NON-HOLONOMIC SYSTEM IN A SPACE OF HIGHER ORDER I. ON THE OPERATIONS OF EXTENSORS

By

Yoshie KATSURADA

Introduction. In the previous papers [1][1] the present author has treated the theory of certain non-holonomic spaces of line-elements and of the non-holonomic system depending on line-elements. The principal purposes of the present paper are to generalize the concept "non-holonomic system" into a space of higher order and to find in the generalized non-holonomic system the structure of several operations of extensors considered by Prof. A. KAWAGUCHI [2]. The former is stated in § 2 and the latter in §§ 4–9. As the preparation for these purposes, § 1 is devoted to the exposition of the notations employed and of the definition of extensors introduced first by H. V. CRAIG [3] and in § 3 we treat upon the transformations of the non-holonomic systems. Since there are three kinds of the previous operations, any two of these produce their products. In the last chapter, we discuss therefore the commutativity of these products.

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§ 1. Notations and preliminaries. In the present paper we shall employ at most two holonomic coordinate systems x and $\bar{x}$ and so far as the quantities that bear indices are concerned, we shall distinguish them whenever feasible by restricting the choice of indicial letters. Specifically, letters at the first of the alphabet a, b, c, d, e shall serve to denote the system $\bar{x}$, while $i, j, k, l, m,$ will be correlated to the system x. Thus x^i is the i-th coordinate variable of the system x, while $\bar{x}^a$ is a variable of number a of the system $\bar{x}$. Differentiation with respect

(1) Number in brackets refer to the references at the end of the paper.

to the parameter t of a parameterized arc will be indicated by primes and Greek indices. To illustrate,

$$x'^i = \frac{dx^i}{dt}, \quad x^{(\alpha)i} = \frac{d^\alpha x^i}{dt^\alpha}, \quad X^{\rho i}_{\alpha a} = X^{(\rho)i}_{(\alpha)a} = \frac{\partial x^{(\rho)i}}{\partial \bar{x}^{(\alpha)a}}, \quad X^{\alpha a}_{\rho i} = \frac{\partial \bar{x}^{(\alpha)a}}{\partial x^{(\rho)i}},$$

$$f_{(\alpha)i} = \frac{\partial f}{\partial x^{(\alpha)i}}, \quad f_{(\alpha)i}^{\cdot\,\cdot\,(\beta)} = \frac{d^\beta}{dt^\beta}\left(\frac{\partial f}{\partial x^{(\alpha)i}}\right), \quad f^{(\beta)}_{\cdot\,(\alpha)i} = \frac{\partial}{\partial x^{(\alpha)i}}\left(\frac{d^\beta f}{dt^\beta}\right).$$

Summation convention. Repeated lower case Latin indices call for summation 1 to N, while the summations indicated by repeated lower case Greek indices are from zero (unless the contrary is specified) to some terminal value usually G or $G+1$. Repeated Capital Greek indices do not generate sums, thus $\binom{M}{\alpha} X^{(M-\alpha)}_{\alpha}$ with α not summed would be written $\binom{M}{A} X^{(M-A)}_{\alpha}$.

At last, we shall use indices with primes, i. e. $a', b', c', \cdots, i', j', k', \cdots$ and $\alpha', \beta', \gamma', \cdots\cdots$, referred to non-holonomic systems.

Extensor. We shall consider a space of line-elements of higher order. For example, an element of the space of line-elements of order M will be denoted with $x^i, x^{(1)i}, \cdots\cdots\cdots, x^{(M)i}$ and it is called an extended point (or expoint) following H. V. CRAIG [3].

The coordinate transformation: $\bar{x}^a = \bar{x}^a(x^i)\,(a, i = 1, \cdots, N)$, which is assumed to be of class M and regular [4], gives rise upon successive differentiations with respect to the parameter t of a parameterized arc to the "coordinate transformation of extended point":

$$\bar{x}^a = \bar{x}^a(x^i), \quad \bar{x}^{(1)a} = \frac{\partial \bar{x}^a}{\partial x^i} x^{(1)i},$$

$$\text{(1. 1)} \qquad \bar{x}^{(2)a} = \frac{\partial \bar{x}^a}{\partial x^i} x^{(2)i} + \frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^j} x^{(1)i} x^{(1)j},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\bar{x}^{(M)a} = \frac{\partial \bar{x}^a}{\partial x^i} x^{(M)i} + M \frac{\partial^2 \bar{x}^a}{\partial x^i \partial x^j} x^{(1)i} x^{(M-1)j} + \cdots\cdots\cdots.$$

Then the last relationship suggests the formulas ([2], p. 17)

$$\frac{\partial \bar{x}^{(\alpha)a}}{\partial x^{(\beta)i}} = \frac{\alpha!\,(\beta-\gamma)!}{\beta!\,(\alpha-\gamma)!} \cdot \frac{\partial \bar{x}^{(\alpha-\gamma)a}}{\partial x^{(\beta-\gamma)i}}$$

for $\alpha, \beta \geqq \gamma > 0$, and for $\gamma = \beta$

$$\text{(1. 2)} \qquad \frac{\partial \bar{x}^{(\alpha)a}}{\partial x^{(\beta)i}} = \binom{\alpha}{\beta} \frac{\partial \bar{x}^{(\alpha-\beta)a}}{\partial x^i}, \quad \frac{\partial \bar{x}^{(\alpha)a}}{\partial x^i} = \left(\frac{\partial \bar{x}^{(\alpha-\beta)a}}{\partial x^i}\right)^{(\beta)} = \left(\frac{\partial \bar{x}^a}{\partial x^i}\right)^{(\alpha)}.$$

Especially $\dfrac{\partial \bar{x}^{(\alpha)a}}{\partial x^{(\beta)i}} = \dfrac{\partial \bar{x}^a}{\partial x^i}$ for $\alpha = \beta$, and $\dfrac{\partial \bar{x}^{(\alpha)a}}{\partial x^{(\beta)i}} = 0$ for $\alpha < \beta$.

The notion of extensor is merely that of tensor relative to this extended transformation. Its components will be denoted by means of symbols bearing Greek-Latin doublet indices such as ρi or αa, i.e., if $\prod_{\lambda=1}^{A}(G_\lambda + 1)\prod_{\mu=1}^{B}(G'_\mu + 1) N^{A+B}$ quantities $T^{\gamma_1 i_1 \cdots \gamma_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B}$ $(i_\lambda, j_\mu = 1, 2, \cdots, N;\ \gamma_\lambda = 0, 1, \cdots, G_\lambda;\ \delta_\mu = 0, 1, \cdots, G_\mu)$ are related to the quantities $T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B}$ belonging to any other coordinate system $\bar{x}$ according to the transformation equation

$$T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B} = \Delta^{-k} T^{\gamma_1 i_1 \cdots \gamma_A i_A}{}_{\delta_1 j_1 \cdots \delta_B j_B} \prod_{\lambda=1}^{A} X^{(\alpha_\lambda) a_\lambda}_{(\gamma_\lambda) i_\lambda} \prod_{\mu=1}^{B} X^{(\delta_\mu) j_\mu}_{(\beta_\mu) b_\mu}$$

where the symbol Δ denotes the determinant $\left| \dfrac{\partial \bar{x}^a}{\partial x^i} \right|$, then we shall speak of these quantities as the components of a mixed extensor of order $A + B$, range G and weight k —excontravariant of order A, excovariant of order B. If the quantities T are functions of the sets of variables $x, x^{(1)}, \cdots, x^{(M)}$ $(M \leqq P)$, then we shall employ the term extensor field of function order M. In case that the Greek indices have different ranges, the range of the extensor will be said to be G, the maximum of separate ranges. Following Prof. A. Kawaguchi ([2], p. 21) this extensor will be called to be of characteristic $(A+B, k, G, M)$.

§ 2. The concept of non-holonomic system in a space of line-elements of order M. Let us consider a space $K_N^{(M)}$ of line-elements of order M and a set of $(G+1)\ N$ excovariant extensors $\lambda_{\alpha}^{\alpha' i'}{}_{i}$ $(x, x^{(1)}, \cdots, x^{(M)})$ $(\alpha' = 0, 1, \cdots, G;\ i' = 1, \cdots, N)$ of characteristic $(1, 0, G, M)$ (where $G \leqq M$), associated with each expoint $x^{(\alpha)i}$ of the space of line-elements of order M: $K_N^{(M)}$, and put the conditions that

$$\lambda_{\alpha}^{\alpha' i'}{}_{i} = 0 \qquad \text{for } \alpha' < \alpha \tag{2.1}$$

and that $(G+1)N$-rowed determinant λ constructed from $\lambda_{\alpha}^{\alpha' i'}{}_{i}$ with respect to the pairs $(\alpha' i')$, (αi) does not vanish in the considered domain of $K_N^{(M)}$. This determinant λ becomes as follows:

$$\lambda = |\lambda_{0}^{0 i'}{}_{i}| \times |\lambda_{1}^{1 i'}{}_{i}| \times \cdots\cdots \times |\lambda_{G}^{G i'}{}_{i}|, \tag{2.2}$$

here we can define uniquely the reciprocal excontravariant extensors $\lambda_{\beta}^{\beta}{}_{j'}^{j}$ of the excovariant extensors $\lambda_{\alpha}^{\alpha' i'}{}_{i}$ such that they satisfy $\sum_{\alpha=0}^{G} \lambda_{\alpha}^{\alpha' i'}{}_{i} \lambda_{\beta'}^{\alpha}{}_{j'}^{i} = \delta_{\beta'}^{\alpha'} \delta_{j'}^{i'}$, where symbol δ denotes the Kronecker's delta.

Remark. Let Q be a certain positive integer less than G, and consider the $(Q+1)N$ excovariant extensors $\lambda^{\alpha' i'}_{\alpha\ i}$ $(\alpha'=0,\cdots,Q: i'=1,\cdots,N)$ that are a part of the set of $(G+1)N$ excovariant extensors $\lambda^{\alpha' i'}_{\alpha\ i}$ $(\alpha'=0,\cdots,G;\ i'=1,\cdots,N)$, then their reciprocal excontravariant extensors $\lambda^{*\alpha\ i}_{\beta' j'}$ $(\beta'=0,\cdots,Q;\ j'=1,\cdots,N)$ will be obtain from the equations $\sum_{\alpha=0}^{Q}\lambda^{\alpha' i'}_{\alpha\ i}\lambda^{*\alpha\ i}_{\beta' j'}=\delta^{\alpha'}_{\beta'}\delta^{i'}_{j'}$ $(\alpha',\beta'=0,\cdots,Q;\ i',j'=1,\cdots,N)$. It is verified without difficulty from (2. 2) that such quantities $\lambda^{*\alpha\ i}_{\beta' j'}$ coincide with $\lambda^{\alpha\ i}_{\beta' j'}$ $(\beta'=0,\cdots,Q;\ j'=1,\cdots,N)$ which are a part of the set of the $(G+1)N$ quantities $\lambda^{\alpha\ i}_{\beta' j'}$ $(\beta'=0,\cdots,G;\ j'=1,\cdots,N)$.

Definition. Under "non-holonomic system" in the extensor space consisted of all extensors at any expoint of $K^{(M)}_N$ with range not greater than G, we understand two sets of $N(G+1)$ mutually independent excovariant extensors $\lambda^{\alpha' i'}_{\alpha\ i}$ and excontravariant extensors $\lambda^{\alpha\ i}_{\alpha' i'}$ which are called the base extensors of the non-holonomic system. Then we shall define the components of the extensor $T^{\gamma_1 i_1\cdots\gamma_A i_A}{}_{\delta_1 j_1\cdots\delta_B j_B}$ of the characteristic $(A+B,\ k,\ R,\ D)$ in the non-holonomic system, under the restriction $R\leqq G$ and $P\geqq D\geqq M$, as follows:

$$T^{\gamma_1' i_1'\cdots\gamma_A' i_A'}{}_{\delta_1' j_1'\cdots\delta_B' j_B'}=\prod_{\theta=0}^{R}\varDelta^{*-k_\theta}_{\theta}\ T^{\gamma_1 i_1\cdots\gamma_A i_A}{}_{\delta_1 j_1\cdots\delta_B j_B}\times\prod_{\lambda=1}^{A}\lambda^{\gamma_\lambda' i_\lambda'}_{\gamma_\lambda\ i_\lambda}\prod_{\mu=1}^{B}\lambda^{\delta_\mu\ j_\mu}_{\delta_\mu' j_\mu'}\tag{2. 3}$$

where the symbol $\varDelta^*_\theta$ indicates the N-rowed determinants constructed from $\lambda^{\theta i'}_{\theta\ i}$ (not summing on θ) and $\sum_{\theta=0}^{R}k_\theta$ is equal to k.

§3. Non-holonomic transformations of non-holonomic systems. Now, we consider infinite collection of the set of base extensors $\lambda^{\alpha' i'}_{\alpha\ i}$ (or $\lambda^{\alpha\ i}_{\alpha' i'}$), and between two sets of base extensors the following relation is assumed to exist:

$$\lambda^{\gamma' i'}_{\gamma\ i}=N^{\gamma' i'}_{\alpha' a'}\lambda^{\alpha' a'}_{\gamma\ i}\tag{3. 1}$$

where the quantities $N^{\gamma' i'}_{\alpha' a'}$ satisfy the conditions that

$$N^{\gamma' i'}_{\alpha' a'}=0\qquad\text{for }\gamma'<\alpha',\tag{3. 2}$$

and that $(G+1)N$-rowed determinant N constructed from $N^{\gamma' i'}_{\alpha' a'}$ with respect to the pairs $(\gamma' i')$, $(\alpha' a')$ does not vanish in considered domain of $K^{(M)}_N$. When (3.1) is considered as a transformation of the base extensors, these transformations form the group, that depends on $\frac{1}{2}(G+1)\times(G+2)N^2$ arbitrary functions of variables $x,\ x^{(1)},\cdots,\ x^{(M)}$. Then we shall call such transformations (3.1) as the *non-holonomic transformations of the non-holonomic system* and such a transformation group as *the non-holonomic transformation group.* The determinant N is written by

$$N = |N^{0i'}_{ca'}| \times \cdots\cdots \times |N^{Gi'}_{Ga'}|$$

from (3.2) and the functions $N^{\alpha'a'}_{\delta' j'}$ corresponding to the inverse transformation of (3.1) are obtained as the solutions satisfying the equations $N^{\gamma' i'}_{\alpha' a'} N^{\alpha' a'}_{\delta' j'} = \delta^{\gamma'}_{\delta} \delta^{i'}_{j'}$ where concerning with these functions $N^{\alpha' a'}_{\delta' j'}$ we should remember the same remark as that for $\lambda^{\gamma}_{\delta}{}^{i}_{j'}$ in §2.

Definition. If there is at a certain point of a parameterized arc of class P in our space one set of $\prod_{\lambda=1}^{A}(R_\lambda+1)\prod_{\mu=1}^{B}(R'_\mu+1)\,N^{A+B}$ quantities which are functions of the set of variables x, $x^{(1)}, \cdots, x^{(D)}$ $(M \leqq D \leqq P)$: $T^{\gamma_1' i_1' \cdots \gamma_A' i_A'}{}_{\delta_1' j_1' \cdots \delta_B' j_B'}$ $(i'_\lambda, j'_\mu = 1, \cdots, N;\ \gamma'_\lambda = 0, 1, \cdots, R_\lambda;\ \delta'_\mu = 0, 1, \cdots, R'_\mu;\ R_\lambda, R'_\mu \leqq G)$ for each non-holonomic system and if the quantities $T^{\gamma_1' i_1' \cdots \gamma_A' i_A'}{}_{\delta_1' j_1' \cdots \delta_B' j_B'}$ associated with any one system are related to the quantities $T^{\alpha_1' a_1' \cdots \alpha_A' a_A'}{}_{\beta_1' b_1' \cdots \beta_B' b_B'}$ belonging to any other system according to the transformation equation

$$T^{\alpha_1 a_1 \cdots \alpha_A a_A}{}_{\beta_1 b_1 \cdots \beta_B b_B} = \prod_{\theta'=0}^{R} \Delta^{*k_{\theta'}}_{\theta'}\, T^{\gamma_1' i_1' \cdots \gamma_A' i_A'}{}_{\delta_1' j_1' \cdots \delta'_B j'_B} \times \prod_{\lambda=1}^{A} N^{\alpha_\lambda' a_\lambda'}_{\gamma_\lambda' i_\lambda'} \prod_{\mu=1}^{B} N^{\delta_\mu' j_\mu'}_{\beta_\mu' b_\mu'} \tag{3.3}$$

where the symbol $\Delta^{*}_{\theta'}$ denotes the N–rowed determinant constructed from $N^{\theta' a'}_{\theta' i'}$ (not summing on θ') and $\sum_{\theta'=0}^{R} k_{\theta'} = k$, then we shall speak of these quantities as the non-holonomic components of a mixed extensors of order $A+B$, range R and weight $\sum_{\theta'=0}^{R} k_{\theta'} = k$, R being the maximum of R_λ and R'_μ, and shall denote their characteristic with the symbol $(A+B, \sum_{\theta'=0}^{R} k_{\theta'} = k, R, D)$. In virtue of this definition, it will be seen easily that the characteristic of the extensor is not changed by the non-holonomic transformations. Henceforth, we shall go to show that non-holonomic components of various extensors obtained from extensors by differential operations can be expressed in terms of non-holonomic components of the original extensor.

It is well known that if v^i is a contravariant vector of functional order M and its necessary derivatives exist, then quantities $v^{(\gamma)i}$ $(\gamma =$

$0, 1, \cdots, L; M+L \leqq P)$ are the components of an extensor of characteristic $(1, 0, L, M+L)$ ([2], p. 22). At first we seek the structure of components (in our system) of this extensor $v^{(\tau)i}$.

