A REMARK TO A COVARIANT DIFFERENTIATION PROCESS

By

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In one of his papers CRAIG[(1)] has introduced a covariant derivative

$$(1) \qquad T^{\alpha\cdots}_{\beta\cdots x(m-1)\gamma} - m T^{\alpha\cdots}_{\beta\cdots x(m)\lambda} \left\{ \begin{matrix} \lambda \\ \gamma \end{matrix} \right\}$$

from a tensor $T^{\alpha\cdots}_{\beta\cdots}$ whose components depend on n coordinates x and their m derivatives $x, x', x'', \dots, x^{(m)}$, that is, are of order m, where

$$\left\{ \begin{matrix} \lambda \\ \gamma \end{matrix} \right\} \equiv x'^{\alpha} \Gamma^{\lambda}_{\gamma\alpha} + \frac{1}{2} x''^{\beta} f_{\gamma\delta\beta} f^{\delta\lambda},$$

which was obtained by TAYLOR[(2)], and partial differentiation was denoted by the subscript. Thereafter, extending this process, JOHNSON[(3)] has introduced the following covariant derivative

$$(2) \quad T^{\alpha\cdots}_{\beta\cdots x(m-2)\gamma} - (m-1) T^{\alpha\cdots}_{\beta\cdots x(m-1)\lambda} \left\{ \begin{matrix} \lambda \\ \gamma \end{matrix} \right\} - \frac{m(m-1)}{2} T^{\alpha\cdots}_{\beta\cdots x(m)\lambda} \left| \begin{matrix} \lambda \\ \gamma \end{matrix} \right|,$$

where

$$\left| \begin{matrix} \lambda \\ \gamma \end{matrix} \right| \equiv Q^{\lambda}_{x\gamma} - Q^{\lambda}_{x'\alpha} \left\{ \begin{matrix} \alpha \\ \gamma \end{matrix} \right\} + Q^{\alpha} \dot{\Lambda}^{\lambda}_{\alpha\gamma}, \quad Q^{\alpha} \equiv x''^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} x'^{\beta} x'^{\gamma},$$

$$\Lambda^{\lambda}_{\alpha\gamma} \equiv \Gamma^{\lambda}_{\alpha\gamma} - \frac{1}{2} f^{\lambda\beta} \left(f_{\gamma\beta\tau} \left\{ \begin{matrix} \tau \\ \alpha \end{matrix} \right\} + f_{\beta\alpha\tau} \left\{ \begin{matrix} \tau \\ \gamma \end{matrix} \right\} - f_{\alpha\gamma\tau} \left\{ \begin{matrix} \tau \\ \beta \end{matrix} \right\} \right).$$

To obtain this result, she eliminated $\partial x''^{\beta} / \partial y^{k}$ making use of a very complicated calculation. Accordingly it seems almost impossible to extend her method further.

In the present paper it is proposed to introduce a general covariant differentiation process which involves (1) and (2) as special cases.

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(1) H. V. CRAIG, On a covariant differentiation process, Bull. of the Amer. Math. Soc., Vol. 37 (1931), p. 731-734.

(2) J. H. TAYLOR, A generalization of LEVI-CIVITA's parallelism and the FRENET formulas, Trans. of the Amer. Math. Soc., Vol. 27 (1925), p. 246-264.

(3) M. M. JOHNSON, An extension of a covariant differentiation process, Bull. of the Amer. Math. Soc., Vol. 46 (1940), p. 269-271.

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In a KAWAGUCHI space[1] of order m and dimension n a covariant differentiation of a contravariant vector X^i is described in the form

$$\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + \Gamma^i_j X^j .$$

The notations

$$X^{(0)i} = X^i , \quad X^{(s)i} = \frac{d^s X^i}{dt^s} \qquad (s = 1, 2, \dots)$$

will be adopted for simplicity.

