frac{1}{2}\frac{\partial^2 C^2}{\partial p_\lambda \partial p_\mu},$$

then by EULER's theorem we get

$$\frac{\partial g^{\lambda\mu}}{\partial p_\nu} p_\lambda p_\mu = \frac{1}{2}\frac{\partial^3 C^2}{\partial p_\lambda \partial p_\mu \partial p_\nu} p_\lambda p_\mu = 0 .$$

Hence from (16), we have

$$(19) \qquad \frac{dx^\lambda}{dt} = hg^{\lambda\mu} p_\mu , \qquad \frac{dp_\lambda}{dt} = -\frac{h}{2}\frac{\partial g^{\nu\mu}}{\partial x^\lambda} p_\mu p_\nu ,$$

where $h^{-1} = C$.

From (1) and the first set of (19),

$$\frac{dp_\lambda}{dt} = hg^{\nu\mu} \omega_{\lambda\mu} p_\nu .$$

If we define arbitrary functions $\omega_{\lambda\mu}$ by the following equations:

$$(20) \qquad \omega_{\lambda\sigma} = -\frac{1}{2} g_{\nu\sigma} \frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu ,$$

then equations (1) are reduced to the second set of (19).

We shall now define a *linear displacement* for contravariant and covariant vectors v^ν and w_λ:

$$(21) \qquad \begin{cases} \delta v^\nu = dv^\nu + \Gamma^\nu_{\lambda\mu} v^\lambda dx^\mu + \Lambda^{\nu\sigma}_\lambda v^\lambda dp_\sigma , \\ \delta w_\nu = dw_\nu - \Gamma^\lambda_{\nu\mu} w_\lambda dx^\mu - \Lambda^{\lambda\sigma}_\nu w_\lambda dp_\sigma , \end{cases}$$

where $\Gamma^\nu_{\lambda\mu}$ and $\Lambda^{\nu\sigma}_\lambda$ are the functions of x's as well as p's. If the linear displacement is taken along the base paths satisfying (1), we get from the above equations

$$\text{(22)} \qquad \begin{cases} \delta v^\nu = dv^\nu + \overset{*}{\Gamma}{}^\nu_{\lambda\mu} v^\lambda dx^\mu \,, \\ \nabla_\mu v^\nu = \dfrac{\partial v^\nu}{\partial x^\mu} + \dfrac{\partial v^\nu}{\partial p_\sigma}\omega_{\sigma\mu} + \overset{*}{\Gamma}{}^\nu_{\lambda\mu} v^\lambda \end{cases}$$

and

$$\text{(22')} \qquad \begin{cases} \delta w_\nu = dw_\nu - \overset{*}{\Gamma}{}^\lambda_{\nu\mu} w_\lambda dx^\mu \,, \\ \nabla_\mu w_\nu = \dfrac{\partial w_\nu}{\partial x^\mu} + \dfrac{\partial w_\nu}{\partial p_\sigma} w_{\sigma\mu} - \overset{*}{\Gamma}{}^\lambda_{\nu\mu} w_\lambda \,, \end{cases}$$

where

$$\overset{*}{\Gamma}{}^\nu_{\lambda\mu} = \Gamma^\nu_{\lambda\mu} + \Lambda^{\nu\sigma}_\lambda \omega_{\sigma\mu} \,.$$

In order that $\nabla_\mu v^\nu$ may be the components of a mixed tensor, $\overset{*}{\Gamma}{}^\nu_{\lambda\mu}$ must satisfy the following equation:

$$\text{(23)} \qquad \frac{\partial u^\nu_\lambda}{\partial x^\mu} + \frac{\partial u^\nu_\lambda}{\partial p_\sigma}\omega_{\sigma\mu} + {}'\overset{*}{\Gamma}{}^\nu_{\varkappa\sigma} u^\varkappa_\lambda u^\sigma_\mu = u^\nu_\varkappa \overset{*}{\Gamma}{}^\varkappa_{\lambda\mu} \,,$$

where $\overset{*}{\Gamma}{}^\lambda_{\mu\nu}$ are functions of x's as well as p's and ${}'\Gamma^\lambda_{\mu\nu}$ of $'x$'s as well as $'p$'s.