In accordance with the definition (2.3), the components (in our system) of the vector v^i and of the extensor $v^{(\tau)i}$ are given by $v^{i'} = \lambda^{0i'}_{0i} v^i$ and $v^{\tau' i'} = \lambda^{\tau' i'}_{\tau i} v^{(\tau)i}$ respectively. Multiplying the first equation by $\lambda^{0i}_{0i'}$, summing and then differentiating the both members of the resulted equation r times with respect to t, we have $v^{(\tau)i} = \sum_{\rho'=0}^{\tau} \binom{\tau}{\rho'} \lambda^{0i\,(\tau-\rho')}_{0i'} v^{i'(\rho')}$, where $\binom{\tau}{\rho'}$ is a binomial coefficient. If we put

$$\sum_{\tau=\rho'}^{\tau'} \binom{\tau}{\rho'} \lambda^{\tau' i'}_{\tau i} \lambda^{0i\,(\tau-\rho')}_{0j'} \equiv C^{\tau' i'}_{\rho' j'} \qquad \begin{array}{l} \gamma' = 0, \cdots\cdots, L \\ \rho' = 0, \cdots\cdots, \gamma' \end{array} \tag{3. 4}$$

$$= 0 \qquad \text{for} \quad \gamma' < \rho',$$

then the quantities $v^{\tau' i'}$ will be written as follows:

$$v^{\tau' i'} = \sum_{\rho'=0}^{\tau'} C^{\tau' i'}_{\rho' j'} v^{0j'(\rho')}. \tag{3. 5}$$

Thus the structure of $v^{\tau' i'}$ is obtained by the right member of the last equation. Here, such the new quantities $C^{\tau' i'}_{\rho' j'}$ in the non-holonomic system are called *extended coefficients of a vector* in the non-holonomic system. Consequently, we have the following

Theorem 1: *If $v^{0i'}$ are components in the non-holonomic system of a vector of the characteristic (1, 0, 0, M) and their necessary derivatives exist, then the quantities*

$$v^{\tau' i'} = \sum_{\rho'=0}^{\tau'} C^{\tau' i'}_{\rho' j'} v^{0j'(\rho')}, \qquad \gamma' = 0, \cdots\cdots, L$$

are the components in the non-holonomic system of an extensor of characteristic $(1, 0, L, M+L(\leqq P))$.

Next, we shall find properties of the new quantities $C^{\tau' i'}_{\rho' j'}$.

Theorem 2: *If our system is a holonomic system, then it follows that $C^{\tau' i'}_{\rho' j'} = \delta^{\tau'}_{\rho'} \delta^{i'}_{j'}$.*

Proof. In a holonomic system, the quantities $\lambda^{\tau' i'}_{\tau i}$ must be indicated by $X^{(\tau')i'}_{(\tau)i}$, accordingly (3.4) become as follows:

$$C^{\tau' i'}_{\rho' j'} = \sum_{\tau=\rho'}^{\tau'} \binom{\tau}{\rho'} \lambda^{\tau' i'}_{\tau i} \lambda^{0i\,(\tau-\rho')}_{0j'} = \sum_{\tau=\rho'}^{\tau'} X^{(\tau')i'}_{(\tau)i} X^{(\tau)i}_{(\rho')j'} = \delta^{\tau'}_{\rho'} \delta^{i'}_{j'}.$$

Theorem 3: *The quantities $C^{\tau' i'}_{\rho' j'}$ are invariant under transformations of holonomic coordinate systems of the base space $K^{(M)}_N$.*

Proof. Consider a holonomic coordinate transformation of expoints (1.1), then the relations between these quantities $\lambda^{\tau' i'}_{\tau i}$ and $\lambda^{\tau' i'}_{\alpha a}$, and between $\lambda^{ci}_{0j'}$ and $\lambda^{ca}_{0j'}$, can be put in the forms $\lambda^{\gamma' i'}_{\gamma\ i} = \sum_{a=\gamma}^{\gamma'} X^{(a)a}_{(\gamma)i} \lambda^{\gamma' i'}_{a\ a}$ and $\lambda^{0i}_{0j'} = X^{i}_{a} \lambda^{ca}_{0j'}$ respectively. Consequently, we have

$$\begin{aligned} C^{\gamma' i'}_{\rho' j'} &= \sum_{\gamma=\rho'}^{\gamma'} \binom{\gamma}{\rho'} \lambda^{\gamma' i'}_{\gamma\ i} \lambda^{0i(\gamma-\rho')}_{0j'} \\ &= \sum_{\gamma=\rho'}^{\gamma'} \sum_{a=\gamma}^{\gamma'} \sum_{\delta'=0}^{\gamma-\rho'} \binom{\gamma}{\rho'} \binom{\gamma-\rho'}{\delta'} X^{(a)a}_{(\gamma)i} X^{i(\gamma-\rho'-\delta')}_{b} \lambda^{\gamma' i'}_{a\ a} \lambda^{0b(\delta')}_{0j'} \\ &= \sum_{\delta'=0}^{\gamma'-\rho'} \sum_{a=\delta'+\rho'}^{\gamma'} \sum_{\gamma=\delta'+\rho'}^{a} \binom{\rho'+\delta'}{\rho'} X^{(a)a}_{(\gamma)i} X^{(\gamma)i}_{(\rho'+\delta')b} \lambda^{\gamma' i'}_{a\ a} \lambda^{0b(\delta')}_{0j'} \\ &= \sum_{a=\rho'}^{\gamma'} \binom{a}{\rho'} \lambda^{\gamma' i'}_{a\ a} \lambda^{0a(a-\rho')}_{0j'}\,. \end{aligned}$$

Q. E. D.

Theorem 4: *The quantities* $C^{\gamma' i'}_{\rho' j'}$ *are given by*

$$C^{\gamma' i'}_{\rho' j'} = \sum_{\beta'=\rho'}^{\gamma'} \sum_{a'=\beta'}^{\gamma'} \binom{\beta'}{\rho'} N^{\gamma' i'}_{a' a'} N^{0b'(\beta'-\rho')}_{0j'} C^{a' a'}_{\beta' b'} \tag{3. 6}$$

under non-holonomic transformations (3.1).

Proof. By virtue of the non-holonomic transformation of the non-holonomic systems: $\lambda^{\gamma' i'}_{\gamma\ i} = \sum_{a'=\gamma}^{\gamma'} N^{\gamma' i'}_{a' a'} \lambda^{a' a'}_{\gamma\ i}$ and $\lambda^{0i}_{0j'} = N^{cb'}_{0j'} \lambda^{ci}_{0b'}$, it follows that

$$\begin{aligned} C^{\gamma' i'}_{\rho' j'} &= \sum_{\gamma=\rho'}^{\gamma'} \binom{\gamma}{\rho'} \lambda^{\gamma' i'}_{\gamma\ i} \lambda^{0i(\gamma-\rho')}_{0j'} \\ &= \sum_{\gamma=\rho'}^{\gamma'} \sum_{a'=\gamma}^{\gamma'} \sum_{\delta'=0}^{\gamma-\rho'} \binom{\gamma}{\rho'} \binom{\gamma-\rho'}{\delta'} N^{\gamma' i'}_{a' a'} N^{0b'(\delta')}_{0j'} \lambda^{a' a'}_{\gamma\ i} \lambda^{ci(\gamma-\rho'-\delta')}_{0b'} \\ &= \sum_{\delta'=0}^{\gamma'-\rho'} \sum_{a'=\delta'+\rho'}^{\gamma'} \binom{\rho'+\delta'}{\rho'} N^{\gamma' i'}_{a' a'} N^{0b'(\delta')}_{0j'} \sum_{\gamma=\delta'+\rho'}^{a'} \binom{\gamma}{\delta'+\rho'} \lambda^{a' a'}_{\gamma\ i} \lambda^{0i(\gamma-\rho'-\delta')}_{cb'} \\ &= \sum_{\beta'=\rho'}^{\gamma'} \sum_{a'=\beta'}^{\gamma'} \binom{\beta'}{\rho'} N^{\gamma' i'}_{a' a'} N^{cb'(\beta'-\rho')}_{0j'} C^{a' a'}_{\beta' b'} \quad (\text{putting } \delta'+\rho'=\beta'). \end{aligned}$$

Remark. If we confine ourselves to consider functions $N^{\gamma' i'}_{a' a'}$ of ex-points with same properties as functions $X^{(a)a}_{(\gamma)i}$ induced from (1.1), i.e., $N^{\gamma' i'}_{a' a'} = \binom{\gamma'}{a'} N^{ci'(\gamma'-a')}_{0a'}$ *etc.*, then, because of

$$C^{\gamma' i'}_{\rho' j'} = \sum_{\beta'=\rho'}^{\gamma'} \sum_{a'=\beta'}^{\gamma'} N^{\gamma' i'}_{a' a'} N^{\beta' b'}_{\rho' j'} C^{a' a'}_{\beta' b'}$$

we can see that the quantities $C^{\gamma' i'}_{\rho' j'}$ $(\gamma', \rho' = 0, \cdots, G)$ are the components of a non-holonomic extensor of characteristic $(2, 0, G, M+G)$ (that is, one supercript, one subscript, weight zero, rang G and functional order $M+G$).

§4. The $\mathfrak{S}^H$-operation of an excontravariant extensors in a non-holonomic system. The $\mathfrak{S}^H$-operation of the excontravariant extensor $v^{\gamma i}$ is defined by

$$\mathfrak{S}^{H} v^{\Upsilon i} = \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} v^{\Upsilon+\lambda\, i\,(H-\lambda)}\,, \quad \begin{matrix} \Upsilon = 0, \cdots\cdots, R-H, \\ H = 0, \cdots\cdots, R \end{matrix} \quad \text{([2], p. 28).}$$

By virtue of definition (2.3), the components of the extensor $\mathfrak{S}^{H} v^{\Upsilon i}$ are given in a non-holonomic system by

$$\mathfrak{S}^{H} v^{\Upsilon' i'} = \sum_{\Upsilon=0}^{\Upsilon'} \lambda_{\Upsilon\, i}^{\Upsilon' i'}\, \mathfrak{S}^{H} v^{\Upsilon i}\,, \qquad \Upsilon' = 0, \cdots\cdots, R-H\,.$$

Differentiating the equation $v^{\Upsilon+\lambda i} = \sum_{\Upsilon'=0}^{\Upsilon+\lambda} \lambda_{\Upsilon'\, j'}^{\Upsilon+\lambda\, i}\, v^{\Upsilon' j'}$ and following the LEIBNITZ's rule for differentiation of products, we observe the equations

$$v^{\Upsilon+\lambda\, i\,(H-\lambda)} = \sum_{\Upsilon'=0}^{\Upsilon+\lambda} \sum_{\rho'=0}^{H-\lambda} \binom{H-\lambda}{\rho'} \lambda_{\Upsilon'\, j'}^{\Upsilon+\lambda\, i\,(H-\lambda-\rho')}\, v^{\Upsilon' j'(\rho')}\,.$$

Consequently, it follows that

$$\begin{aligned} \mathfrak{S}^{H} v^{\Upsilon' i'} &= \sum_{\Upsilon=0}^{\Upsilon'} \lambda_{\Upsilon\, i}^{\Upsilon' i'} \sum_{\lambda=0}^{H} (-1)^{H-\lambda} \binom{H}{\lambda} \sum_{\delta'=0}^{\Upsilon+\lambda} \sum_{\rho'=0}^{H-\lambda} \binom{H-\lambda}{\rho'} \lambda_{\delta'\, j'}^{\Upsilon+\lambda\, i\,(H-\lambda-\rho')}\, v^{\delta' j'(\rho')} \\ &= \sum_{\Upsilon=0}^{\Upsilon'} \lambda_{\Upsilon\, i}^{\Upsilon' i'} \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\Upsilon+H-\rho'} (-1)^{\rho'} \binom{H}{\rho'} v^{\delta' j'(\rho')} \\ &\qquad \times \sum_{\lambda=0}^{H-\rho'} (-1)^{H-\lambda-\rho'} \binom{H-\rho'}{\lambda} \lambda_{\delta'\, j'}^{\Upsilon+\lambda\, i\,(H-\lambda-\rho')}\,. \end{aligned}$$

As

$$\sum_{\lambda=0}^{H-\rho'} (-1)^{H-\lambda-\rho'} \binom{H-\rho'}{\lambda} \lambda_{\delta'\, j'}^{\Upsilon+\lambda\, i\,(H-\lambda-\rho')} = \mathfrak{S}^{H-\rho'} \lambda_{\delta' j'}^{\Upsilon\, i}\,,$$

we get the following results

$$\mathfrak{S}^{H} v^{\Upsilon' i'} = \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\Upsilon'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} v^{\delta' j'(H-\rho')} \sum_{\Upsilon=0}^{\Upsilon'} \lambda_{\Upsilon\, i}^{\Upsilon' i'}\, \mathfrak{S}^{\rho'} \lambda_{\delta' j'}^{\Upsilon\, i}\,.$$

By putting

$$\sum_{\Upsilon=0}^{\Upsilon'} \lambda_{\Upsilon\, i}^{\Upsilon' i'}\, \mathfrak{S}^{\rho'} \lambda_{\delta' j'}^{\Upsilon\, i} \equiv C^{\rho'}{}_{\delta' j'}^{\Upsilon' i'}(\mathfrak{S})\,, \tag{4.2}$$

we have

$$\mathfrak{S}^{H} v^{\Upsilon' i'} = \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\Upsilon'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} v^{\delta' j'(H-\rho')}\, C^{\rho'}{}_{\delta' j'}^{\Upsilon' i'}(\mathfrak{S})\,, \quad \begin{matrix} \Upsilon' = 0, \cdots, R-H, \\ H = 0, \cdots, R\,. \end{matrix} \tag{4.3}$$

The right member of the above equation shows the structure of $\mathfrak{S}^{H} v^{\Upsilon' i'}$, where the quantities $C^{\rho'}{}_{\delta' j'}^{\Upsilon' i'}(\mathfrak{S})$ are called as *$\mathfrak{S}$–operation coefficients of excontravariant extensors* in the non-holonomic system. Hence we shall get

Theorem 5: *If $v^{\Upsilon' i'}$ are components in the non-holonomic system of an excontravariant extensor of characteristic (1, 0, R, M), while the necessary derivatives exist, then the quantities*

$$\mathfrak{S}^{H} v^{\Upsilon' i'} = \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\Upsilon'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} v^{\delta' j'(H-\rho')}\, C^{\rho'}{}_{\delta' j'}^{\Upsilon' i'}(\mathfrak{S})\,, \quad \begin{matrix} \Upsilon' = 0, \cdots, R-H, \\ H = 0, \cdots, R \end{matrix}$$

are the components of an excontravariant extensor of characteristic $(1,\ 0,\ R-H,\ M+H(\leqq P))$.

$C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S})$ has the properties stated in following theorems 6–8.

Theorem 6: *If our system is a holonomic system, then it follows that* $C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S}) = \delta^{\gamma'}_{\delta'-\rho'}\delta^{i'}_{j'}$.

Proof. In a holonomic system, the quantities $\lambda^{\gamma' i'}_{\gamma\ i}$ must be indicated by $X^{(\gamma')i'}_{(\gamma)i}$, accordingly (4.2) becomes as follows:

$$C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S}) = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{S}^{\rho'} \lambda^{\gamma\ i}_{\delta' j'} \tag{4. 4}$$

$$= \sum_{\gamma=0}^{\gamma'} X^{(\gamma')i'}_{(\gamma)i} \sum_{\lambda=0}^{\rho'} (-1)^{\rho'-\lambda} \binom{\rho'}{\lambda} \binom{\gamma+\lambda}{\delta'} X^{i\,(\rho'+\gamma-\delta')}_{j'} .$$

Also, by using the fact that

$$\sum_{\lambda=0}^{\rho'} (-1)^{\rho'-\lambda} \binom{\rho'}{\lambda} \binom{\gamma+\lambda}{\delta'} = \sum_{\mu=0}^{\rho'} (-1)^{\mu} \binom{\rho'}{\mu} \binom{\lambda+\rho'-\mu}{\delta'} \quad \text{(putting } \rho'-\lambda=\mu)$$

$$= 0 \qquad \text{for} \quad \delta' < \rho'$$

$$= \binom{\gamma}{\delta'-\rho'} \qquad \text{for} \quad \delta' > \rho' ,$$

then (4.4) becomes the following results: $C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S}) = \delta^{\gamma'}_{\delta'-\rho'}\delta^{i'}_{j'}$. Q.E.D. .

Theorem 7: *The quantities* $C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S})$ *are invariant under transformations of holonomic coordinate systems of the base space* $K^{(M)}_N$.

Proof. For a holonomic coordinate transformation of expoint (1.1), the relations between the quantities $\lambda^{\gamma' i'}_{\gamma\ i}$ and $\lambda^{\gamma' i'}_{\alpha\ a}$, and between $\mathfrak{S}^{\rho'}\lambda^{\gamma\ i}_{\delta' j'}$ and $\mathfrak{S}^{\rho'}\lambda^{\alpha\ a}_{\delta' j'}$, are written in the forms $\lambda^{\gamma' i'}_{\gamma\ i} = \sum_{\alpha=\gamma}^{\gamma'} X^{(\alpha)a}_{(\gamma)i} \lambda^{\gamma' i'}_{\alpha\ a}$ and $\mathfrak{S}^{\rho'}\lambda^{\gamma\ i}_{\delta' j'} = \sum_{\beta=0}^{\gamma} X^{(\gamma)i}_{(\beta)b} \mathfrak{S}^{\rho'} \lambda^{\beta\ b}_{\delta' j'}$ respectively, consequently we have

$$C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S}) = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{S}^{\rho'} \lambda^{\gamma\ i}_{\delta' j'} = \sum_{\gamma=0}^{\gamma'} \sum_{\alpha=\gamma}^{\gamma'} \sum_{\beta=0}^{\gamma} X^{(\alpha)a}_{(\gamma)i} X^{(\gamma)i}_{(\beta)b} \lambda^{\gamma' i'}_{\alpha\ a} \mathfrak{S}^{\rho'} \lambda^{\beta\ b}_{\delta' j'}$$

$$= \sum_{\alpha=0}^{\gamma'} \sum_{\beta=0}^{\alpha} \sum_{\gamma=\beta}^{\alpha} X^{(\alpha)a}_{(\gamma)i} X^{(\gamma)i}_{(\beta)b} \lambda^{\gamma' i'}_{\alpha\ a} \mathfrak{S}^{\rho'} \lambda^{\beta\ b}_{\delta' j'} = \sum_{\alpha=0}^{\gamma'} \lambda^{\gamma' i'}_{\alpha\ a} \mathfrak{S}^{\rho'} \lambda^{\alpha\ a}_{\delta' j'}. \quad \text{Q.E.D.}$$

Theorem 8: *The quantities* $C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S})$ *are given by*

$$C^{\rho'\gamma' i'}_{\delta' j'}(\mathfrak{S}) = \sum_{\alpha'=0}^{\gamma'} \sum_{\theta'=0}^{\rho'} \sum_{\beta'=\delta'}^{\alpha'-\rho'+\theta'} (-1)^{\theta'} \binom{\rho'}{\theta'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b'(\theta')}_{\delta' j'} C^{\rho'\theta' \alpha' a'}_{\beta' b'}(\mathfrak{S})$$

under non-holonomic transformations (3.1).