Let X^i be any contravariant vector, then its r-th covariant derivative is expressed by a linear combination of X^i and its ordinary derivatives with respect to t

$$(3) \qquad \frac{\delta^r X^i}{dt^r} = \sum_{s=0}^{r} \binom{r}{s} \overset{(s)}{\Gamma^i_j} X^{(r-s)j} ,$$

where

$$\overset{(0)}{\Gamma^i_j} = \delta^i_j , \quad \overset{(1)}{\Gamma^i_j} = \Gamma^i_j , \quad \overset{(s)}{\Gamma^i_j} = \frac{d\overset{(s-1)}{\Gamma^i_j}}{dt} + \Gamma^i_k \overset{(s-1)}{\Gamma^k_j} \quad (s = 2, 3, \dots , r).$$

This can be proved by mathematical induction. That is, if (3) is correct for r,

$$\frac{\delta^{r+1} X^i}{dt^{r+1}} = \sum_{s=0}^{r} \binom{r}{s} \left(\overset{(s)}{\Gamma^i_j} X^{(r+1-s)j} + \frac{d\overset{(s)}{\Gamma^i_j}}{dt} X^{(r-s)j} + \Gamma^i_k \overset{(s)}{\Gamma^k_j} X^{(r-s)j} \right)$$

$$= \sum_{s=0}^{r} \binom{r}{s} \left(\overset{(s)}{\Gamma^i_j} X^{(r+1-s)j} + \overset{(s+1)}{\Gamma^i_j} X^{(r-s)j} \right)$$

$$= \sum_{s=0}^{r+1} \binom{r+1}{s} \overset{(s)}{\Gamma^i_j} X^{(r+1-s)j} .$$

And (3) is primitive for $r = 1$. Therefore (3) is true for all r.

Inversely :

Let X^i be any contravariant vector, then its r-th ordinary derivative with respect to t is expressed by a linear combination of X^i and its covariant derivatives

(1) A. KAWAGUCHI, Theory of connections in a KAWAGUCHI space of higher order, Proc. of the Imp. Acad., Tokyo, Vol. 13 (1937), p. 237-240.

(4) $$X^{(r)i} = \sum_{s=0}^{r} \binom{r}{s} \overset{(s)}{\Pi}{}^i_j \frac{\delta^{r-s} X^j}{dt^{r-s}},$$

where

(5) $$\overset{(0)}{\Pi}{}^i_j = \delta^i_j, \qquad \overset{(1)}{\Pi}{}^i_j = \Pi^i_j = -\Gamma^i_j,$$

$$\overset{(s)}{\Pi}{}^i_j = \frac{d\overset{(s-1)}{\Pi}{}^i_j}{dt} + \overset{(s-1)}{\Pi}{}^i_k \Pi^k_j \qquad (s = 2, 3, \ldots, r).$$

In one of his papers Prof. A. KAWAGUCHI has introduced the covariant derivation(1)

(6) $$\overset{p-\rho}{D}_{ij}(T) X^j = \sum_{\alpha=\rho}^{p} \binom{\alpha}{\rho} T_{i(\alpha)j} X^{(\alpha-\rho)j} \qquad (\rho = 1, 2, \ldots, p)$$

along a curve, T_i being any covariant vector of order p and X^i an arbitrary contravariant vector of any order. Use of (4) leads the right member of (6) to

$$\overset{p-\rho}{D}_{ij}(T) X^j = \sum_{\alpha=\rho}^{p} \binom{\alpha}{\rho} T_{i(\alpha)k} \sum_{\lambda=0}^{\alpha-\rho} \binom{\alpha-\rho}{\lambda} \overset{(\lambda)}{\Pi}{}^k_j \frac{\delta^{\alpha-\rho-\lambda} X^j}{dt^{\alpha-\rho-\lambda}}$$

$$= \sum_{\alpha=\rho}^{p} \binom{\alpha}{\rho} T_{i(\alpha)k} \sum_{\lambda=\rho}^{\alpha} \binom{\alpha-\rho}{\alpha-\lambda} \overset{(\alpha-\lambda)}{\Pi}{}^k_j \frac{\delta^{\lambda-\rho} X^j}{dt^{\lambda-\rho}}$$

$$= \sum_{\lambda=\rho}^{p} \binom{\lambda}{\rho} \sum_{\alpha=\lambda}^{p} \binom{\alpha}{\lambda} T_{i(\alpha)k} \overset{(\alpha-\lambda)}{\Pi}{}^k_j \frac{\delta^{\lambda-\rho} X^j}{dt^{\lambda-\rho}}.$$