In the same manner as that of the general linear displacements,[1] we can calculate the *curvature tensor*:

$$R_{\nu\mu\rho}^{\cdot\cdot\cdot\lambda} = \frac{\partial \overset{*}{\Gamma}{}^\lambda_{\rho\nu}}{\partial x^\mu} - \frac{\partial \overset{*}{\Gamma}{}^\lambda_{\rho\mu}}{\partial x^\nu} + \overset{*}{\Gamma}{}^\lambda_{\omega\mu} \overset{*}{\Gamma}{}^\omega_{\rho\nu} - \overset{*}{\Gamma}{}^\lambda_{\omega\nu} \overset{*}{\Gamma}{}^\omega_{\rho\mu} + \frac{\partial \overset{*}{\Gamma}{}^\lambda_{\rho\nu}}{\partial p_\tau}\omega_{\tau\mu} - \frac{\partial \overset{*}{\Gamma}{}^\lambda_{\rho\mu}}{\partial p_\tau}\omega_{\tau\nu} \,.$$

When the p's are such that $C \neq 0$ and $h = \text{const.}$, we can normalize $C = 1$, by replacing p_λ by $h^{-1} p_\lambda$. Since C is homogeneous of degree one in the p's. Hence from (19) we get

$$\text{(24)} \qquad \frac{dx^\lambda}{dt} = g^{\lambda\mu} p_\mu \,, \qquad \frac{dp_\lambda}{dt} = -\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\lambda} p_\mu p_\nu \,.$$

When the rank of the hessian of C with respect to p's is $n-1$, the first set of the above equations can be solved with respect to p's as

(1) T. HOSOKAWA: On the Various Linear Displacements in the Berwald-Finsler's Manifold, Science Reports, Tôhoku Imp. University, vol. 19 (1930), p. 37–51.

functions of the x's and $\dot{x}$'s, where $\dot{x}^\lambda = \frac{dx^\lambda}{dt}$. We denote by $\hat{C}$ the function resulting from the substitution in C of these expressions for p_λ. Then we get

$$(25) \qquad \hat{g}_{\lambda\mu} = \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial \dot{x}^\lambda \partial \dot{x}^\mu}, \qquad p_\lambda = \hat{g}_{\lambda\mu}\dot{x}^\mu .$$

From the second set of (24), we have

$$(26) \qquad \frac{d^2x^\lambda}{dt^2} + \left\{ {\lambda \atop \mu\nu} \right\} \frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 ,$$

where $\left\{ {\lambda \atop \mu\nu} \right\}$ are CHRISTOFFEL's symbol with respect to $\hat{g}_{\mu\lambda}$. Thus the paths defined by (24) are the geodesics of BERWALD-FINSLER's manifold.(1) In assumption (20), the parallelism defined by equations (1) is reduced to that by (26). Accordingly from the first set of (25), we have

$$dp_\lambda = -\left\{ {\nu \atop \lambda\mu} \right\} \hat{g}_{\nu\sigma}\dot{x}^\sigma dx^\mu .$$

Hence from (1), we obtain

$$\omega_{\lambda\mu} = -\left\{ {\nu \atop \lambda\mu} \right\} \hat{g}_{\nu\sigma}\dot{x}^\sigma .$$

Consequently from the linear displacement (22) and (22′) we can reduce the connections which has already been studied by the present author.(2)

(1) M. S. KNEBELMEN: Collineations and Motions in Generalized Space, American Journal of M thematics, vol. 51 (1928), p. 527–564.

(2) T. HOSOKAWA: loc. cit., (1930), p. 42.