Proof. By virtue of the non-holonomic transformation $\lambda^{\gamma' i'}_{\gamma\ i} = \sum_{\alpha'=\gamma}^{\gamma'} N^{\gamma' i'}_{\alpha' a'} \lambda^{\alpha' a'}_{\gamma\ i}$ and $\lambda^{\gamma+\lambda\ i}_{\delta' j'} = \sum_{\beta'=\delta'}^{\gamma+\lambda} N^{\beta' b'}_{\delta' j'} \lambda^{\gamma+\lambda\ i}_{\beta' b'}$, we get as follows:

$$\mathfrak{S}^{\rho'} \lambda^{\gamma\ i}_{\delta' j'} = \sum_{\lambda=0}^{\rho'} (-1)^{\rho'-\lambda} \binom{\rho'}{\lambda} \lambda^{\gamma+\lambda\ i\,(\rho'-\lambda)}_{\delta'\ j'}$$

$$= \sum_{\lambda=0}^{\rho'} \sum_{\beta'=\delta'}^{\gamma+\lambda} \sum_{\mu=0}^{\rho'-\lambda} (-1)^{\rho'-\lambda} \binom{\rho'}{\lambda}\binom{\rho'-\lambda}{\mu} N^{\beta'b'(\mu)}_{\delta'j'} \lambda^{\gamma+\lambda}_{\beta'}{}^{i}_{b'}{}^{(\rho'-\lambda-\mu)}$$

$$= \sum_{\mu=0}^{\rho'} \sum_{\beta'=\delta'}^{G} \sum_{\lambda=0}^{\rho'-\mu} (-1)^{\rho'-\lambda-\mu} \binom{\rho'-\mu}{\lambda} \lambda^{\gamma+\lambda}_{\beta'}{}^{i}_{b'}{}^{(\rho'-\lambda-\mu)} (-1)^{\mu} \binom{\rho'}{\mu} N^{\beta'b'(\mu)}_{\delta'j'},$$

because $\sum_{\beta'=\delta'}^{\gamma+\lambda} N^{\beta'b'(\mu)}_{\delta'j'} \lambda^{\gamma+\lambda}_{\beta'}{}^{i}_{b'}{}^{(\rho'-\lambda-\mu)} = \sum_{\beta'=\delta'}^{G} N^{\beta'b'(\mu)}_{\delta'j'} \lambda^{\gamma+\lambda}_{\beta'}{}^{i}_{b'}{}^{(\rho'-\lambda-\mu)}$ by $\lambda^{\gamma+\lambda}_{\beta'}{}^{i}_{b'} = 0$ for $\beta' > \gamma + \lambda$. Consequently, we have

$$\mathfrak{S}^{\rho'} \lambda^{\gamma}_{\delta'}{}^{i}_{j'} = \sum_{\mu=0}^{\rho'} \sum_{\beta'=\delta'}^{G} (-1)^{\mu} \binom{\rho'}{\mu} N^{\beta'b'(\mu)}_{\delta'j'} \mathfrak{S}^{\rho'-\mu} \lambda^{\gamma}_{\beta'}{}^{i}_{b'} ,$$

hence it follows that

$$C^{\rho'\gamma'i'}_{\delta'j'}(\mathfrak{S}) = \sum_{\alpha'=0}^{\gamma'} \sum_{\mu=0}^{\rho'} \sum_{\beta'=\delta'}^{G} (-1)^{\mu} \binom{\rho'}{\mu} N^{\gamma'i'}_{\alpha'a'} N^{\beta'b'(\mu)}_{\delta'j'} \sum_{\gamma=0}^{\alpha'} \lambda^{\alpha'a'}_{\gamma\, i} \mathfrak{S}^{\rho'-\mu} \lambda^{\gamma}_{\beta'}{}^{i}_{b'}$$

$$= \sum_{\alpha'=0}^{\gamma'} \sum_{\theta'=0}^{\alpha'} \sum_{\beta'=\delta'}^{\alpha'-\rho'+\theta'} (-1)^{\theta'} \binom{\rho'}{\theta'} N^{\gamma'i'}_{\alpha'a'} N^{\beta'b'(\theta')}_{\delta'j'} C^{\rho'-\theta'\alpha'a'}_{\beta'b'} .$$

Theorem 9: *For non-holonomic components of an excontravariant extensor $v^{\gamma'i'}$ of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ and a scalar f of weight $\theta_1 + \cdots\cdots + \theta_R$, the quantities $\mathfrak{S}^{*H} v^{\gamma'i'} \equiv f \mathfrak{S}^{H}(v^{\gamma'i'}/f)$,*

$$(4.5) \qquad \mathfrak{S}^{*H} v^{\gamma'i'} = \mathfrak{S}^{H} v^{\gamma'i'} + \sum_{\mu'=1}^{H} \sum_{\lambda'=0}^{H-\mu'} (-1)^{\lambda'+\mu'} \binom{H-\mu'}{\lambda'} \binom{H}{\mu'} \left(\frac{1}{f}\right)^{(\mu')} f^{(\lambda')} \times \mathfrak{S}^{*H-\mu'-\lambda'} v^{\gamma'i'}$$

are the components of an excontravariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R-H, M + H (\leqq P))$.

Proof. Since $v^{\gamma'i'}/f$ are non-holonomic components of an excontravariant extensor of characteristic $(1, 0, R, M)$, it follows that

$$\mathfrak{S}^{H}(v^{\gamma'i'}/f) = \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\gamma'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} (v^{\delta'j'}/f)^{(H-\rho')} C^{\rho'\gamma'i'}_{\delta'j'}(\mathfrak{S})$$

$$= \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\gamma'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} \sum_{\mu'=0}^{H-\rho'} \binom{H-\rho'}{\mu'} \left(\frac{1}{f}\right)^{(\mu')} v^{\delta'j'(H-\rho'-\mu')} C^{\rho'\gamma'i'}_{\delta'j'}(\mathfrak{S})$$

and using the fact that

$$\sum_{\rho'=0}^{H} \sum_{\delta'=0}^{\gamma'+\rho'} (-1)^{H-\rho'} \binom{H}{\rho'} v^{\delta'j'(H-\rho')} C^{\rho'\gamma'i'}_{\delta'j'}(\mathfrak{S}) \equiv \mathfrak{S}^{H} v^{\gamma'i'} ,$$

$$\binom{H}{\rho'}\binom{H-\rho'}{\mu'} = \binom{H-\mu'}{\rho'}\binom{H}{\mu'}, \text{ and } \binom{a}{b} = 0 \text{ if } a < b,$$

then

$$\mathfrak{S}^{*H} v^{\gamma'i'} = \mathfrak{S}^{H} v^{\gamma'i'} - f \sum_{\mu'=1}^{H} \sum_{\rho'=0}^{H-\mu'} \sum_{\delta'=0}^{\gamma'+\rho'} (-1)^{H-\rho'} \binom{H-\mu'}{\rho'} \binom{H}{\mu'} \left(\frac{1}{f}\right)^{(\mu')} v^{\delta'j'(H-\rho'-\mu')} \times C^{\rho'\gamma'i'}_{\delta'j'}(\mathfrak{S}) .$$

Replacing the term $v^{\delta'j'(H-\rho'-\mu')}$ in the right member of the last equation with $(v^{\delta'j'}/f \times f)^{(H-\rho'-\mu')}$, we shall obtain (4.5). In fact,

$$\mathfrak{S}^{*H}v^{\gamma'\iota'} = \mathfrak{S}^{H}v^{\gamma'\iota'} - f\sum_{\mu'=1}^{H}\sum_{\lambda'=0}^{H-\mu'}(-1)^{\lambda'+\mu'}\binom{H-\mu'}{\lambda'}\binom{H}{\mu'}\left(\frac{1}{f}\right)^{(\mu')}f^{(\lambda')}$$
$$\times\sum_{\rho'=0}^{H-\lambda'-\mu'}\sum_{\delta'=0}^{\gamma'+\rho'}(-1)^{H-\mu'-\lambda'-\rho'}\binom{H-\mu'-\lambda'}{\rho'}\left(\frac{v^{\delta'\jmath'}}{f}\right)^{(H-\rho'-\mu'-\lambda')}C^{\rho'\gamma'\iota'}_{\delta'\jmath'}(\mathfrak{S})$$
$$= \mathfrak{S}^{H}v^{\gamma'\iota'} + \sum_{\mu'=1}^{H}\sum_{\lambda'=0}^{H-\mu'}(-1)^{\lambda'+\mu'}\binom{H-\mu'}{\lambda'}\binom{H}{\mu'}\left(\frac{1}{f}\right)^{(\mu')}f^{(\lambda')}\mathfrak{S}^{*H-\mu'-\lambda'}v^{\gamma'\iota'},$$

where we have used $\binom{H-\mu'}{\rho'}\binom{H-\rho'-\mu'}{\lambda'} = \binom{H-\mu'-\lambda'}{\rho'}\binom{H-\mu'}{\lambda'}$.

Theorem 10: *The $\mathfrak{S}^H$–operation of an excontravariant extensor in a non-holonomic system holds the commutative and linear associative law, i. e., for two positive integers H and H′ we have the following results:*

$$\text{(4. 6)}\qquad \begin{aligned}\mathfrak{S}^{H'}\mathfrak{S}^{H}v^{\gamma'\iota'} &= \mathfrak{S}^{H}\mathfrak{S}^{H'}v^{\gamma'\iota'} = \mathfrak{S}^{H+H'}v^{\gamma'\iota'}\ ,\\ \mathfrak{S}^{H}(v^{\gamma'\iota'} + u^{\gamma'\iota'}) &= \mathfrak{S}^{H}v^{\gamma'\iota'} + \mathfrak{S}^{H}u^{\gamma'\iota'}\ .\end{aligned}$$

Proof. By virtue of the definition (2.3), we see

$$\begin{aligned}\mathfrak{S}^{H+H'}v^{\gamma'\iota'} &= \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H+H'}v^{\gamma\iota} && \text{for}\ \ \gamma = 0,\cdots\cdots, R-(H+H'),\\ &= \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H'}(\mathfrak{S}^{H}v^{\gamma\iota}) && \text{for}\ \ \gamma = 0,\cdots\cdots,(R-H)-H'\ ,\\ &= \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H}(\mathfrak{S}^{H'}v^{\gamma\iota}) && \text{for}\ \ \gamma = 0,\cdots\cdots,(R-H')-H\ .\end{aligned}$$

According to the fact that $\mathfrak{S}^{H'}\mathfrak{S}^{H}v^{\gamma\iota} = \mathfrak{S}^{H}\mathfrak{S}^{H'}v^{\gamma\iota} = \mathfrak{S}^{H+H'}v^{\gamma\iota}$ in the holonomic system ([2], p. 33), we obtain the relationship $\mathfrak{S}^{H'}\mathfrak{S}^{H}v^{\gamma'\iota'} = \mathfrak{S}^{H}\mathfrak{S}^{H'}\times v^{\gamma'\iota'} = \mathfrak{S}^{H+H'}v^{\gamma'\iota'}$. Further, using that $\mathfrak{S}^{H}(v^{\gamma\iota} + u^{\gamma\iota}) = \mathfrak{S}^{H}v^{\gamma\iota} + \mathfrak{S}^{H}u^{\gamma\iota}$ in the holonomic system ([2], p. 33), we can go on as follows:

$$\begin{aligned}\mathfrak{S}^{H}(v^{\gamma'\iota'} + u^{\gamma'\iota'}) &= \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H}(v^{\gamma\iota} + u^{\gamma\iota}) = \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H}v^{\gamma\iota} + \lambda^{\gamma'\iota'}_{\gamma\ \iota}\,\mathfrak{S}^{H}u^{\gamma\iota}\\ &= \mathfrak{S}^{H}v^{\gamma'\iota'} + \mathfrak{S}^{H}u^{\gamma'\iota'}\ .\end{aligned}$$

Q. E. D.

§ 5. The $\mathfrak{S}^H$–operation of an excovariant extensors in a non-holonomic system.

Under the $\mathfrak{S}^H$–operation of the excovariant extensor $w_{\gamma\iota}$ of characteristic $(1,\ 0,\ R,\ M)$ we understand

$$\text{(5. 1)}\qquad \mathfrak{S}^{H}w_{\gamma\iota} = H!\sum_{\nu=0}^{H}(-1)^{\nu}\binom{\gamma+\nu}{\nu}\binom{R-\gamma-\nu}{R-H-\gamma}w_{\gamma+\nu\,\iota}{}^{(\nu)},\quad H = 0,\cdots\cdots, R$$

([2], p. 31).

The non-holonomic components of this extensor $\mathfrak{S}^{H}w_{\gamma\iota}$ will be expressed in terms of the non-holonomic quantities in the following way.

By using $w_{\gamma+\nu\,\iota}{}^{(\nu)} = \sum_{\delta'=\gamma+\nu}^{R}\sum_{\rho=0}^{\nu}\binom{\nu}{\rho}\lambda_{\gamma+\nu\ \iota}^{\ \delta'\jmath'(\nu-\rho)}w_{\delta'\jmath'}{}^{(\rho)}$ which follows from $w_{\gamma+\nu\,\iota} = \sum_{\delta'=\gamma+\nu}^{R}\lambda_{\gamma+\nu\ \iota}^{\ \delta'\jmath'}w_{\delta'\jmath'}$, the equation of definition $\mathfrak{S}^{H}w_{\gamma'\iota'} = \sum_{\gamma=\gamma'}^{R-H}\lambda^{\gamma\ \iota}_{\gamma'\iota'}\,\mathfrak{S}^{H}\times w_{\gamma\iota}$ $(\gamma' = 0,\cdots\cdots,\ R-H)$ leads us, by virtue of (5.1), to

$$\mathfrak{S}^{H}w_{\gamma'\iota'} = \sum_{\gamma=\gamma'}^{R-H}\lambda^{\gamma\ \iota}_{\gamma'\iota'}\left\{H!\sum_{\nu=0}^{H}(-1)^{\nu}\binom{\gamma+\nu}{\gamma}\binom{R-\gamma-\nu}{R-H-\gamma}w_{\gamma+\nu\,\iota}{}^{(\nu)}\right\}$$

$$= \sum_{\gamma=\gamma'}^{R-H} \lambda_{\gamma' i'}^{\gamma\ i} \{ H! \sum_{\nu=0}^{H} \sum_{\delta'=0}^{R} \sum_{\rho=0}^{\nu} (-1)^{\nu} \binom{\gamma+\nu}{\gamma+\rho} \binom{\gamma+\rho}{\rho} \binom{R-\gamma-\nu}{R-H-\gamma} \lambda_{\gamma+\nu\, i}^{\delta'\ j'(\nu-\rho)} w_{\delta' j'}^{(\rho)} \}$$

$$= \sum_{\gamma=\gamma'}^{R-H} \lambda_{\gamma' i'}^{\gamma\ i} \{ \frac{H!}{(H-\rho)!} \sum_{\rho=0}^{H} \sum_{\delta'=0}^{R} (-1)^{\rho} \binom{\gamma+\rho}{\rho} w_{\delta' j'}^{(\rho)} (H-\rho)!$$

$$\times \sum_{\lambda=0}^{H-\rho} (-1)^{\lambda} \binom{\gamma+\rho+\lambda}{\gamma+\rho} \binom{R-(\gamma+\rho)-\lambda}{R-(H-\rho)-(\gamma+\rho)} \lambda_{\gamma+\rho+\lambda\, i}^{\delta'\ j'(\lambda)} \} \text{ (putting } \nu-\rho=\lambda).$$

Since $\lambda_{\gamma\ i}^{\gamma' i'}$ in the last member may be considered to be of range R, we can put

$$(5.\ 2) \qquad (H-\rho)! \sum_{\lambda=0}^{H-\rho} (-1)^{\lambda} \binom{\gamma+\rho+\lambda}{\gamma+\rho} \binom{R-(\gamma+\rho)-\lambda}{R-(H-\rho)-(\gamma+\rho)} \lambda_{\gamma+\rho+\lambda\, i}^{\delta'\ j'(\lambda)} = \mathfrak{S}^{H-\rho} \lambda_{\gamma+\rho\, i}^{\delta'\ j'}$$

following the definition of the $\mathfrak{S}^{H}$*-operation*. Then it follows that

$$(5.\ 3) \qquad \mathfrak{S}^{H} w_{\gamma' i'} = \sum_{\rho=0}^{H} \sum_{\delta'=0}^{R} \frac{H!}{(H-\rho)!} (-1)^{\rho} w_{\delta' j'}^{(\rho)} \sum_{\gamma=\gamma'}^{R-H} \binom{\gamma+\rho}{\rho} \lambda_{\gamma' i'}^{\gamma\ i} \mathfrak{S}^{H-\rho} \lambda_{\gamma+\rho\, i}^{\delta'\ j'}$$

$$= \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{R} H! \, (-1)^{\rho'} \binom{\delta'}{\rho'} \binom{R-\delta'}{H-\rho'} w_{\delta' j'}^{(\rho')} C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}) ,$$

$$\gamma' = 0, \cdots\cdots, R-H; \quad H = 0, \cdots\cdots, R,$$

putting

$$(5.\ 4) \qquad \{(H-\rho)! \binom{\delta'}{\rho'} \binom{R-\delta'}{H-\rho'}\}^{-1} \sum_{\gamma=\gamma'}^{R-H} \binom{\gamma+\rho'}{\rho'} \lambda_{\gamma' i'}^{\gamma\ i} \mathfrak{S}^{H-\rho'} \lambda_{\gamma+\rho'\, i}^{\delta'\ j'} = C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}^{H}) .$$

(5. 3) shows the structure of $\mathfrak{S}^{H} w_{\gamma' i'}$ and $C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}^{H})$ are called $\mathfrak{S}^{H}$–*operation coefficients of excovariant extensors having range* R in the non-holonomic system. From this we have

Theorem 11: *In the non-holonomic system the quantities*

$$\mathfrak{S}^{H} w_{\gamma' i'} = \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{R} H! \, (-1)^{\rho'} \binom{\delta'}{\rho'} \binom{R-\delta'}{H-\rho'} w_{\delta' j'}^{(\rho')} C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}^{H}) \quad \begin{matrix} \gamma' = 0, \cdots\cdots, R-H, \\ H = 0, \cdots\cdots, R \end{matrix}$$

are the components of an excovariant extensor of characteristic $(1,\ 0,\ R-H,\ M+H\ (\leqq P))$, *while* $w_{\gamma' i'}$ *are components of an excovariant extensor of characteristic* $(1,\ 0,\ R,\ M)$.