Therefore it is obtained covariant tensors

(7) $$\sum_{\alpha=\lambda}^{p} \binom{\alpha}{\lambda} T_{i(\alpha)k} \overset{(\alpha-\lambda)}{\Pi}{}^k_j \qquad (\lambda = 1, 2, \ldots, p).$$

Let any tensor $T^{l\cdots}_{i\cdots}$ *be considered, then*

(8) $$\sum_{\alpha=\lambda}^{p} \binom{\alpha}{\lambda} T^{l\cdots}_{i\cdots(\alpha)k} \overset{(\alpha-\lambda)}{\Pi}{}^k_j \qquad (\lambda = 1, 2, \ldots, p)$$

are tensors whose indices are one more than those of $T^{l\cdots}_{i\cdots}$, *where* $\overset{(s)}{\Pi}{}^k_j$ $(s = 0, 1, \ldots, p)$ *are defined by* (5).

For $p = m$, $\lambda = m-1$, one obtains

(9) $$T^{l\cdots}_{i\cdots(m-1)k} \overset{(0)}{\Pi}{}^k_j + m T^{l\cdots}_{i\cdots(m)k} \overset{(1)}{\Pi}{}^k_j,$$

(1) A. KAWAGUCHI, Some intrinsic derivations in a generalized space, Proc. of the Imp. Acad., Tokyo, Vol. 12 (1936), p. 149-151.

and, for $p = m$, $\lambda = m-2$, one obtains

$$(10) \quad T^{l}_{i\cdots(m-2)k}\overset{(0)}{\Pi}{}^{k}_{j} + (m-1)T^{l}_{i\cdots(m-1)k}\overset{(1)}{\Pi}{}^{k}_{j} + \frac{m(m-1)}{2}T^{l}_{i\cdots(m)k}\overset{(2)}{\Pi}{}^{k}_{j}$$

from (8).

CRAIG has used $\left\{ {i \atop j} \right\}$ in place of our Γ^{i}_{j}, and (9) reduces to his covariant differentiation (1). (10) is different partly from JOHNSON's. But one can reconcile the present process to hers, as follows. Making use of

$$(11) \qquad \frac{\delta dx^{i}}{dt} = dx'^{i} + \left\{ {i \atop j} \right\} dx^{j},$$

one has

$$df_{ij} = \left(f_{ij(0)k} - f_{ij(1)l}\left\{ {l \atop k} \right\}\right)dx^{k} + f_{ij(1)l}\frac{\delta dx^{l}}{dt},$$

and in the same way

$$df_{jk} = \left(f_{jk(0)i} - f_{jk(1)l}\left\{ {l \atop i} \right\}\right)dx^{i} + f_{jk(1)l}\frac{\delta dx^{l}}{dt},$$

$$df_{ki} = \left(f_{ki(0)j} - f_{ki(1)l}\left\{ {l \atop j} \right\}\right)dx^{j} + f_{ki(1)l}\frac{\delta dx^{l}}{dt}.$$

From these it is found that Λ^{i}_{jk} are transformed just as well as the parameters of an affine connection under a coordinate transformation. Therefore one can introduce the following base connection

$$(12) \qquad \delta Q^{i} = dQ^{i} + \Lambda^{i}_{jk}Q^{j}dx^{k} = dx''^{i} + \left| {i \atop j} \right| dx^{j} + Q^{i}_{x',k}\frac{\delta dx^{k}}{dt}.$$

From (11), (12) and

$$T^{l}_{i\cdots(m-2)j}dx^{j} + (m-1)T^{l}_{i\cdots(m-1)j}dx'^{j} + \frac{m(m-1)}{2}T^{l}_{i\cdots(m)j}dx''^{j},$$

which is a tensor obtained immediately by using the vector dx^{i} instead of X^{i} in (6), her covariant derivative (2) is obtained.

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