On the coefficients $C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}^{H})$, we can state Theorems 12–14.

Theorem 12: *For a holonomic system it follows that* $C^{\rho' \delta' j'}_{\gamma' i'} (\mathfrak{S}^{H}) = \delta_{\gamma'}^{\delta' - \rho'} \delta_{i'}^{j'}$.

Proof. In a holonomic system the quantities $\lambda_{\gamma' i'}^{\gamma\ i}$ must be $X_{(\gamma') i'}^{(\gamma)\ i}$. Hence it follows that

$$\mathfrak{S}^{H-\rho'} \lambda_{\gamma+\rho'\, i}^{\delta'\ j'} = (H-\rho')! \sum_{\lambda=0}^{H-\rho'} (-1)^{\lambda} \binom{\delta'}{\gamma+\rho'} \binom{R-(\gamma+\rho')-\lambda}{R-(H-\rho')-(\gamma+\rho')} \binom{\delta'-\gamma-\rho'}{\lambda} X_{i}^{j'(\delta'-\gamma-\rho')}.$$

By reason of

$$\binom{\delta'}{\gamma+\rho'} \sum_{\lambda=0}^{H-\rho'} (-1)^{\lambda} \binom{R-(\gamma+\rho')-\lambda}{R-(H-\rho')-(\gamma+\rho')} \binom{\delta'-\gamma-\rho'}{\lambda} = \binom{\delta'}{\gamma+\rho'} \binom{R-\gamma-\rho'}{H-\rho'} \binom{R-\gamma-\rho'}{R-\delta'}^{-1}$$

$$\times \sum_{\lambda=0}^{H-\rho'} (-1)^{\lambda} \binom{H-\rho'}{\lambda} \binom{R-\gamma-\rho'-\lambda}{R-\delta'} ,$$

$$\begin{aligned}\sum_{\lambda=0}^{H-\rho'} (-1)^{\lambda} \binom{H-\rho'}{\lambda} \binom{R-\gamma-\rho'-\lambda}{R-\delta'} &= \sum_{\lambda=0}^{H-\rho'} (-1)^{\lambda} \binom{H-\rho'}{\lambda} \binom{R-\gamma-H+H-\rho'-\lambda}{R-\delta'} \\ &= 0 \qquad \text{for} \quad R-H+\rho' < \delta' \\ &= \binom{R-\gamma-H}{R-H+\rho'-\delta'} \text{ for } \quad R-H+\rho' \geqq \delta' ,\end{aligned}$$

we have the following results

$$\mathfrak{S}^{H-\rho'} \lambda_{\gamma+\rho'\, i}^{\delta'\, j'} = (H-\rho')!\, \binom{\delta'}{\gamma+\rho'} \binom{R-\delta'}{H-\rho'} X_{i}^{j'(\delta'-\gamma-\rho')} .$$

Hence (5.4) goes into

$$\begin{aligned} C^{\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H}) &= \frac{1}{K} \sum_{\gamma=\gamma'}^{R-H} \binom{\gamma+\rho'}{\rho'} (H-\rho')!\, \binom{\delta'}{\gamma+\rho'} \binom{R-\delta'}{H-\rho'} X^{\gamma\, i}_{\gamma' i'} X_{i}^{j'(\delta'-\gamma-\rho')} \\ &= \sum_{\gamma=\gamma'}^{R-H} \binom{\delta'-\rho'}{\gamma} X^{\gamma\, i}_{\gamma' i'} X_{i}^{j'(\delta'-\rho'-\gamma)} = \sum_{\gamma=\gamma'}^{R-H} X^{\gamma\, i}_{\gamma' i'} X^{\delta'-\rho' j'}_{\gamma\, i} = \delta^{\delta'-\rho'}_{\gamma'} \delta^{j'}_{i'} ,\end{aligned}$$

putting $K \equiv (H-\rho')!\, \binom{\delta'}{\rho'} \binom{R-\delta'}{H-\rho'}$.

Theorem 13: *The quantities $C^{\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H})$ are invariant under transformations of holonomic coordinate systems of the base space $K_N^{(M)}$.*

The method of proof is similar as that of Theorem 7.

Theorem 14: *The quantities $C^{*\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H}) \equiv \binom{\delta'}{\rho'} \binom{R-\delta'}{H-\rho'} C^{\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H})$ are given by*

$$C^{*\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H}) = \sum_{\alpha'=\gamma'}^{R-H} \sum_{\beta'=0}^{\delta'} \sum_{\mu'=0}^{H-\rho'} (-1)^{\mu'} \binom{\rho'+\mu'}{\rho'} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu')}_{\beta' b'} C^{*\rho'+\mu'\, \beta' b'}_{\alpha' a'}(\mathfrak{S}^{H})$$

under non-holonomic transformations of non-holonomic systems (3.1).

Proof. By virtue of the non-holonomic transformation of the non-holonomic systems, we have

$$\begin{aligned}\mathfrak{S}^{H-\rho'} \lambda_{\gamma+\rho'\, i}^{\delta'\, j'} = (H-\rho')!\sum_{\nu'=0}^{H-\rho'} \sum_{\beta'=\gamma+\rho'+\nu'}^{\delta'} \sum_{\mu'=0}^{\nu'} (-1)^{\nu'} \binom{\gamma+\rho'+\nu'}{\gamma+\rho'} \\ \times \binom{R-(\gamma+\rho')-\nu'}{R-(H-\rho')-(\gamma+\rho')} \binom{\nu'}{\mu'} N^{\delta' j'(\mu')}_{\beta' b'} \lambda_{\gamma+\rho'+\nu'\, i'}^{\beta'\quad b'(\nu'-\mu')} ,\end{aligned}$$

consequently it follows that

$$\begin{aligned} C^{*\rho'\delta' j'}_{\gamma' i'}(\mathfrak{S}^{H}) &= \frac{1}{(H-\rho)!} \sum_{\gamma=\gamma'}^{R-H} \binom{\gamma+\rho'}{\rho'} \lambda^{\gamma\, i}_{\gamma' i'} \mathfrak{S}^{H-\rho} \lambda_{\gamma+\rho\, i}^{\delta'\, j'} \\ &= \sum_{\gamma=\gamma'}^{R-H} \sum_{\alpha'=\gamma'}^{\gamma} \binom{\gamma+\rho'}{\rho'} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu')}_{\beta' b'} \lambda^{\gamma\, i}_{\alpha' a'} (-1)^{\mu'} \binom{\gamma+\rho'+\mu'}{\mu'} \frac{1}{(H-\rho'-\mu')!} \\ &\quad \times \sum_{\nu'=0}^{H-\rho'} \sum_{\beta'=\gamma+\rho'+\nu'}^{\delta'} \sum_{\mu'=0}^{\nu'} (H-\rho'-\mu')!\,(-1)^{\nu'-\mu'} \binom{\gamma+\rho'+\mu'+\nu'-\mu'}{\gamma+\rho'+\mu'} \\ &\quad \times \binom{R-(\gamma+\rho'+\mu')-(\nu'-\mu')}{R-(H-\rho'-\mu')-(\gamma+\rho'+\mu')} \lambda_{\gamma+\rho'+\nu'\, i}^{\beta'\quad b'(\nu'-\mu')} .\end{aligned}$$

Since we see $\sum_{\nu'=0}^{H-\rho'} \sum_{\mu'=0}^{\nu'} = \sum_{\mu'=0}^{H-\rho'} \sum_{\nu'=\mu'}^{H-\rho'}$ and $\sum_{\beta'=\gamma+\rho'+\nu'}^{\delta'} N^{\delta' j'(\mu')}_{\beta' b'} \lambda_{\gamma+\rho'+\nu'\, i}^{\beta'\quad b'(\nu'-\mu')}$ $= \sum_{\beta'=0}^{\delta'} N^{\delta' j'(\mu')}_{\beta' b'} \lambda_{\gamma+\rho'+\nu'\, i}^{\beta'\quad b'(\nu'-\mu')}$ as $\lambda_{\gamma+\beta'+\nu'\, i}^{\beta'\quad b'(\nu'-\mu')} = 0$ for $\beta' < \gamma+\rho'+\nu'$, we have

$$C^{*\rho'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{S}^H) = \sum_{\alpha'=\gamma'}^{R-H} \sum_{\gamma=\alpha'}^{R-H} \sum_{\beta'=0}^{\delta'} \sum_{\mu'=0}^{H-\rho'} \binom{\rho'+\mu'}{\rho'} (-1)^{\mu'} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu')}_{\beta' b'} \frac{1}{(H-\rho'-\mu')!}$$
$$\times \binom{\gamma+\rho'+\mu'}{\rho'+\mu'} \lambda^{\gamma\ i}_{\alpha' a'} \mathfrak{S}^{H-\rho'-\mu'} \lambda_{\gamma+\rho'+\mu' i}^{\beta'\ b'}$$
$$= \sum_{\alpha'=\gamma'}^{R-H} \sum_{\beta'=0}^{\delta'} \sum_{\mu'=0}^{H-\rho'} \binom{\rho'+\mu'}{\rho'} (-1)^{\mu'} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu)}_{\beta' b'} C^{*\rho'+\mu'}{}^{\beta' b'}_{\alpha' a'}(\mathfrak{S}^H) .$$

Q. E. D.

Theorem 15: *If $w_{\gamma' i'}$ are non-holonomic components of an excovariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ and f is a scalar of weight $\theta_1 + \cdots\cdots + \theta_R$, then the quantities $\mathfrak{S}^{*H} w_{\gamma' i'} \equiv f \cdot \mathfrak{S}^H(w_{\gamma' i'}/f)$:*

$$\mathfrak{S}^{*H} w_{\gamma' i'} = \mathfrak{S}^H w_{\gamma' i'} + f \sum_{\rho'=0}^{H} \sum_{\delta'=0}^{R} H!\,(-1)^{\rho'} \sum_{\lambda=1}^{\rho'} \binom{\rho'}{\lambda} \left(\frac{1}{f}\right)^{(\lambda)} w_{\delta' j'}{}^{(\rho'-\lambda')}$$
$$\times\ C^{*\rho'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{S}^H)$$

are the components of an excovariant extensor of characteristic $(1, \theta_1 + \theta_2 + \cdots\cdots + \theta_R, R-H, M+H (\leqq P))$.

The theorem is proved without difficulty.

We know in the holonomic system that if $w_{\gamma i}$ is an excovariant extensor of characteristic $(1, 0, R, M)$, then $\mathfrak{S}^H w_{R-H i}$ $(H = 0, \cdots\cdots, R)$ is a vector ([2], p. 32), but this fact does not hold in the non-holonomic system. But between the quantities $\mathfrak{S}^H w_{R-H i'} = \lambda^{R-H\ i}_{R-H\ i'} \mathfrak{S}^H w_{R-H i}$ and the components $\lambda^i_{i'} \mathfrak{S}^H w_{R-H i}$ $(\equiv \tilde{\mathfrak{S}}^H w_{R-H i'})$ in the non-holonomic system of the vector $\mathfrak{S}^H w_{R-H i}$, the following relation holds good:

$$\tilde{\mathfrak{S}}^H w_{R-H i'} = \mathfrak{S}^H w_{H-R j'} C^{R-H}{}^{j'}_{i'} \tag{5. 5}$$

where $C^{R-H}{}^{j'}_{i'} = \lambda^i_{i'} \lambda^{R-H\ j'}_{R-H\ i}$. Accordingly, we have

Theorem 16: *If $w_{\gamma' i'}$ are non-holonomic components of an excovariant extensor of characteristic $(1, 0, R, M)$, then the quantities $C^{R-H}{}^{j'}_{i'} \mathfrak{S}^H w_{R-H j'}$ $(\equiv \tilde{\mathfrak{S}}^H w_{R-H i'})$ are non-holonomic components of a vector.*

For example, from a scalar function F of order M, we have the excovariant extensor $w_{\gamma i} = \dfrac{\partial F(x, \cdots, x^{(M)})}{\partial x^{(\gamma) i}}$ $(\gamma = 0, \cdots\cdots, M)$ and vectors of Synge $E^K_i = \dfrac{1}{(M-K)!}(-1)^K \mathfrak{S}^{M-K} F_{(K) i}$ $(K = 0, \cdots\cdots, M)$. In the non-holonomic system, vectors of Synge are then written in

$$\tilde{E}^K_{i'} = C^{K}{}^{j'}_{i'} \frac{1}{(M-K)!} (-1)^K \mathfrak{S}^{M-K} \tilde{F}_{K j'}$$

because of (5.5), where the quantities $\tilde{F}_{K j'}$ are the components of $F_{(\gamma) i}$ in the non-holonomic system, i. e. $\tilde{F}_{K j'} = \sum_{\gamma=K}^{M} \lambda^{\gamma\ i}_{K j'} F_{(\gamma i)}$.

Theorem 17: *If our system is holonomic, then $C^{K}{}^{j'}_{i} = \delta^{j'}_{i'}$.*

As in a holonomic system, the quantities $\lambda^{Kj'}_{Ki}$ is equal to $\partial x^{j'}/\partial x^i$, this theorem will follow.

Theorem 18: *The quantities $C^{Kj'}_{i'}$ are invariant under transformations of holonomic coordinate systems.*

Theorem 19: *The quantities $C^{Kj'}_{i'}$ are changed by*

$$C^{Kj'}_{i'} = N^{0a'}_{0i'} N^{Kj'}_{Kb'} C^{Kb'}_{a'}$$

under non-holonomic transformations of non-holonomic systems.

Proof. The non-holonomic transformation of the non-holonomic systems $\lambda^{0i}_{0i'} = N^{0a'}_{0i'} \lambda^{0i}_{0a'}$ and $\lambda^{Kj'}_{Ki} = N^{Kj'}_{Kb'} \lambda^{Kb'}_{Ki}$ shows that $C^{Kj'}_{i'} = N^{0a'}_{0i'} N^{Kj'}_{Kb'} C^{Kb'}_{a'}$.

Q. E. D.

Theorem 20: *The $\mathfrak{S}^H$-operation of an excovariant extensor in a non-holonomic system holds the commutative and linearly associative law.*

The method of proof is essentially that of Theorem 10.

§ 6. The $\mathfrak{Z}^H$-operation of an excontravariant extensors in a non-holonomic system. The quantities

$$\text{(6. 1)} \qquad \mathfrak{Z}^H v^{\gamma i} = \sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} 2^{H\lambda} (1-2^H)^{\gamma-\lambda} v^{\lambda i(\gamma-\lambda)}, \qquad H = 0, \cdots\cdots, R$$

are the components of an excontravariant extensor, being $v^{\gamma i}$ an excontravariant extensor ([2], p. 37). Put

$$v^{\lambda i(\gamma-\lambda)} = \Big(\sum_{\delta'=0}^{\lambda} \lambda^{\lambda}_{\delta'}{}^{i}_{j'} v^{\delta' j'}\Big)^{(\gamma-\lambda)} = \sum_{\delta'=0}^{\lambda} \sum_{\mu=0}^{\gamma-\lambda} \binom{\gamma-\lambda}{\mu} \lambda^{\lambda}_{\delta'}{}^{i}_{j'}{}^{(\gamma-\lambda-\mu)} v^{\delta' j'(\mu)}$$

into (6.1), then the components of the extensor $\mathfrak{Z}^H v^{\gamma' i'} = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \mathfrak{Z}^H v^{\gamma i}$ ($\gamma' = 0, \cdots\cdots, R$) in the non-holonomic system have the forms

$$\begin{aligned} \mathfrak{Z}^H v^{\gamma' i'} &= \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \sum_{\lambda=0}^{\gamma} \sum_{\delta'=0}^{\lambda} \sum_{\mu=0}^{\gamma-\lambda} 2^{H\lambda} (1-2^H)^{\gamma-\lambda} \binom{\gamma}{\lambda} \binom{\gamma-\lambda}{\mu} \lambda^{\lambda}_{\delta'}{}^{i}_{j'}{}^{(\gamma-\lambda-\mu)} v^{\delta' j'(\mu)} \\ &= \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \sum_{\mu=0}^{\gamma} \sum_{\delta'=0}^{\gamma-\mu} \sum_{\lambda=\delta'}^{\gamma-\mu} 2^{H\lambda} (1-2^H)^{\gamma-\mu-\lambda} \binom{\gamma-\mu}{\lambda} \\ &\qquad \times \lambda^{\lambda}_{\delta'}{}^{i}_{j'}{}^{(\gamma-\lambda-\mu)} (1-2^H)^{\mu} \binom{\gamma}{\mu} v^{\delta' j'(\mu)} . \end{aligned}$$

Since we see, using the fact that $\lambda^{\lambda}_{\delta'}{}^{i}_{j'} = 0$ for $\lambda < \delta'$,

$$\begin{aligned} \sum_{\lambda=\delta'}^{\gamma-\mu} 2^{H\lambda} (1-2^H)^{\gamma-\mu-\lambda} \binom{\gamma-\mu}{\lambda} \lambda^{\lambda}_{\delta'}{}^{i}_{j'}{}^{(\gamma-\lambda-\mu)} &= \sum_{\lambda=0}^{\gamma-\mu} \binom{\gamma-\mu}{\lambda} 2^{H\lambda} (1-2^H)^{\gamma-\mu-\lambda} \lambda^{\lambda}_{\delta'}{}^{i}_{j'}{}^{(\gamma-\lambda-\mu)} \\ &= \mathfrak{Z}^H \lambda^{\gamma-\mu}_{\delta'}{}^{i}_{j'} , \end{aligned}$$

the following result is obtained:

$$\begin{aligned} \mathfrak{Z}^H v^{\gamma' i'} &= \sum_{\gamma=0}^{\gamma'} \sum_{\mu=0}^{\gamma} \sum_{\delta'=0}^{\gamma-\mu} \lambda^{\gamma' i'}_{\gamma\ i} \mathfrak{Z}^H \lambda^{\gamma-\mu}_{\delta'}{}^{i}_{j'} \binom{\gamma}{\mu} (1-2^H)^{\mu} v^{\delta' j'(\mu)} \\ &= \sum_{\mu=0}^{\gamma'} \sum_{\delta'=0}^{\gamma'-\mu} \sum_{\gamma=\delta'+\mu}^{\gamma'} \binom{\gamma}{\mu} \lambda^{\gamma' i'}_{\gamma\ i} \mathfrak{Z}^H \lambda^{\gamma-\mu}_{\delta'}{}^{i}_{j'} (1-2^H)^{\mu} v^{\delta' j'(\mu)} . \end{aligned}$$

Here by putting

(6. 2) $$\{2^{H\delta'}\binom{\delta'+\mu'}{\delta'}\}^{-1}\sum_{\gamma=\delta'+\mu}^{\gamma'}\binom{\gamma}{\mu'}\lambda^{\gamma' i'}_{\gamma i}\,\mathfrak{Z}^H\lambda^{\gamma-\mu' i}_{\delta' j'} = C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)\ ,$$

we can see the relation

(6. 3) $$\mathfrak{Z}^H v^{\gamma' i'} = \sum_{\mu'=0}^{\gamma'}\sum_{\delta'=0}^{\gamma'-\mu'}\binom{\delta'+\mu'}{\delta'}2^{H\delta'}(1-2^H)^{\mu'}v^{\delta' j'(\mu')}C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)\ .$$

The quantities $C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)$ in the non-holonomic system are said the *$\mathfrak{Z}^H$-operation coefficients of excontravariant extensors* in the non-holonomic system, and we can state

Theorem 21: *Let $v^{\gamma' i'}$ be components of an excontravariant extensor of characteristic* (1, 0, R, M) *in the non-holonomic system, then the quantities*

$$\mathfrak{Z}^H v^{\gamma' i'} = \sum_{\mu'=0}^{\gamma'}\sum_{\delta'=0}^{\gamma'-\mu'}\binom{\delta'+\mu'}{\delta'}2^{H\delta'}(1-2^H)^{\mu'}v^{\delta' j'(\mu')}C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)$$

are the components of an excontravariant extensor of characteristic (1, 0, R, $M+R\leqq P$).

Theorem 22: *For a holonomic system* $C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H) = \delta^{\gamma'-\mu'}_{\delta'}\delta^{i'}_{j'}$.

Proof. Since in a holonomic system the quantities $\lambda^{\gamma' i'}_{\gamma i}$ are equal to $X^{(\gamma')i'}_{(\gamma)i}$, (6.2) may be written in the form

$$\begin{aligned} KC^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H) &= \sum_{\gamma=\delta'+\mu'}^{\gamma'}\binom{\gamma}{\mu'}\lambda^{\gamma' i'}_{\gamma i}\,\mathfrak{Z}^H\lambda^{\gamma-\mu' i}_{\delta' j'} \\ &= \sum_{\gamma=\delta'+\mu'}^{\gamma'}\sum_{\lambda=\delta'}^{\gamma-\mu'} X^{(\gamma')i'}_{(\gamma)i}\binom{\gamma}{\mu'}\binom{\gamma-\mu'}{\lambda}\binom{\lambda}{\delta'}2^{H\lambda}(1-2^H)^{\gamma-\mu'-\lambda}X^{i}_{j'}{}^{(\gamma-\mu'-\delta')} \\ &= \sum_{\gamma=\delta'+\mu'}^{\gamma'}\sum_{\lambda=\delta'}^{\gamma-\mu'} X^{(\gamma')i'}_{(\gamma)i}\binom{\gamma}{\mu'+\delta'}\binom{\mu'+\delta'}{\delta'}\binom{\gamma-\mu'-\delta'}{\lambda-\delta'}2^{H\lambda}(1-2^H)^{\gamma-\mu'-\lambda}X^{i}_{j'}{}^{(\gamma-\mu'-\delta')} \\ &= \sum_{\gamma=\delta'+\mu'}^{\gamma'}\binom{\mu'+\delta'}{\delta'}X^{\gamma' i'}_{\gamma i}X_{\mu'+\delta' j'}^{\gamma\ i}\sum_{\lambda=\delta'}^{\gamma-\mu'}\binom{\gamma-\mu'-\delta'}{\lambda-\delta'}2^{H\lambda}(1-2^H)^{\gamma-\mu'-\lambda}\ , \end{aligned}$$

putting $K=\binom{\delta'+\mu'}{\delta'}2^{H\delta'}_{-}$. Furthermore applying the relation $\sum_{\lambda=\delta'}^{\gamma-\mu'}\binom{\gamma-\mu'-\delta'}{\lambda-\delta'}\times 2^{H\lambda}(1-2^H)^{\gamma-\mu'-\lambda} = 2^{H\delta'}$,

$$C^{\mu'\gamma' i'}_{\delta' j'} = \frac{1}{K}\sum_{\gamma=\delta'+\mu'}^{\gamma'}\binom{\mu'+\delta'}{\delta'}X^{\gamma' i'}_{\gamma i}X_{\mu'+\delta' j'}^{\gamma\ i}2^{H\delta'} = \delta_{\mu'+\delta'}^{\gamma'}\delta^{i'}_{j'}. \qquad \text{Q. E. D.}$$

Theorem 23: *The quantities $C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)$ are invariant under transformations of holonomic coordinate systems.*

Theorem 24: *The quantities $C^{*\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)\equiv 2^{H\delta'}\binom{\delta'+\mu'}{\delta'}C^{\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H)$ are changed in the form*

$$C^{*\mu'\gamma' i'}_{\delta' j'}(\mathfrak{Z}^H) = \sum_{\alpha'=\mu'+\delta'}^{\gamma'}\sum_{\rho'=0}^{\alpha'-\delta'-\mu'}\sum_{\beta'=\delta'}^{\alpha'-\rho'-\mu'}\binom{\mu'+\rho'}{\rho'}(1-2^H)^{\rho'}N^{\gamma' i'}_{\alpha' a'}N^{\beta' b'(\rho')}_{\delta' j'} \times C^{*\rho'+\mu' \alpha' a'}_{\beta' b'}(\mathfrak{Z}^H)$$

under non-holonomic transformations of non-holonomic systems.

Proof. According to the non-holonomic transformation $\lambda^{\gamma' i'}_{\gamma\, i} = \sum_{\alpha'=\gamma}^{\gamma'} N^{\gamma' i'}_{\alpha' a'} \lambda^{\alpha' a'}_{\gamma\, i}$ and $\lambda^{\lambda\, i}_{\delta' j'} = \sum_{\beta'=\delta'}^{\lambda} N^{\beta' b'}_{\delta' j'} \lambda^{\lambda\, i}_{\beta' b'}$, we have

$$
\begin{aligned}
\mathfrak{Z}^H \lambda^{\gamma-\mu'\, i}_{\delta'\; j'} &= \sum_{\lambda=\delta'}^{\gamma-\mu'} 2^{H\lambda} (1-2^H)^{\gamma-\mu'-\lambda} \binom{\gamma-\mu'}{\lambda} \lambda^{\lambda\, i\, (\gamma-\lambda-\mu')}_{\delta' j'} \\
&= \sum_{\lambda=\delta'}^{\gamma-\mu'} \sum_{\beta'=\delta'}^{\lambda} \sum_{\rho'=0}^{\gamma-\lambda-\mu'} 2^{H\lambda} (1-2^H)^{\gamma-\mu'-\lambda} \binom{\gamma-\mu'}{\lambda} \binom{\gamma-\lambda-\mu'}{\rho'} N^{\beta' b' (\rho')}_{\delta' j'} \lambda^{\lambda\, i\, (\gamma-\lambda-\mu'-\rho')}_{\beta' b'} \\
&= \sum_{\beta'=\delta'}^{\gamma-\mu'} \sum_{\rho'=0}^{\gamma-\mu'-\rho'} (1-2^H)^{\rho'} \binom{\gamma-\mu'}{\rho'} N^{\beta' b' (\rho')}_{\delta' j'} \\
&\qquad \times \sum_{\lambda=\beta'}^{\gamma-\mu'-\rho'} 2^{H\lambda} (1-2^H)^{\gamma-\mu'-\rho'-\lambda} \binom{\gamma-\mu'-\rho'}{\lambda} \lambda^{\lambda\, i\, (\gamma-\mu'-\rho'-\lambda)}_{\beta' b'} \\
&= \sum_{\beta'=\delta'}^{\gamma-\mu'} \sum_{\rho'=0}^{\gamma-\mu'-\rho'} (1-2^H)^{\rho'} \binom{\gamma-\mu'}{\rho'} N^{\beta' b' (\rho')}_{\delta' j'} \mathfrak{Z}^H \lambda^{\gamma-\mu'-\rho'\, i}_{\beta'\; b'} .
\end{aligned}
$$

Consequently, the following relations are introduced:

$$
\begin{aligned}
C^{*\mu' \gamma' i'}_{\;\;\delta' j'}(\mathfrak{Z}^H) &= \sum_{\gamma=\delta'+\mu'}^{\gamma'} \sum_{\alpha'=\gamma}^{\gamma'} \sum_{\beta'=\delta}^{\gamma-\mu'} \sum_{\rho'=0}^{\gamma-\mu'-\rho'} \binom{\gamma}{\mu'} N^{\gamma' i'}_{\alpha' a'} \lambda^{\alpha' a'}_{\gamma\, i} (-2^H)^{\rho'} \\
&\qquad \times \binom{\gamma-\mu'}{\rho'} N^{\beta' b' (\rho')}_{\delta' j'} \mathfrak{Z}^H \lambda^{\gamma-\mu'-\rho'\, i}_{\beta'\; b'} \\
&= \sum_{\alpha'=\delta'+\mu'}^{\gamma'} \sum_{\beta'=\delta'}^{\alpha'-\mu'} \sum_{\rho'=0}^{\alpha'-\mu'-\beta'} \binom{\rho'+\mu'}{\mu'} (1-2^H)^{\rho'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b' (\rho')}_{\delta' j'} \\
&\qquad \times \sum_{\gamma=\rho'+\mu'+\beta'}^{\alpha'} \binom{\gamma}{\rho'+\mu'} \lambda^{\alpha' a'}_{\gamma\, i} \mathfrak{Z}^H \lambda^{\gamma-\mu'-\rho'\, i}_{\beta'\; b'} \\
&= \sum_{\alpha'=\delta'+\mu'}^{\gamma'} \sum_{\beta'=\delta'}^{\alpha'-\mu'} \sum_{\rho'=0}^{\alpha'-\mu'-\beta'} \binom{\rho'+\mu'}{\mu'} (1-2^H)^{\rho'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b' (\rho')}_{\delta' j'} C^{*\rho'+\mu'\, \alpha' a'}_{\;\;\beta' b'}(\mathfrak{Z}^H) .
\end{aligned}
$$

Theorem 25: *For non-holonomic components of an excontravariant extensor $v^{\gamma' i'}$ of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ and a scalar f of weight $\theta_1 + \cdots\cdots + \theta_R$, the quantities $\mathfrak{Z}^{*H} v^{\gamma' i'} \equiv f \cdot \mathfrak{Z}^H (v^{\gamma' i'}/f)$:*

$$
\mathfrak{Z}^{*H} v^{\gamma' i'} = \mathfrak{Z}^H v^{\gamma' i'} + f \sum_{\mu'=0}^{\gamma'} \sum_{\delta'=0}^{\gamma'-\mu'} (1-2^H)^{\mu'} C^{*\mu' \gamma' i'}_{\;\;\delta' j'}(\mathfrak{Z}^H) \sum_{\nu=0}^{\mu'-1} \binom{\mu'}{\nu} \left(\frac{1}{f}\right)^{(\mu'-\nu)} v^{\delta' j' (\nu)}
$$

are the components of an excontravariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R, M + R \leqq P)$.

Remark. Take the non-holonomic components of $x^{(\gamma+1)i}$ $(\gamma = 0, \cdots\cdots, M-1)$: $x^{\gamma'+1 i'}$ instead of $v^{\gamma' i'}$ in Theorem 21, then it follows that $\mathfrak{Z}^H x^{\gamma'+1 i'} = x^{\gamma'+1\, i'}$ $(\gamma' = 0, \cdots\cdots, M)$.

§ 7. The $\mathfrak{Z}^H$-operation of an excovariant extensor in a non-holonomic system.

The quantities

$$
(7.1) \qquad \mathfrak{Z}^H w_{\gamma i} = \sum_{\nu=0}^{R-H-\gamma} (-1)^{\nu} \binom{H-1+\nu}{H-1} w_{\gamma+H+\nu\; i}{}^{(\nu)} , \qquad \gamma = 0, \cdots\cdots, R-H
$$

are the components of an excovariant extensor, being $w_{\gamma i}$ an excovariant extensor ([2], p. 40). Put

$$w_{\gamma+H+\nu i}{}^{(\nu)} = \Big(\sum_{\delta'=\gamma+H+\nu}^{R} \lambda_{\gamma+H+\nu}{}^{\delta'}{}_{i}{}^{j'} w_{\delta' j'}\Big)^{(\nu)} = \sum_{\delta'=\gamma+H+\nu}^{R} \sum_{\lambda=0}^{\nu} \binom{\nu}{\lambda} \lambda_{\gamma+H+\nu}{}^{\delta'}{}_{i}{}^{j'(\nu-\lambda)} w_{\delta' j'}{}^{(\lambda)}$$

into (7.1), then the components of the extensor $\mathfrak{Z}^H w_{\gamma' i'} = \sum_{\gamma=\gamma'}^{R-H} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^H w_{\gamma i}$ in the non-holonomic system have the form

$$\begin{aligned}
\mathfrak{Z}^H w_{\gamma' i'} &= \sum_{\gamma=\gamma'}^{R-H} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \sum_{\nu=0}^{R-H-\gamma} \sum_{\lambda=0}^{\nu} \sum_{\delta'=\gamma+H+\nu}^{R} (-1)^{\nu} \binom{H-1+\nu}{H-1} \binom{\nu}{\lambda} \lambda_{\gamma+H+\nu}{}^{\delta'}{}_{i}{}^{j'(\nu-\lambda)} w_{\delta' j'}{}^{(\lambda)} \\
&= \sum_{\gamma=\gamma'}^{R-H} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \sum_{\lambda=0}^{R-H-\gamma} \sum_{\delta'=\gamma+H+\lambda}^{R} (-1)^{\lambda} \binom{H-1+\lambda}{H-1} w_{\delta' j'}{}^{(\lambda)} \\
&\quad \times \sum_{\nu-\lambda=0}^{\delta'-(\gamma+H+\lambda)} (-1)^{\nu-\lambda} \binom{H+\lambda-1+\nu-\lambda}{H+\lambda-1} \lambda_{\gamma+H+\lambda-(\nu-\lambda)}{}^{\delta'}{}_{i}{}^{j'(\nu-\lambda)} .
\end{aligned}$$

From the fact that $\lambda_{\omega}{}^{\theta'}{}_{i}{}^{j'} = 0$ *for* $\omega > \theta'$, we have the equation

$$\begin{aligned}
(7.2)\qquad & \sum_{\nu=\lambda}^{R-(\gamma+H)} (-1)^{\nu-\lambda} \binom{H+\lambda-1+\nu-\lambda}{H+\lambda-1} \lambda_{\gamma+H+\lambda+(\nu-\lambda)}{}^{\delta'}{}_{i}{}^{j'(\nu-\lambda)} \\
&= \sum_{\nu=\lambda}^{\delta'-(\gamma+H)} (-1)^{\nu-\lambda} \binom{H+\lambda-1+\nu-\lambda}{H+\lambda-1} \lambda_{\gamma+H+\lambda+(\nu-\lambda)}{}^{\delta'}{}_{i}{}^{j'(\nu-\lambda)} = \mathfrak{Z}^{H+\lambda} \lambda_{\gamma}{}^{\delta'}{}_{i}{}^{j'}
\end{aligned}$$

from which we shall know that even if the range of $\lambda_{\gamma}{}^{\delta'}{}_{i}{}^{j'}$ is either R or δ', the corresponding $\mathfrak{Z}^H$*-operations* of the base extensors can not be distinguished. Then it follows that

$$\begin{aligned}
\mathfrak{Z}^H w_{\gamma' i'} &= \sum_{\gamma=\gamma'}^{R-H} \sum_{\lambda=0}^{R-H-\gamma} \sum_{\delta'=\gamma+H+\lambda}^{R} (-1)^{\lambda} \binom{H-1+\lambda}{H-1} w_{\delta' j'}{}^{(\lambda)} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H+\lambda} \lambda_{\gamma}{}^{\delta'}{}_{i}{}^{j'} \\
&= \sum_{\lambda=0}^{R-H-\gamma'} \sum_{\delta'=\gamma'+H+\lambda}^{R} (-1)^{\lambda} \binom{H-1+\lambda}{H-1} w_{\delta' j'}{}^{(\lambda)} \sum_{\gamma=\gamma'}^{\delta'-H-\lambda} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H+\lambda} \lambda_{\gamma}{}^{\delta'}{}_{i}{}^{j'} ,
\end{aligned}$$

and by putting

$$(7.3)\qquad \sum_{\gamma=\gamma'}^{\delta'-H-\lambda'} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H+\lambda'} \lambda_{\gamma}{}^{\delta'}{}_{i}{}^{j'} = C^{\lambda'}{}_{\gamma'}{}^{\delta'}{}_{i'}{}^{j'} (\mathfrak{Z}^H) ,$$

$$(7.4)\qquad \mathfrak{Z}^H w_{\gamma' i'} = \sum_{\lambda'=0}^{R-H-\gamma'} \sum_{\delta'=\gamma'+H+\lambda'}^{R} (-1)^{\lambda'} \binom{H-1+\lambda'}{H-1} w_{\delta' j'}{}^{(\lambda')} C^{\lambda'}{}_{\gamma'}{}^{\delta'}{}_{i'}{}^{j'} (\mathfrak{Z}^H) .$$

The quantities $C^{\lambda'}{}_{\gamma'}{}^{\delta'}{}_{i'}{}^{j'} (\mathfrak{Z}^H)$ in the last equation are called $\mathfrak{Z}^H$*–operation coefficients of excovariant extensors* in the non-holonomic system. Furthermore we can state

Theorem 26: *The quantities*

$$\mathfrak{Z}^H w_{\gamma' i'} = \sum_{\lambda'=0}^{R-H-\gamma'} \sum_{\delta'=\gamma'+H+\lambda'}^{R} (-1)^{\lambda'} \binom{H-1+\lambda'}{H-1} w_{\delta' j'}{}^{(\lambda')} C^{\lambda'}{}_{\gamma'}{}^{\delta'}{}_{i'}{}^{j'} (\mathfrak{Z}^H)$$

are components of an excovariant extensor of characteristic (1, 0, $R-H$, $M+R-H\,(\leqq P)$), *when* $w_{\gamma' i'}$ *are components of an excovariant extensor of charac-*

teristic (1, 0, R, M) *in the non-holonomic system.*

Theorem 27: *When our system is holonomic, the relations* $C^{\lambda'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^H) = \delta^{\delta'-\lambda'}_{\gamma'}{}^{-H}\,\delta^{j'}_{i'}$ *hold good.*

Proof. The quantities $\lambda^{\gamma\,i}_{\gamma' i'}$ must be $X^{(\gamma)i}_{(\gamma)i'}$ in a holonomic system. Accordingly, (7.2) is calculated as follows:

$$\mathfrak{Z}^{H+\lambda}\lambda^{\delta' j'}_{\gamma\ i} = \sum_{\nu=0}^{R-\gamma-H-\lambda}(-1)^{\nu}\binom{H+\lambda-1+\nu}{H+\lambda-1}\lambda_{\gamma+H+\lambda+\nu}{}^{\delta'}{}_{i}^{j'(\nu)}$$

$$= \sum_{\nu=0}^{\delta'-\gamma-H-\lambda}(-1)^{\nu}\binom{H+\lambda-1+\nu}{H+\lambda-1}\binom{\delta'}{\gamma+H+\lambda+\nu}X_{i}^{j'(\delta'-\gamma-H-\lambda)},$$

which becomes $\mathfrak{Z}^{H+\lambda}\lambda^{\delta' j'}_{\gamma\ i} = X^{\delta'-(H+\lambda)j'}_{\gamma\qquad\quad i}$, because $\sum_{\nu=0}^{\delta'-\gamma-H-\lambda}(-1)^{\nu}\binom{H+\lambda-1+\nu}{H+\lambda-1}$ $\times\binom{\delta'}{\gamma+H+\lambda+\nu} = \binom{\delta'-H-\lambda}{\gamma}$. From it we can prove

Theorem 28: *The quantities* $C^{\lambda'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^H)$ *are invariant under transformations of holonomic coordinate systems.*

Theorem 29: *The quantities* $C^{\lambda'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^H)$ *are given by*

$$C^{\lambda'}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^H) = \sum_{\alpha=\gamma'}^{\delta'-\lambda'-H}\ \sum_{\mu'=0}^{\delta'-\alpha'-\lambda'-H}\ \sum_{\beta'=\alpha'+H+\lambda'+\mu'}^{\delta'}(-1)^{\mu'}\binom{H+\lambda'+\mu'-1}{H+\lambda'-1}N^{\alpha' a'}_{\gamma' i'}N^{\delta' j'(\mu')}_{\beta' b'}$$
$$\times\, C^{\lambda'+\mu'}{}^{\beta' b'}_{\alpha' a'}(\mathfrak{Z}^H)$$

under a non-holonomic transformation of non-holonomic systems.

Proof. By virtue of the non-holonomic transformation of the non-holonomic systems $\lambda^{\gamma\,i}_{\gamma' i'} = \sum_{\alpha'=\gamma'}^{\gamma} N^{\alpha' a'}_{\gamma' i'}\lambda^{\gamma\,i}_{\alpha' a'}$, $\lambda_{\gamma+H+\lambda+\nu}{}^{\delta'}{}_{i}^{j'} = \sum_{\beta'=\gamma+H+\lambda+\nu}^{\delta'} N^{\delta' j'}_{\beta' b'}$ $\times\,\lambda_{\gamma+H+\lambda+\nu}{}^{\beta'}{}_{i}^{b'}$, we have

$$\mathfrak{Z}^{H+\lambda}\lambda^{\delta' j'}_{\gamma\ i} = \sum_{\nu=0}^{\delta'-\gamma-H-\lambda}\ \sum_{\beta'=\gamma+H+\lambda+\nu}^{\delta'}\ \sum_{\mu=0}^{\nu}(-1)^{\nu}\binom{H+\lambda-1+\nu}{H+\lambda-1}\binom{\nu}{\mu}N^{\delta' j'(\mu)}_{\beta' b'}\lambda_{\gamma+H+\lambda+\nu}{}^{\beta'}{}_{i}^{b'(\nu-\mu)}$$

$$= \sum_{\mu=0}^{\delta'-\gamma-H-\lambda}\ \sum_{\beta'=\gamma+H+\lambda+\nu}^{\delta'}\ \sum_{\nu=\mu}^{\beta'-\gamma-H-\lambda}(-1)^{\nu-\mu}\binom{H+\lambda+\mu-1+\nu-\mu}{\nu-\mu}$$
$$\times\,\lambda_{\gamma+H+\lambda+\nu}{}^{\beta'}{}_{i}^{b'(\nu-\mu)}(-1)^{\mu}\binom{H+\lambda+\mu-1}{H+\lambda-1}N^{\delta' j'(\mu)}_{\beta' b'}$$

$$= \sum_{\mu=0}^{\delta'-\gamma-H-\lambda}\ \sum_{\beta'=\gamma+H+\lambda+\mu}^{\delta'}(-1)^{\mu}\binom{H+\lambda+\mu-1}{H+\lambda-1}N^{\delta' j'(\beta)}_{\mu' b'}\mathfrak{Z}^{H+\lambda+\mu}\lambda^{\beta' b'}_{\gamma\ i},$$

from which it follows that

$$C^{\lambda}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^H) = \sum_{\gamma=\gamma'}^{\delta'-\lambda'-H}\ \sum_{\alpha'=\gamma'}^{\gamma}\ \sum_{\mu'=0}^{\delta'-\gamma-\lambda-H}\ \sum_{\beta'=\gamma+H+\lambda+\mu}^{\delta'} N^{\alpha' a'}_{\gamma' i'}N^{\delta' j'(\mu)}_{\beta' b'}(-1)^{\mu}\binom{H+\lambda+\mu-1}{H+\lambda'-1}$$
$$\times\,\lambda^{\gamma\,i}_{\alpha' a'}\mathfrak{Z}^{H+\lambda+\mu}\lambda^{\beta' b'}_{\gamma\ i}$$

$$= \sum_{\alpha'=\gamma'}^{\delta'-\lambda-H}\ \sum_{\mu=0}^{\delta'-\alpha'-\lambda-H}\ \sum_{\beta'=\alpha'+H+\lambda+\mu}^{\delta'}(-1)^{\mu}\binom{H+\lambda+\mu-1}{H+\lambda-1}N^{\alpha' a'}_{\gamma' i'}N^{\delta' j'(\mu)}_{\beta' b'}$$
$$\times\, C^{\lambda+\mu}{}^{\beta' b'}_{\alpha' a'}(\mathfrak{Z}^H).$$

Theorem 30: *If* $w_{\gamma' i'}$ *are non-holonomic components of an excovariant extensor of characteristic* $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ *and* f *is a scalar of weight* $\theta_1 + \cdots\cdots + \theta_R$, *then the quantities* $\mathfrak{Z}^{*H} w_{\gamma' i'} \equiv f \cdot \mathfrak{Z}^H (w_{\gamma' i'}/f)$, $\mathfrak{Z}^{*H} w_{\gamma' i'} = \mathfrak{Z}^H w_{\gamma' i'} + \sum_{\mu=1}^{R-H-\gamma'} \sum_{\nu=0}^{R-(H+\mu)-\gamma'} (-1)^{\mu+\nu} \binom{H+\mu-1}{H-1} \binom{H+\mu+\nu-1}{H+\mu-1} \left(\frac{1}{f}\right)^{(\mu)} f^{(\nu)} \mathfrak{Z}^{*H+\mu+\nu} w_{\gamma' i'}$ *are components of an excovariant extensor of characteristic* $(1, \theta_1 + \cdots\cdots + \theta_R, R-H, M+R-H (\leqq P))$.

The method of proof is essentially same as that of Theorem 9.

Theorem 31: *A necessary and sufficient condition for that a function* $f dt$ *of an expoint* $(x, x^{(1)}, \cdots, x^{(M)})$ *be invariant in functional form under a transformation of parameter* t *is given by the equations referred to a non-holonomic system:* $\mathfrak{Z} f_{\gamma' i'} x^{\gamma'+1 i'} = f$, $\mathfrak{Z}^H f_{\gamma' i'} x^{\gamma'+1 i'} = 0$ *for* $H \geqq 2$.

Proof. From the definition, we can observe the relations

$$\mathfrak{Z}^H f_{\gamma' i'} = \sum_{\gamma=\gamma'}^{M-H} \lambda^{\gamma\ i}_{\gamma' i'} \mathfrak{Z}^H f_{(\gamma) i} \text{ and } x^{\gamma'+1 i'} = \sum_{\delta=0}^{\gamma'} \lambda^{\gamma' i'}_{\delta\ j} x^{(\delta+1) j} \quad (H = 1, \cdots\cdots M).$$

Consequently, we have

$$\begin{aligned}
\sum_{\gamma=\gamma'}^{M-H} \mathfrak{Z}^H f_{\gamma' i'} x^{\gamma'+1 i'} &= \sum_{\gamma'=0}^{M-H} \sum_{\gamma=\gamma'}^{M-H} \sum_{\delta=0}^{\gamma'} \lambda^{\gamma\ i}_{\gamma' i'} \lambda^{\gamma' i'}_{\delta\ j} \mathfrak{Z}^H f_{(\gamma) i} \cdot x^{(\delta+1) j} \\
&= \sum_{\gamma=\delta}^{M-H} \sum_{\delta=0}^{M-H} \sum_{\gamma'=\delta'}^{\gamma} (\lambda^{\gamma\ i}_{\gamma' i'} \lambda^{\gamma' i'}_{\delta\ j}) \mathfrak{Z}^H f_{(\gamma) i} \cdot x^{(\delta+1) j} \\
&= \sum_{\delta=0}^{M-H} \mathfrak{Z}^H f_{(\delta) j} \cdot x^{(\delta+1) j} = f \qquad \text{for} \quad H = 1 \\
&= 0 \qquad \text{for} \quad H \geqq 2
\end{aligned}$$

from the similar theorem in a holonomic system ([2], p. 41).

Remark. In a holonomic system, we have the following relation

$$\begin{aligned}
&(\alpha) \quad \mathfrak{Z}^H w_{\gamma-1 i} = \mathfrak{Z}^{H-1} w_{\gamma i} - (\mathfrak{Z}^H w_{\gamma i})^{(1)}, \\
&(\beta) \quad \mathfrak{Z}^H w_{R-H i} = \mathfrak{Z}^{H-1} w_{R-H+1 i}
\end{aligned} \qquad \gamma = 0, 1, \cdots\cdots, R-H$$

([2], p. 42).

Next, we shall consider whether such the properties exist or not for the $\mathfrak{Z}^H$*-operation* of excovariant extensors in the non-holonomic system.

(α). From the equations $\mathfrak{Z}^H w_{\gamma' i'} = \sum_{\gamma=\gamma'}^{R-H} \lambda^{\gamma\ i}_{\gamma' i'} \mathfrak{Z}^H w_{\gamma i}$, we obtain the following results:

$$\begin{aligned}
\mathfrak{Z}^{H-1} w_{\gamma' i'} - (\mathfrak{Z}^H w_{\gamma' i'})^{(1)} &= \sum_{\gamma=\gamma'}^{R-H+1} \lambda^{\gamma\ i}_{\gamma' i'} \mathfrak{Z}^{H-1} w_{\gamma i} \\
&\quad - \sum_{\gamma=\gamma'}^{R-H} \{ (\lambda^{\gamma\ i}_{\gamma' i'})^{(1)} \mathfrak{Z}^H w_{\gamma i} + \lambda^{\gamma\ i}_{\gamma' i'} (\mathfrak{Z}^H w_{\gamma i})^{(1)} \} \\
&= \lambda^{R-H+1 i}_{\gamma'\ i'} \mathfrak{Z}^{H-1} w_{R-H+1 i} + \sum_{\gamma=\gamma'}^{R-H} \lambda^{\gamma\ i}_{\gamma' i'} \{ \mathfrak{Z}^{H-1} w_{\gamma i} - (\mathfrak{Z}^H w_{\gamma i})^{(1)} \} - \sum_{\gamma=\gamma'}^{R-H} (\lambda^{\gamma\ i}_{\gamma' i'})^{(1)} \mathfrak{Z}^H w_{\gamma i}
\end{aligned}$$

$$= \lambda^{R-H+1}{}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H-1} w_{R-H+1i} + \sum_{\gamma=\gamma'}^{R-H} \{ \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H} w_{\gamma-1\,i} - (\lambda^{\gamma}_{\gamma'}{}^{i}_{i'})^{(1)} \mathfrak{Z}^{H} w_{\gamma i} \}$$

$$= \sum_{\gamma=\gamma'}^{R-H+1} \lambda^{\gamma}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H} w_{\gamma-1\,i} - \sum_{\gamma=\gamma'}^{R-H} (\lambda^{\gamma}_{\gamma'}{}^{i}_{i'})^{(1)} \mathfrak{Z}^{H} w_{\gamma i}$$

$$= \sum_{\gamma=\gamma'}^{R-H} \{ \lambda^{\gamma+1}_{\gamma'}{}^{i}_{i'} - (\lambda^{\gamma}_{\gamma'}{}^{i}_{i'})^{(1)} \} \mathfrak{Z}^{H} w_{\gamma i} + \lambda^{\gamma'}_{\gamma'}{}^{i}_{i'} \mathfrak{Z}^{H} w_{\gamma'-1\,i} \quad \text{(not summing on } \gamma').$$

Using symbols $\mu_{\gamma}{}^{\gamma}_{\gamma'-1}{}^{i}_{i'}$ and $\mu^{\gamma'}_{\gamma'}{}^{-1}_{-1}{}^{i}_{i'}$ for $\{\lambda^{\gamma+1}_{\gamma'}{}^{i}_{i'} - (\lambda^{\gamma}_{\gamma'}{}^{i}_{i'})^{(1)}\}$ and $\lambda^{\gamma'}_{\gamma'}{}^{i}_{i'}$ respectively, and putting $\sum_{\gamma=\gamma'-1}^{\delta'} \mu_{\gamma}{}^{\gamma}_{\gamma'-1}{}^{i}_{i'} \lambda^{\delta'}_{\gamma}{}^{j'}_{i} = C^{*}{}_{\gamma'}{}^{\delta'}_{-1}{}^{j'}_{i'}$ $(= \delta_{\gamma'}{}^{\delta'}_{-1} \delta^{j'}_{i'}$ in the case of the holonomic system), we have

$$\mathfrak{Z}^{H-1} w_{\gamma' i'} - (\mathfrak{Z}^{H} w_{\gamma' i'})^{(1)} = \sum_{\delta'=\gamma'-1}^{R-H} C^{*}{}_{\gamma'}{}^{\delta'}_{-1}{}^{j'}_{i'} \mathfrak{Z}^{H} w_{\delta' j'} .$$

(β). Also from that

$$\mathfrak{Z}^{H} w_{R-H\,i'} = \lambda^{R-H}_{R-H}{}^{i}_{i'} \mathfrak{Z}^{H} w_{R-H\,i} \text{ and } \mathfrak{Z}^{H-1} w_{R-H+1\,i'} = \lambda^{R-H+1}_{R-H+1}{}^{i}_{i'} \mathfrak{Z}^{H-1} w_{R-H+1\,i},$$

it follows that

$$\mathfrak{Z}^{H} w_{R-H\,i'} - \mathfrak{Z}^{H-1} w_{R-H+1\,i'} = (\lambda^{R-H}_{R-H}{}^{i}_{i'} - \lambda^{R-H+1}_{R-H+1}{}^{i}_{i'}) \mathfrak{Z}^{H} w_{R-H\,i}$$

and further putting $(\lambda^{R-H}_{R-H}{}^{i}_{i'} - \lambda^{R-H+1}_{R-H+1}{}^{i}_{i'}) \lambda^{R-H}_{R-H}{}^{j'}_{i} = C^{R-H}_{*}{}^{j'}_{i'}$ $(= 0$ in the holonomic system), in the non-holonomic system, the following results are obtained:

$$(\alpha') \qquad \mathfrak{Z}^{H-1} w_{\gamma' i'} - (\mathfrak{Z}^{H} w_{\gamma' i'})^{(1)} = \sum_{\delta'=\gamma'-1}^{R-H} C^{*}{}_{\gamma'}{}^{\delta'}_{-1}{}^{j'}_{i'} \mathfrak{Z}^{H} w_{\delta' j'} ,$$

$$(\beta') \qquad \mathfrak{Z}^{H} w_{R-H\,i'} - \mathfrak{Z}^{H-1} w_{R-H+1i'} = C^{R-H}_{*}{}^{j'}_{i'} \mathfrak{Z}^{H} w_{R-H\,j'} .$$

Theorem 32: *The $\mathfrak{Z}^{H}$–operation in the non-holonomic system holds the commutative and linearly associative law.*

§8. The $\mathfrak{Y}^{H}$–operation of an excontravariant extensors in a non-holonomic system. The $\mathfrak{Y}^{H}$–operation for an excontravariant extensor is defined by

$$(8.1) \qquad \mathfrak{Y}^{H} v^{\gamma i} = H! \sum_{\nu=0}^{H} \binom{\gamma}{\nu} \binom{R^{*}+H-\gamma}{H-\nu} v^{\gamma-\nu\, i\, (\nu)} \quad \text{for } \gamma = H, H+1, \cdots, R^{*}$$

$$= H! \sum_{\nu=0}^{\gamma} \binom{\gamma}{\nu} \binom{R^{*}+H-\gamma}{H-\nu} v^{\gamma-\nu\, i\, (\nu)} \quad \text{for } \gamma = 0, 1, \cdots\cdots\cdots, H$$

$$= H! \sum_{\nu=\gamma-R^{*}}^{H} \binom{\gamma}{\nu} \binom{R^{*}+H-\gamma}{H-\nu} v^{\gamma-\nu i (\nu)} \text{ for } \gamma = R^{*}, R^{*}+1, \cdots, R^{*}+H$$

$$(R^{*} \leqq R, \ H = 1, 2, \cdots\cdots, R^{*})$$

where $v^{\gamma i}$ is an excontravariant extensor ([2], p. 44). We shall confine ouselves to consider only R^{*} and H satisfying the relation $R^{*} + H \leqq G$ in this chapter. (8.1) can be written simply in an equation

(8. 1)′ $\mathfrak{Y}^H v^{\gamma i} = H! \sum_{\nu=0}^{\gamma} \binom{\gamma}{\nu} \binom{R^*+H-\gamma}{H-\nu} v^{\gamma-\nu\, i(\nu)}, \qquad \gamma = 0, \cdots\cdots, R^*+H,$

reading $\binom{\lambda}{\mu}$ for $0<\lambda<\mu$ or $\mu<0$ as zero. Using (8.1)′ and

$$v^{\gamma-\nu\, i(\nu)} = \Big(\sum_{\delta'=0}^{\gamma-\nu} \lambda^{\gamma-\nu}{}_{\delta'}{}^{i}{}_{j'}\, v^{\delta' j'}\Big)^{(\nu)} = \sum_{\delta'=0}^{\gamma-\nu} \sum_{\rho=0}^{\nu} \binom{\nu}{\rho} \lambda^{\gamma-\nu}{}_{\delta'}{}^{i(\nu-\rho)}_{j'}\, v^{\delta' j'(\rho)},$$

it follows that

$$\begin{aligned}
\mathfrak{Y}^H v^{\gamma' i'} &= \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{Y}^H v^{\gamma i} \qquad (\gamma' = 0, \cdots\cdots, R^*+H) \\
&= \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, H! \sum_{\nu=0}^{\gamma} \binom{\gamma}{\nu} \binom{R^*+H-\gamma}{H-\nu} \sum_{\delta'=0}^{\gamma-\nu} \sum_{\rho=0}^{\nu} \binom{\nu}{\rho} \lambda^{\gamma-\nu}{}_{\delta'}{}^{i(\nu-\rho)}_{j'}\, v^{\delta' j'(\rho)} \\
&= \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \frac{H!}{(H-\rho)!} \sum_{\delta'=0}^{\gamma} \sum_{\rho=0}^{\gamma-\delta'} \binom{\gamma}{\rho} v^{\delta' j'(\rho)} \\
&\quad \times \sum_{\nu=\rho}^{\gamma-\delta'} (H-\rho)! \binom{\gamma-\rho}{\nu-\rho} \binom{R^*+H-\rho-(\gamma-\rho)}{H-\rho-(\nu-\rho)} \lambda^{\gamma-\rho-(\nu-\rho)}{}_{\delta'}{}^{i(\nu-\rho)}_{j'}.
\end{aligned}$$

On the other hand, we can see, because of $\lambda^{\gamma-\rho-(\nu-\rho)}{}_{\delta'}{}^{i}{}_{j'} = 0$ for $\nu-\rho > \gamma-\delta'-\rho$

(8. 2) $\sum_{\nu=\rho}^{\gamma-\delta'} (H-\rho)! \binom{\gamma-\rho}{\nu-\rho} \binom{R^*+H-\rho-(\gamma-\rho)}{H-\rho-(\nu-\rho)} \lambda^{\gamma-\rho-(\nu-\rho)}{}_{\delta'}{}^{i(\nu-\rho)}_{j'}$

$$= \sum_{\nu=\rho}^{\gamma} (H-\rho)! \binom{\gamma-\rho}{\nu-\rho} \binom{R^*+H-\rho-(\gamma-\rho)}{H-\rho-(\nu-\rho)} \lambda^{\gamma-\rho-(\nu-\rho)}{}_{\delta'}{}^{i(\nu-\rho)}_{j'} = \mathfrak{Z}^{H-\rho} \lambda^{\gamma-\rho}{}_{\delta'}{}^{i}{}_{j'}$$

which depends on the range R of $\lambda^{\gamma-\rho}{}_{\delta'}{}^{i}{}_{j'}$. Consequently, we have

$$\mathfrak{Y}^H v^{\gamma' i'} = \sum_{\delta'=0}^{\gamma'} \sum_{\rho=0}^{\gamma'-\delta'} \sum_{\gamma=\rho+\delta'}^{\gamma'} \frac{H!}{(H-\rho)!} \binom{\gamma}{\rho} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{Y}^{H-\rho} \lambda^{\gamma-\rho}{}_{\delta'}{}^{i}{}_{j'}\, v^{\delta' j'(\rho)}.$$

By putting

(8. 3) $\{(H-\rho)!\, \binom{\rho+\delta'}{\rho} \binom{R^*+H-\gamma'}{H-\rho}\}^{-1} \sum_{\gamma=\rho'+\delta'} \binom{\gamma}{\rho} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{Y}^{H-\rho} \lambda^{\gamma-\rho}{}_{\delta'}{}^{i}{}_{j'} = C^{\rho}{}^{\gamma' i'}_{\delta' j'}(\mathfrak{Y}^H)$,

it follows that

(8. 4) $\mathfrak{Y}^H v^{\gamma' i'} = \sum_{\delta'=0}^{\gamma'} \sum_{\rho'=0}^{\gamma'-\delta'} H!\, \binom{\rho'+\delta'}{\rho'} \binom{R^*+H-\gamma'}{H-\rho'} v^{\delta' j'(\rho')} C^{\rho'}{}^{\gamma' i'}_{\delta' j'}(\mathfrak{Y}^H)$

where the quantities $C^{\rho'}{}^{\gamma' i'}_{\delta' j'}(\mathfrak{Y}^H)$ are called the *$\mathfrak{Y}^H$-operation coefficients of excontravariant extensors of the range R.* Here we can see

Theorem 33: *If $v^{\gamma' i'}$ are components in the non-holonomic system of an excontravariant extensor of characteristic* $(1, 0, R, M)$, *then the quantities*

$$\mathfrak{Y}^H v^{\gamma' i'} = \sum_{\delta'=0}^{\gamma'} \sum_{\rho'=0}^{\gamma'-\delta'} H!\, \binom{\rho'+\delta'}{\rho'} \binom{R^*+H-\gamma'}{H-\rho'} v^{\delta' j'(\rho')} C^{\rho'}{}^{\gamma' i'}_{\delta' j'}(\mathfrak{Y}^H)$$

$$(\gamma' = 0, \cdots\cdots, R^*+H, \quad H = 1, 2, \cdots\cdots, R^*;\ R^* \leqq R)$$

are components of an excontravariant extensor of characteristic $(1, 0, R^*+H, M+H (\leqq P))$.

Corollary. *The $\mathfrak{D}^H$-operation of an excontravariant extensor $v^{\gamma' i'}$ in a non-holonomic system has the property $\mathfrak{D}(\mathfrak{D}^H v^{\gamma' i'}) = \mathfrak{D}^{H+1} v^{\gamma' i'}$.*

Since $\mathfrak{D}(\mathfrak{D}^H v^{\gamma' i'}) = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{D}(\mathfrak{D}^H v^{\gamma i})$ and $\mathfrak{D}^{H+1} v^{\gamma' i'} = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i}\, \mathfrak{D}^{H+1} v^{\gamma i}$ from the definition and also $\mathfrak{D}\,\mathfrak{D}^H v^{\gamma i} = \mathfrak{D}^{H+1} v^{\gamma i}$ in the holonomic system ([2], p. 45), the corollary follows.

Remark. When we replace $v^{\gamma' i'}$ in Theorem 33 with the non-holonomic components of $x^{(\gamma+1)i}$ $(\gamma = 0, \cdots, M-1)$: $x^{\gamma'+1\, i'}$ $(\gamma' = 0, \cdots, M-1)$, it is established that $\mathfrak{D}^H x^{\gamma'+1\, i'} = H!\binom{M^*+H}{H} x^{\gamma'+1\, i'}$ $(\gamma' = 0, \cdots, M^* + H \leqq M-1,\ H = 1, \cdots\cdots, M^*;\ M^* \leqq M)$.

Theorem 34: *The quantities $C^{\rho' \gamma' i'}_{\ \ \delta' j'}(\mathfrak{D}^H)$ become $\delta_{\rho'+\delta'}^{\gamma'}\delta^{i'}_{j'}$, if our system is holonomic.*

Proof. In this case, (8.3) may be written in the form

$$\begin{aligned} C^{\rho'\gamma' i'}_{\ \ \delta' j'}(\mathfrak{D}^H) &= K^{-1} \sum_{\gamma=\rho'+\delta'}^{\gamma'} \binom{\gamma}{\rho'} X^{(\gamma')i'}_{(\gamma)\ i} \\ &\quad\times \sum_{\nu=0}^{\gamma-\rho'} (H-\rho')!\binom{R^*+H-\rho'-(\gamma-\rho')}{H-\rho'-\nu}\binom{\gamma-\rho'}{\nu} X^{(\gamma-\rho'-\nu)\ i\ (\nu)}_{\qquad\ \ \delta'\ j'} \\ &= K^{-1} \sum_{\gamma=\rho'+\delta'}^{\gamma'} \binom{\gamma}{\rho'} X^{(\gamma')i'}_{(\gamma)\ i} \\ &\quad\times \sum_{\nu=0}^{\gamma-\rho'} (H-\rho')!\binom{R^*+H-\rho'-(\gamma-\rho')}{H-\rho'-\nu}\binom{\gamma-\rho'-\delta'}{\nu}\binom{\gamma-\rho'}{\delta'} X^{i\ (\gamma-\rho'-\delta')}_{j'} . \end{aligned}$$

putting $K = (H-\rho')!\binom{\rho'+\delta'}{\rho'}\binom{R^*+H-\gamma'}{H-\rho'}$. Since

$$\sum_{\nu=0}^{\gamma-\rho'} \binom{R^*+H-\rho'-(\gamma-\rho')}{H-\rho'-\nu}\binom{\gamma-\rho'-\delta'}{\nu} = \binom{R^*+H-\rho'-\delta'}{H-\rho'},$$

it follows

$$\begin{aligned} C^{\rho'\gamma' i'}_{\ \ \delta' j'}(\mathfrak{D}^H) &= K^{-1} \sum_{\gamma=\rho'+\delta}^{\gamma'} \binom{\gamma}{\rho'}\binom{\gamma-\rho'}{\delta'}(H-\rho')!\binom{R^*+H-\rho'-\delta'}{H-\rho'} X^{(\gamma')i'}_{(\gamma)\ i} X^{i\ (\gamma-\rho'-\delta')}_{j'} \\ &= \sum_{\gamma=\rho'+\delta'}^{\gamma'} X^{(\gamma')i'}_{(\gamma)\ i} X_{(\rho'+\delta')}{}^{(\gamma)}{}^{i}_{j'} = \delta_{\rho'+\delta'}^{\gamma'}\delta^{i'}_{j'} . \end{aligned}$$

Q. E. D.

Theorem 35: *The quantities $C^{\rho'\gamma' i'}_{\ \ \delta' j'}(\mathfrak{D}^H)$ are invariant under transformations of holonomic coordinate systems.*

Theorem 36: *The quantities $C^{*\rho'\gamma' i'}_{\ \ \ \delta' j'}(\mathfrak{D}^H) \equiv \binom{\rho'+\delta'}{\rho'}\binom{R+H+\gamma'}{H-\rho'} C^{\rho'\gamma' i'}_{\ \ \delta' j'}(\mathfrak{D}^H)$ are given by*

$$C^{*\rho'\gamma' i'}_{\ \ \ \delta' j'}(\mathfrak{D}^H) = \sum_{\alpha'=\rho'+\delta'}^{\gamma'} \sum_{\mu'=0}^{\alpha'-\rho'-\delta'} \sum_{\beta'=\delta'}^{\alpha'-\rho'-\mu'} \binom{\rho'+\mu'}{\mu'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b' (\mu')}_{\delta' j'} C^{*\rho'+\mu'\,\alpha' a'}_{\qquad\ \beta' b'}(\mathfrak{D}^H)$$

under non-holonomic transformations of the non-holonomic systems.

Proof. From a non-holonomic transformation of the non-holonomic systems $\lambda^{\gamma' i'}_{\gamma\ i} = \sum_{\alpha'=\gamma}^{\gamma'} N^{\gamma' i'}_{\alpha' a'} \lambda^{\alpha' a'}_{\gamma\ i}$, $\lambda^{\gamma-\rho-\nu\, i}_{\delta'\ \ j'} = \sum_{\beta'=\delta'}^{\gamma-\rho-\nu} N^{\beta' b'}_{\delta' j'} \lambda^{\gamma-\rho-\nu\, i}_{\beta'\ \ b'}$, we can calculate as follows:

$$\mathfrak{Y}^{H-\rho} \lambda^{\gamma-\rho\, i}_{\delta'\ j'} = \sum_{\nu=0}^{\gamma-\rho} (H-\rho)!\, \binom{R^*+H-\rho-(\gamma-\rho)}{H-\rho-\nu} \binom{\gamma-\rho}{\nu} \lambda^{\gamma-\rho-\nu\, i(\nu)}_{\delta'\ \ j'}$$

$$= \sum_{\nu=0}^{\gamma-\rho-\delta'} \sum_{\beta'=\delta'}^{\gamma-\rho-\nu} \sum_{\mu=0}^{\nu} (H-\rho)!\, \binom{\gamma-\rho}{\nu} \binom{R^*+H-\rho-(\gamma-\rho)}{H-\rho-\nu} \binom{\nu}{\mu} N^{\beta' b'(\mu)}_{\delta' j'} \lambda^{\gamma-\rho-\nu\, i(\nu-\mu)}_{\beta'\ \ b'}$$

$$= \sum_{\mu=0}^{\gamma-\rho-\delta'} \sum_{\beta'=\delta'}^{\gamma-\rho-\mu} \frac{(H-\rho)!}{(H-\rho-\mu)!} \binom{\gamma-\rho}{\mu} N^{\beta' b'(\mu)}_{\delta' j'} \mathfrak{Y}^{H-\rho-\mu} \lambda^{\gamma-\rho-\mu\, i}_{\beta'\ \ b'}.$$

From this, it follows that

$$C^{*\rho' \gamma' i'}_{\ \ \ \delta' j'}(\mathfrak{Y}^H) = \sum_{\gamma=\rho'+\delta'}^{\gamma'} \sum_{\alpha'=\gamma}^{\gamma'} \sum_{\mu'=0}^{\gamma-\rho'-\delta'} \sum_{\beta'=\delta'}^{\gamma-\rho'-\mu'} \binom{\gamma}{\rho'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b'(\mu')}_{\delta' j'} \binom{\gamma-\rho'}{\mu'} \{(H-\rho'-\mu')!\}^{-1}$$
$$\times \lambda^{\alpha' a'}_{\gamma\ i} \mathfrak{Y}^{H-\rho'-\mu'} \lambda^{\gamma-\rho'-\mu'\, i}_{\beta'\ \ \ b'}$$

$$= \sum_{\alpha'=\rho'+\delta'}^{\gamma'} \sum_{\mu'=0}^{\alpha'-\rho'-\delta'} \sum_{\beta'=\delta'}^{\alpha'-\rho'-\mu'} \binom{\rho'+\mu'}{\mu'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b'(\mu')}_{\delta' j'} \{(H-\rho'-\mu')!\}^{-1}$$
$$\times \sum_{\gamma=\beta'+\rho'+\mu'}^{\alpha'} \binom{\gamma}{\rho'+\mu'} \lambda^{\alpha' a'}_{\gamma\ i} \mathfrak{Y}^{H-\rho'-\mu'} \lambda^{\gamma-\rho'-\mu'\, i}_{\beta'\ \ \ b'}$$

$$= \sum_{\alpha'=\rho'+\delta'}^{\gamma'} \sum_{\mu'=0}^{\alpha'-\rho'-\delta'} \sum_{\beta'=\delta'}^{\alpha'-\rho'-\mu'} \binom{\rho'+\mu'}{\mu'} N^{\gamma' i'}_{\alpha' a'} N^{\beta' b'(\mu')}_{\delta' j'} C^{*\rho'+\mu'\, \alpha' a'}_{\ \ \ \ \ \ \ \beta' b'}(\mathfrak{Y}^H).$$

Theorem 37: *When $v^{\gamma' i'}$ are non-holonomic components of an excontravariant extensor of characteristic* $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ *and f is a scalar of weight* $\theta_1 + \cdots\cdots + \theta_R$, *the quantities* $\mathfrak{Y}^{*H} v^{\gamma' i'} \equiv f \mathfrak{Y}^H (v^{\gamma' i'}/f)$:

$$\mathfrak{Y}^{*H} v^{\gamma' i'} = \mathfrak{Y}^H v^{\gamma' i'} + f \sum_{\delta'=0}^{\gamma'} \sum_{\rho=0}^{\gamma'-\delta'} \sum_{\mu=0}^{\rho-1} \binom{\rho}{\mu} \left(\frac{1}{f}\right)^{(\rho-\mu)} v^{\delta' j'(\mu)}\, H!\, C^{*\rho \gamma' i'}_{\ \ \ \delta' j'}(\mathfrak{Y}^H)$$
$$(H = 1, 2, \cdots\cdots, R^*, \qquad R^* \leqq R)$$

are components of an excontravariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R^* + H, M + H (\leqq P))$.

§ 9. The $\mathfrak{Y}^H$-operation of excovariant extensors in a non-holonomic system. Next, we shall proceed to consider non-holonomic components of an extensor $\mathfrak{Y}^H w_{\gamma i}$ which is defined by

(9. 1) $$\mathfrak{Y}^H w_{\gamma i} = \sum_{\nu=0}^{H} \binom{H}{\nu} w_{\gamma-\nu\ i}^{\ \ \ \ (H-\nu)} \qquad \text{for } \gamma = H, H+1, \cdots\cdots, R$$
$$= \sum_{\nu=0}^{\gamma} \binom{H}{\nu} w_{\gamma-\nu\ i}^{\ \ \ \ (H-\nu)} \qquad \text{for } \gamma = 0, 1, \cdots\cdots\cdots\cdots, H$$
$$= \sum_{\nu=\gamma-R}^{H} \binom{H}{\nu} w_{\gamma-\nu\ i}^{\ \ \ \ (H-\nu)} \qquad \text{for } \gamma = R, \cdots\cdots\cdots, R+H$$
$$(H = 1, 2, \cdots\cdots, R)$$

([2], p. 46).

Let R and H considered satisfy the relation $R + H \leqq G$, if we assume that $w_{\gamma-\nu\iota} = 0$ for $\gamma-\nu > R$, (9.1) is given by the equation:

$$\mathfrak{Y}^H w_{\gamma\iota} = \sum_{\nu=0}^{\gamma} \binom{\gamma}{\nu} w_{\gamma-\nu\,\iota}{}^{(H-\nu)}, \qquad \gamma = 0, \cdots\cdots, R+H,$$

therefore the components of the extensor $\mathfrak{Y}^H w_{\gamma\iota}$ in our system are indicated by $\mathfrak{Y}^H w_{\gamma'\iota'} = \sum_{\gamma=\gamma'}^{R+H} \lambda_{\gamma'\iota'}^{\gamma\iota} \mathfrak{Y}^H w_{\gamma\iota}$ ($\gamma' = 0, \cdots\cdots, R+H$). Since

$$\begin{aligned} w_{\gamma-\nu\,\iota}{}^{(H-\nu)} &= \Big(\sum_{\delta'=\gamma-\nu}^{R} \lambda_{\gamma-\nu\,\iota}^{\delta'\,j'} w_{\delta' j'}\Big)^{(H-\nu)} \\ &= \sum_{\delta'=\gamma-\nu}^{R} \sum_{\rho=0}^{H-\nu} \binom{H-\nu}{\rho} \lambda_{\gamma-\nu\,\iota}^{\delta'\,j'\,(H-\nu-\rho)} w_{\delta' j'}{}^{(\rho)}, \end{aligned}$$

it will be that

$$\begin{aligned} \mathfrak{Y}^H w_{\gamma'\iota'} &= \sum_{\gamma=\gamma'}^{R+H} \lambda_{\gamma'\iota'}^{\gamma\iota} \sum_{\nu=0}^{\gamma} \sum_{\delta'=\gamma-\nu}^{R} \sum_{\rho=0}^{H-\nu} \binom{H}{\nu}\binom{H-\nu}{\rho} \lambda_{\gamma-\nu\,\iota}^{\delta'\,j'\,(H+\nu-\rho)} w_{\delta' j'}{}^{(\rho)} \\ &= \sum_{\gamma=\gamma'}^{R+H} \lambda_{\gamma'\iota'}^{\gamma\iota} \sum_{\delta'=0}^{R} \sum_{\rho=0}^{H} \binom{H}{\rho} w_{\delta' j'}{}^{(\rho)} \sum_{\nu=0}^{\gamma} \binom{H-\rho}{\nu} \lambda_{\gamma-\nu\,\iota}^{\delta'\,j'\,(H-\nu-\rho)} . \end{aligned}$$

On the other hand, we can write

$$(9.2) \qquad \sum_{\nu=0}^{\gamma} \binom{H-\rho}{\nu} \lambda_{\gamma-\nu\,\iota}^{\delta'\,j'\,(H-\nu-\rho)} = \mathfrak{Y}^{H+\rho} \lambda_{\gamma\,\iota}^{\delta' j'},$$

consequently we have

$$(9.3) \qquad \mathfrak{Y}^H w_{\gamma'\iota'} = \sum_{\delta'=0}^{R} \sum_{\rho=0}^{H} \binom{H}{\rho} w_{\delta' j'}{}^{(\rho)} C^{H-\rho\,\delta' j'}_{\gamma'\iota'}(\mathfrak{Y}),$$

by putting

$$(9.4) \qquad \sum_{\gamma=\gamma'}^{R+H} \lambda_{\gamma'\iota'}^{\gamma\iota} \mathfrak{Y}^{H-\rho} \lambda_{\gamma\,\iota}^{\delta' j'} = C^{H-\rho\,\delta' j'}_{\gamma'\iota'}(\mathfrak{Y}),$$

where we shall call the quantities $C^{H-\rho\,\delta' j'}_{\gamma'\iota'}(\mathfrak{Y})$ the *$\mathfrak{Y}$-operation coefficients of excovariant extensors*. That is,

Theorem 38: *When $w_{\gamma'\iota'}$ are components in the non-holonomic system of an excovariant extensor of characteristic* (1, 0, R, M), *the quantities*

$$\mathfrak{Y}^H w_{\gamma'\iota'} = \sum_{\delta'=0}^{R} \sum_{\rho=0}^{H} \binom{H}{\rho} w_{\delta' j'}{}^{(\rho)} C^{H-\rho\,\delta' j'}_{\gamma'\iota'}(\mathfrak{Y}), \qquad \begin{array}{l} \gamma' = 0, \cdots\cdots, R+H \\ H = 1, 2, \cdots, R \end{array}$$

are components of an excovariant extensor of characteristic (1, 0, $R+H$, $M+H(\leq P)$).

Corollary. $\mathfrak{Y}\,\mathfrak{Y}^H w_{\gamma'\iota'} = \mathfrak{Y}^{H+1} w_{\gamma'\iota'}$.

The corollary is similarly proved as that of Theorem 33.

Remark. Applying Theorem 38 to $f_{\gamma'\iota'}$ ($\gamma' = 0, \cdots\cdots, R-1$), it follows that $\mathfrak{Y}^H f_{\gamma'\iota'} = f^H{}_{\gamma'\iota'}$ where $f^H{}_{\gamma'\iota'} = \sum_{\gamma=\gamma'}^{R+H} \lambda_{\gamma'\iota'}^{\gamma\iota} f^{(H)}_{(\gamma)\iota}$.

Theorem 39: *Let $w_{\gamma'\iota'}$ be non-holonomic components of an excovariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R, R, M)$ and f be a scalar of weight $\theta_1 + \cdots\cdots + \theta_R$, then the quantities $\mathfrak{D}^{*H} w_{\gamma'\iota'} \equiv f \cdot \mathfrak{D}^{H}(w_{\gamma'\iota'}/f)$:*

$$\mathfrak{D}^{*H} w_{\gamma'\iota'} = \mathfrak{D}^{H} w_{\gamma'\iota'} + \sum_{\mu=1}^{H}\sum_{\nu=0}^{H-\mu}\binom{H}{\mu}\binom{H-\mu}{\nu}\left(\frac{1}{f}\right)^{(\mu)} f^{(\nu)}\,\mathfrak{D}^{*H-\mu-\nu} w_{\gamma'\iota'}$$

$$(H = 1,\ 2,\ \cdots\cdots,\ R)$$

are components of an excovariant extensor of characteristic $(1, \theta_1 + \cdots\cdots + \theta_R,\ R+H,\ M+H(\leqq P))$.

Theorem 40: *The $\mathfrak{D}^H$-operation in a non-holonomic system conserve the commutative and linearly associative law.*

Theorem 41: $C^{H-\rho}{}^{\delta' j'}_{\gamma' \iota'}(\mathfrak{D}) = \delta^{H-\rho+\delta'}_{\gamma'}\,\delta^{j'}_{\iota'}$ *for the special case that our system is holonomic.*

Proof. (9.2) will be denoted as follows:

$$\mathfrak{D}^{H-\rho}\lambda^{\delta' j'}_{\gamma\ i} = \sum_{\nu=0}^{\gamma}\binom{H-\rho}{\nu}\lambda_{\gamma-\nu}{}^{\delta'}{}_{i}{}^{j'(H-\rho-\nu)}$$

$$= \sum_{\nu=0}^{\gamma}\binom{H-\rho}{\nu}\binom{\delta'}{\gamma-\nu} X^{j'(H-\rho+\delta'-\gamma)}_{i} = X^{(H-\rho+\delta')j'}_{\gamma\ \ i}$$

from the reason that $\sum_{\nu=0}^{\gamma}\binom{H-\rho}{\nu}\binom{\delta'}{\gamma-\nu} = \binom{H-\rho+\delta'}{\gamma}$. Consequently, we obtain

$$C^{H-\rho}{}^{\delta' j'}_{\gamma' \iota'}(\mathfrak{D}) = \sum_{\gamma=\gamma'}^{R+H} X^{\gamma\ i}_{\gamma'\iota'}\, X^{H-\rho+\delta' j'}_{\gamma\ \ i} = \delta^{H-\rho+\delta'}_{\gamma'}\,\delta^{j'}_{\iota'}\,. \qquad \text{Q. E. D.}$$

Theorem 42: *The quantities $C^{H-\rho}{}^{\delta' j'}_{\gamma' \iota'}(\mathfrak{D})$ are invariant under transformations of holonomic coordinate systems.*

Theorem 43: *The quantities $C^{H-\rho}{}^{\delta' j'}_{\gamma' \iota'}(\mathfrak{D})$ are changed as follows:*

$$C^{H-\rho}{}^{\delta' j'}_{\gamma' \iota'} = \sum_{\alpha'=\gamma'}^{\delta'+H-\rho}\ \sum_{\beta'=0}^{\delta'}\ \sum_{\mu=0}^{H-\rho}\binom{H-\rho}{\mu} N^{\alpha' a'}_{\gamma'\iota'}\, N^{\delta' j'(\mu)}_{\beta' b'}\, C^{H-\rho-\mu}{}^{\beta' b'}_{\alpha' a'}$$

under non-holonomic transformations of the non-holonomic systems.

Proof. According to a non-holonomic transformation of the non-holonomic systems $\lambda^{\gamma\ i}_{\gamma'\iota'} = \sum_{\alpha'=\gamma'}^{\gamma} N^{\alpha' a'}_{\gamma'\iota'}\lambda^{\gamma\ i}_{\alpha' a'}$ and $\lambda_{\gamma-\nu}{}^{\delta'}{}_{i}{}^{j'} = \sum_{\beta'=\gamma-\nu}^{\delta'} N^{\delta' j'}_{\beta' b'}\lambda^{\beta' b'}_{\gamma-\nu\, i}$ $= \sum_{\beta'=0}^{\delta'} N^{\delta' j'}_{\beta' b'}\lambda^{\beta' b'}_{\gamma-\nu\, i}$, we have the following results

$$\mathfrak{D}^{H-\rho}\lambda^{\delta' j'}_{\gamma\ i} = \sum_{\nu=0}^{\gamma}\binom{H-\rho}{\nu}\lambda_{\gamma-\nu}{}^{\delta'}{}_{i}{}^{j'(H-\rho-\nu)}$$

$$= \sum_{\beta'=0}^{\delta'}\ \sum_{\mu=0}^{H-\rho}\ \sum_{\nu=0}^{H-\rho-\mu}\binom{H-\rho}{\mu}\binom{H-\rho-\mu}{\nu} N^{\delta' j'(\mu)}_{\beta' b'}\,\lambda_{\gamma-\nu}{}^{\beta'}{}_{i}{}^{b'(H-\rho-\mu-\nu)}$$

$$= \sum_{\beta'=0}^{\delta'}\ \sum_{\mu=0}^{H-\rho}\binom{H-\rho}{\mu} N^{\delta' j'(\mu)}_{\beta' b'}\,\mathfrak{D}^{H-\rho-\mu}\lambda^{\beta' b'}_{\gamma\ i}\,.$$

Consequently, it follows that

$$C^{H-\rho}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Y}) = \sum_{\gamma=\gamma'}^{\delta'+H-\rho} \sum_{\alpha'=\gamma'}^{\gamma} \sum_{\beta'=0}^{\delta'} \sum_{\mu=0}^{H-\rho} \binom{H-\rho}{\mu} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu)}_{\beta' b'} \lambda^{\gamma\ i}_{\alpha' a'} \mathfrak{Y}^{H-\rho-\mu} \lambda^{\beta' b'}_{\gamma\ i}$$
$$= \sum_{\alpha'=0}^{\delta'+H-\rho} \sum_{\beta'=0}^{\delta'} \sum_{\mu=0}^{H-\rho} \binom{H-\rho}{\mu} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu)}_{\beta' b'} \sum_{\gamma=\alpha'}^{\beta'+H-\rho} \lambda^{\gamma\ i}_{\alpha' a'} \mathfrak{Y}^{H-\rho-\mu} \lambda^{\beta' b'}_{\gamma\ i}$$
$$= \sum_{\alpha'=0}^{\delta'+H-\rho} \sum_{\beta'=0}^{\delta'} \sum_{\mu=0}^{H-\rho} \binom{H-\rho}{\mu} N^{\alpha' a'}_{\gamma' i'} N^{\delta' j'(\mu)}_{\beta' b'} C^{H-\rho-\mu}{}^{\beta' b'}_{\alpha' a'}(\mathfrak{Y}) .$$

§ 10. The product operation of $\mathfrak{S}^H$–, $\mathfrak{Z}^K$– and $\mathfrak{Y}^L$–operations in a non-holonomic system. We shall at last consider properties concerning to the product of any two of three kinds of the operations considered in the previous chapters.

Theorem 44: *For non-holonomic components of an excontravariant extensor $v^{\gamma' i'}$ of characteristic (1, 0, R, M), the $\mathfrak{Y}^L$–operation is interchangeable with the $\mathfrak{S}^H$–operation and $\mathfrak{Z}^K$–operation, i. e.,*

$$\mathfrak{Y}^L \mathfrak{S}^H v^{\gamma' i'} = \mathfrak{S}^H \mathfrak{Y}^L v^{\gamma' i'} , \quad \mathfrak{Y}^H \mathfrak{Z}^K v^{\gamma' i'} = \mathfrak{Z}^K \mathfrak{Y}^H v^{\gamma' i'} ,$$

although the $\mathfrak{S}^H$– and $\mathfrak{Z}^K$–operations are not interchangeable but

$$\mathfrak{S}^H \mathfrak{Z}^K v^{\gamma' i'} = 2^{HK} \mathfrak{Z}^K \mathfrak{S}^H v^{\gamma' i'} .$$

Proof. In according to

$$\mathfrak{Y}^L \mathfrak{S}^H v^{\gamma' i'} = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \mathfrak{Y}^L \mathfrak{S}^H v^{\gamma\ i} , \quad \mathfrak{S}^H \mathfrak{Y}^L v^{\gamma' i'} = \sum_{\gamma=0}^{\gamma'} \lambda^{\gamma' i'}_{\gamma\ i} \mathfrak{S}^H \mathfrak{Y}^L v^{\gamma i}$$

and $\mathfrak{Y}^L \mathfrak{S}^H v^{\gamma i} = \mathfrak{S}^H \mathfrak{Y}^L v^{\gamma i}$ in the holonomic system ([2], p. 50), it follows that $\mathfrak{Y}^L \mathfrak{S}^H v^{\gamma' i'} = \mathfrak{S}^H \mathfrak{Y}^L v^{\gamma' i'}$.

The other statements in the theorem are proved in the same way.

Theorem 45: *The $\mathfrak{S}^H$–, $\mathfrak{Z}^K$– and $\mathfrak{Y}^L$–operations for non-holonomic components of an excovariant extensor of characteristic* (1, 0, R, M) *are interchangeable with each other, i. e.,*

$$\mathfrak{S}^H \mathfrak{Z}^K w_{\gamma' i'} = \mathfrak{Z}^K \mathfrak{S}^H w_{\gamma' i'} , \quad \mathfrak{Z}^K \mathfrak{Y}^L w_{\gamma' i'} = \mathfrak{Y}^L \mathfrak{Z}^K w_{\gamma' i'} ,$$
$$\mathfrak{Y}^L \mathfrak{S}^H w_{\gamma' i'} = \mathfrak{S}^H \mathfrak{Y}^L w_{\gamma' i'} .$$

The method of proof is essentially same as that of Theorem 44.

It may be possible to find many new interesting relations among the derivatives of the new quantities $C^{\rho}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{S})$, $C^{\rho}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Z}^K)$ and $C^{\rho}{}^{\delta' j'}_{\gamma' i'}(\mathfrak{Y}^L)$ by virtue of Theorem 44, as well as Theorem 45.

Remark. In the present paper, the quantities $\lambda^{\gamma' i'}_{\gamma\ i}(x, x^{(1)}, \cdots, x^{(M)})$ $(\gamma, \gamma' = 0, \cdots\cdots, G)$ have been considered as functions of $x, x^{(1)}, \cdots\cdots, x^{(M)}$, but if they depend on only $x, x^{(1)}, \cdots\cdots, x^{(\gamma')}$ where $x^{(\gamma')}$ means the γ'–th derivative of x, we can establish more concrete results which are nearer to the holonomic case. But we shall put off treatments of these relations for the present. (January, 1950)

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